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Polynomials and degrees of maps in real normed algebras

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Abstract. Let $\mathcal{A}$ be the algebra of quaternions $\mathbb{H}$ or octonions $\mathbb{O}$. In this manuscript an elementary proof is given, based on ideas of Cauchy and D'Alembert, of the fact that an ordinary polynomial $f(t) \in \mathcal{A}[t]$ has a root in $\mathcal{A}$. As a consequence, the Jacobian determinant $|J(f)|$ is always non-negative in $\mathcal{A}$. Moreover, using the idea of the topological degree we show that a regular polynomial $g(t)$ over $\mathcal{A}$ has also a root in $\mathcal{A}$. Finally, utilizing multiplication $(*)$ in $\mathcal{A}$, we prove various results on the topological degree of products of maps. In particular, if S is the unit sphere in $\mathcal{A}$ and $h_1, h_2 \colon S \to S$ are smooth maps, it is shown that $\deg(h_1 * h_2) = \deg(h_1) + \deg(h_2)$.

1 Introduction

Eilenberg and Niven in [3], proved the fundamental theorem of algebra (FTA) for quaternions using a degree argument. The key ingredient was Lemma 2 whose proof was depended on the positiveness of $|J(t^n)|$ at the roots of the equation $t^n = \mathbf{i}$ along with the fact that $\deg(t^n) = n$ for $t \in S^4$. Since then, there have been many proofs of the FTA for $\mathcal{A}$, [5], [6], [8], [9]. All of them, however, follow the spirit of [3] and are based on homotopy and degree theory.

In this work we first give an elementary proof, based on ideas of analysis rather than of topology, of the fact that an ordinary polynomial $f(t) \in \mathcal{A}[t]$ has a root in $\mathcal{A}$. Further, it is shown that the Jacobian determinant $|J(f)|$ is always non negative in $\mathcal{A}$, a result similar to the one for holomorphic functions in complex analysis. Using this fact, we also prove various results on the topological degree – in the sense of Brower – of products of maps between spheres. In particular, we show that the degree of $h(t) \colon S^{\mathbf{m}} \to S^{\mathbf{m}}, h(t) = t^k, k \in \mathbb{Z}$ is equal to k, where $\mathbf{m}$ is either 3 or 7.

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We now state some preliminaries needed for this work. We begin with the description of the normed algebras of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. For further reading we refer to the work of John Baez, [1].

Quaternions: An element c of $\mathbb{H}$ is of the form $c = c_0 + \mathbf{i}c_1 + \mathbf{j}c_2 + \mathbf{k}c_3$, where $c_i \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The real part of c is $\mathrm{Re}(c) = c_0$ while the imaginary part $\mathrm{Im}(c) = \mathbf{i}c_1 + \mathbf{j}c_2 + \mathbf{k}c_3$. The norm of c, $|c| = \sqrt{c_0^2 + c_1^2 + c_2^2 + c_3^2}$, its conjugate $\bar{c} = c_0 - \mathbf{i}c_1 - \mathbf{j}c_2 - \mathbf{k}c_3$ while its inverse is $c^{-1} = \bar{c} \cdot |c|^{-2}$, provided that $|c| \neq 0$. Element c is called an *imaginary* unit if $\mathrm{Re}(c) = 0$ and $|c| = 1$, and it has the property $c^2 = -1$. In that regard, multiplication in $\mathbb{H}$ is associative but not commutative.

Octonions: An octonion c is an element of the form $c = c_0 + \sum_{k=1}^{7} \mathbf{e}_k\, c_k$, where $c_0, c_k \in \mathbb{R}$, and $\mathbf{e}_k^2 = -1$. To define the algebra structure of octonions, it is enough to specify the multiplication table for the imaginary elements $\mathbf{e}_1, \ldots, \mathbf{e}_7$.[1] For brevity this can be described as follows: write out seven triples of imaginary elements (1) $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$; (2) $\mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_5$; (3) $\mathbf{e}_1, \mathbf{e}_6, \mathbf{e}_7$; (4) $\mathbf{e}_2, \mathbf{e}_6, \mathbf{e}_4$; (5) $\mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_7$; (6) $\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_7$ and (7) $\mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6$. In each triple, we multiply elements just in the same way as in quaternions. For example, in triple (3) we have: $\mathbf{e}_1\mathbf{e}_6 = -\mathbf{e}_6\mathbf{e}_1 = \mathbf{e}_7$, $\mathbf{e}_6\mathbf{e}_7 = -\mathbf{e}_7\mathbf{e}_6 = \mathbf{e}_1$, $\mathbf{e}_7\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_7 = \mathbf{e}_6$. The real part of c is $\mathrm{Re}(c) = c_0$ while the imaginary part $\mathrm{Im}(c) = \sum_{k=1}^{7} \mathbf{e}_k c_k$. The norm of c,

$$|c| = \sqrt{c_0^2 + c_1^2 + \cdots + c_7^2}\,,$$

its conjugate $\bar{c} = c_0 - \sum_{k=1}^{7} \mathbf{e}_k c_k$ while its inverse is $c^{-1} = \bar{c} \cdot |c|^{-2}$, provided that $|c| \neq 0$. Element c is called an *imaginary* unit if $\mathrm{Re}(c) = 0$ and $|c| = 1$, and it has the property $c^2 = -1$. The algebra of octonions is non commutative, non associative, but it is alternative; that is, for every $a, b \in \mathbb{O}$, we have $(ab)b = a(bb)$ and $b(ba) = (bb)(a)$. Moreover, for any octonions a, b, we have

$$(\ldots((a\underbrace{b)b)\ldots b}_{n \text{ times}}) = ab^n.$$

Throughout this note, $\mathcal{A}$ will stand for either $\mathbb{H}$ or $\mathbb{O}$ equipped with multiplication $(*)$ as defined above. Also, $\mathbf{m}$ will either be 3 or 7 and it shall not be confused with m which might be used as an index. An element $c \in \mathcal{A}$ can also be represented via a real matrix $\mathcal{C}$ so that $x\mathcal{C}^t = c * x$, $x = (x_0, x_1, \ldots, x_{\mathbf{m}})$. For example, if $c = c_0 + \mathbf{i}c_1 + \mathbf{j}c_2 + \mathbf{k}c_3 \in \mathbb{H}$, matrix $\mathcal{C}$ has the form:

$$\mathcal{C} = \begin{bmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & -c_3 & c_2 \\ c_2 & c_3 & c_0 & -c_1 \\ c_3 & -c_2 & c_1 & c_0 \end{bmatrix}$$

Notice that $|\mathcal{C}| = |c|^4$; if $c \in \mathbb{O}$, $|\mathcal{C}| = |c|^8$. The following notation will be needed in the sequel:

[1] There are several ways to define multiplication in $\mathbb{O}$ since the vector product of two elements in $\mathbb{R}^7$ is not unique. We opt to choose this way to conform with multiplication in $\mathbb{H}$ – as a *natural* subset of $\mathbb{O}$ – and identify $\mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$.

Definition 1. For any real-valued $(k \times k)$-matrix B ($k = 4, 8$) and $c \in \mathbb{H}, \mathbb{O}$, we define: $cB \equiv \mathcal{C}B$ and $Bc \equiv B\mathcal{C}$, respectively.

The elements $c_1, c_2 \in \mathcal{A}$ are called *similar*, and denoted by $c_1 \sim c_2$, if $c_1\eta = \eta c_2$ for a non zero $\eta \in \mathcal{A}$. Similarity is an equivalence relation and for $c \in \mathcal{A}$ let us denote by $[c]$ its equivalence class. The following is a useful criterion of similarity, [9]:

Proposition 1. *$c_1, c_2 \in \mathcal{A}$ are similar if and only if $\mathrm{Re}(c_1) = \mathrm{Re}(c_2)$ and $|\mathrm{Im}(c_1)| = |\mathrm{Im}(c_2)|$. Furthermore, any $c \in \mathcal{A}$ is similar to the complex number $\alpha + \mathbf{i}\beta$ with $\alpha = \mathrm{Re}(c)$ and $|\beta| = |\mathrm{Im}(c)|$.*

We may identify $\mathcal{A}$ with $\mathbb{R}^{\mathbf{m}+1}$ via the map

$$(x_0 + \sum_{k=1}^{\mathbf{m}} \mathbf{e}_k x_k) \mapsto (x_0, x_1, \dots, x_{\mathbf{m}})\,.$$

Let $f\colon \mathcal{A} \to \mathcal{A}$. In view of this identification, we can also think of f as a map from $\mathbb{R}^{\mathbf{m}+1} \to \mathbb{R}^{\mathbf{m}+1}$. Indeed, if

$$f(x_0 + \sum_{k=1}^{\mathbf{m}} \mathbf{e}_k x_k) = f_0(x_0, x_1, \dots, x_{\mathbf{m}}) + \sum_{k=1}^{\mathbf{m}} \mathbf{e}_k f_k(x_0, x_1, \dots, x_{\mathbf{m}})$$

we define $f\colon \mathbb{R}^{\mathbf{m}+1} \to \mathbb{R}^{\mathbf{m}+1}$ by $f(x_0, x_1, \dots, x_{\mathbf{m}}) = (f_0, f_1, \dots, f_{\mathbf{m}})$. We can also multiply maps in $\mathcal{A}$. For example, if $\mathcal{A} = \mathbb{H}$ and $f, g\colon \mathbb{H} \to \mathbb{H}$, $f = (f_0, f_1, f_2, f_3)$, $g = (g_0, g_1, g_2, g_3)$, we define

$$\begin{aligned} f * g = (&f_0g_0 - f_1g_1 - f_2g_2 - f_3g_3, f_0g_1 + f_1g_0 + f_2g_3 - f_3g_2, \\ &f_0g_2 + f_2g_0 + f_3g_1 - f_1g_3, f_0g_3 + f_3g_3 + f_1g_2 - f_2g_1). \end{aligned}$$

In addition, if $f\colon \mathcal{A} \to \mathcal{A}$, we set in analogy to the real and complex cases.

Definition 2. The Jacobian $J(f)(c)$, $c \in \mathcal{A}$, is the matrix $\left[\frac{\partial f_i}{\partial x_j}\right], i, j = 0, \dots, \mathbf{m}$ evaluated at c. The determinant of $J(f)$ will be denoted by $|J(f)|$.

2 Polynomials over $\mathcal{A}$

Let $n \in \mathbb{N} \cup \{0\}$ and $t, a_i \in \mathcal{A}$. A "monomial" of degree n is defined as

$$\phi_a^n(t) = a_1 t a_2 t \dots a_n t\,.$$

A finite sum of monomials of degree n will be denoted by $\phi^n(t)$. In the above, special care has to be taken if $\mathcal{A} = \mathbb{O}$, where parenthesis are needed to be taken into account in the definition of $\phi_a^n(t)$.

Definition 3. A polynomial $f(t)\colon \mathcal{A} \to \mathcal{A}$ of degree n over $\mathcal{A}$ is a function of the form

$$f(t) = \sum_{k=0}^{n} \phi^k(t)\,.$$

Polynomial $f(t)$ shall be called *regular* if either $n = 0$ or $\lim_{|t|\to\infty} |\phi^n(t)| = \infty$; otherwise $f(t)$ will be called *non-regular*. Furthermore, $f(t)$ is called *ordinary* if it is of the form

(1) $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$ or

(2) $f(t) = t^n a_n + t^{n-1}a_{n-1} + \cdots + a_0$, $a_i \in \mathcal{A}$

In the former case f is called *left* while in the latter *right*.

Example 1. Let $f_1(t) = \mathbf{i}t^2\mathbf{j} + \mathbf{j}t^2\mathbf{i} - 1$, $f_2(t) = \mathbf{i}t\mathbf{j} + \mathbf{j}t\mathbf{k} + \mathbf{k}t\mathbf{i} + 7$ and $f_3(t)$ be an ordinary polynomial. Then, f_1 is non regular and f_2, f_3 are regular.

If $c \in \mathcal{A}$ with $f(c) = 0$, then c is called a *zero* or a *root* of f. For the sake of brevity we shall call in the sequel, unless otherwise stated, an ordinary polynomial simply *polynomial*. For convenience, we will work with *left* polynomials and note that all the results proven hold true for right polynomials as well. To this end we shall give an elementary proof, based on the ideas of Cauchy and D'Alembert [2], of the fact that every ordinary polynomial $f(t)$ of positive degree has a root in $\mathcal{A}$, (Theorem 1). The proof that every regular polynomial of positive degree has also a root in $\mathcal{A}$ will be deferred to the next section.

First we will need the following:

Remark 1. Let $n \in \mathbb{N}$ and $a \in \mathcal{A}$. Then, the equation $t^n - a = 0$ has a solution in $\mathcal{A}$.

Proof. Let $\alpha + \mathbf{i}\beta \in \mathbb{C}$ be similar to a; that is, $a = \eta(\alpha + \mathbf{i}\beta)\eta^{-1}$. Write $\alpha + \mathbf{i}\beta = |\alpha + \mathbf{i}\beta|e^{i\theta}$ and let $b = |\alpha + \mathbf{i}\beta|^{1/n}e^{i\phi}$, where $\phi = \theta/n$. Then, if $\zeta = \eta b \eta^{-1}$ we have $\zeta^n = \eta b^n \eta^{-1} = a$. $\square$

Now we have:

Lemma 1. *If $f(t)$ is a polynomial of positive degree n, then for every $t_0 \in \mathcal{A}$ with $f(t_0) \neq 0$ and for every $r > 0$, there exists $t \in \mathcal{A}$ with $|t - t_0| < r$ so that $|f(t)| < |f(t_0)|$.*

Proof. We argue by contradiction. Assume then that there exists an $r_0 > 0$ so that for each $|t - t_0| \leq r_0$,

$$0 < |f(t_0)| \leq |f(t)|\,.$$

We may assume that $t_0 \in \mathbb{R}$; for if not, replace t with $u = t * t_0$ and thus $f(1) = f(1 * t_0) = f(t_0)$.

We now consider the polynomial $q(t) = \frac{f(t+t_0)}{f(t_0)}$. Then, $q(t)$ has degree n and constant term equal to 1. Also, observe that $1 = q(0) \leq |q(t)|$ for all $|t| \leq r_0$. Now

$$q(t) = 1 + b_k t^k + \cdots + b_n t^n$$

with $b_k \neq 0$ since $t_0 \in \mathbb{R}$. Let ζ be a solution of the equation $t^k = -\frac{|b_k|}{b_k}$. Note that $|\zeta| = 1$. Let $I = \{r\zeta : 0 < r \leq r_0\}$. For $r\zeta \in I$ we have

$$|q(r\zeta)| \leq |1 + b_k r^k \zeta^k| + |b_{k+1} r^{k+1} \zeta^{k+1}| + \cdots + |b_n r^n \zeta^n|\,.$$

Now we have

$$|1 + b_k r^k \zeta^k| = |1 - r^k |b_k|| = 1 - \rho^k |b_k|\,,$$

for some $\rho < r_0$. Thus we get

$$|q(\rho\zeta)| \leq 1 - \rho^k(|b_k| - |b_{k+1}|\rho - \cdots - |b_n|\rho^{n-k}) .$$

Now for perhaps an even smaller $0 < \rho_1 < \rho$ we will have $|q(r\zeta)| < 1$ for $0 < r < \rho_1$, a contradiction to the fact that $|q(t)| \geq 1$ for all $|t| \leq r_0$. □

Theorem 1. *Let $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$, with $n \geq 1$. Then, f has a root in $\mathcal{A}$.*

Proof. First note that $\lim_{|t|\to\infty}|f(t)| = +\infty$ because $\lim_{|t|\to\infty}|a_n t^n| = +\infty$. Let now

$$\gamma = \inf\{|f(t)| : t \in \mathcal{A}\} .$$

Since $\lim_{|t|\to\infty}|f(t)| = +\infty$, there exists an $r > \gamma$ so that

$$\gamma = \inf\{|f(t)| : |t| \leq r\} .$$

Since the closed ball $B = \{t \in \mathcal{A} : |t| \leq r\}$ is compact and the function $t \to |f(t)|$ is continuous, there must be a t_0, $|t_0| \leq r$ with $\gamma = |f(t_0)|$. Finally observe that $|f(t_0)| \leq |f(t)|$ for $|t| \leq r$. But if $f(t_0) \neq 0$ this contradicts Lemma 1. □

If $g(t) = b_m t^m + b_{m-1}t^{m-1} + \cdots + b_0$ is another polynomial, their product $fg(t)$ is defined in the usual way:

$$fg(t) = \sum_{k=0}^{m+n} c_k t^k , \quad \text{where } c_k = \sum_{i=0}^{k} a_i b_{k-i} .$$

Note that in the above setting the multiplication is performed as if the coefficients were chosen in a commutative field. However, due to the non-commutative nature of $\mathcal{A}$, we have that $(fg)(t) \neq f(t) * g(t)$, when

$$(fg)(t) = F_0 + \sum_{i=1}^{\mathbf{m}} \mathbf{e}_i F_i , \quad f = f_0 + \sum_{i=1}^{\mathbf{m}} \mathbf{e}_i f_i , \quad g = g_0 + \sum_{i=1}^{\mathbf{m}} \mathbf{e}_i g_i .$$

According to Theorem 1 of [6] an element $c \in \mathcal{A}$ is a zero of f if and only if there exists a polynomial $g(t)$ such that $f(t) = g(t)(t-c)$. In that way, f can be factored into a product of linear factors $(t - c_i)$, $c_i \in \mathcal{A}$. Indeed, since $f(t) = g(t)(t-c)$ and $g(t)$ has a root, simple induction shows that

$$f(t) = a_n(t - c_n)(t - c_{n-1}) \dots (t - c_1), \quad c_j \in \mathcal{A}$$

A word of caution: In the above factorization, while c_1 is necessarily a root of f, $c_j, j = 2, \dots, n$, might not be roots of f. For example, the polynomial

$$f(t) = (t + \mathbf{k})(t + \mathbf{j})(t + \mathbf{i}) = t^3 + (\mathbf{i} + \mathbf{j} + \mathbf{k})t^2 + (-\mathbf{i} + \mathbf{j} - \mathbf{k})t + 1$$

has only one root, namely $t = -\mathbf{i}$. Theorem 2.1 of [4] provides a more detailed version of the above factorization in the case $\mathcal{A} = \mathbb{H}$.

Roots of f are distinguished into two types: (i) *isolated* and (ii) *spherical*. A root c of f is called spherical if and only if its characteristic polynomial

$$q_c(t) = t^2 - 2t\,\mathrm{Re}(c) + |c|^2$$

divides f; for any such polynomial, call $\alpha_c \pm \mathbf{i}\beta_c$ its complex roots. In that case any $\gamma \in \mathcal{A}$ similar to c, is also a root of f. For example, if $f(t) = t^2 + 1$, any imaginary unit element $c \in \mathcal{A}$ is a root of f.

Remark 2. The polynomial $f(t)$ has a spherical root if and only if it has roots $\alpha + \mathbf{i}\beta$, $\alpha - \mathbf{i}\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$.

If we write f in the form $f(t) = a_0(t) + \sum_{i=1}^{\mathbf{m}} \mathbf{e}_i a_i(t)$ we see that f has no spherical roots if and only if $\gcd(a_j)_{j=0}^{\mathbf{m}} = 1$. Such an f will be called *primitive*. Then, it is easy to see that a primitive $f(t)$ of degree n, has at most n distinct roots in $\mathcal{A}$.

The conjugate $\bar{f}$ of f is defined as $\bar{f} = a_0 - \sum_{i=1}^{\mathbf{m}} \mathbf{e}_i a_i$. Note that

$$\bar{f} f = a_0^2 + a_1^2 + \cdots + a_{\mathbf{m}}^2 \,,$$

which is a real positive polynomial. Observe that if $\alpha + \mathbf{i}\beta$ is a root of $\bar{f}f$, then there exists $c \in \mathcal{A}$, similar to $\alpha + \mathbf{i}\beta$ so that $f(c) = 0$ [6, Theorem 4, p. 221].

Definition 4. Let $\phi(t) \in \mathbb{C}[t]$ and $\zeta \in \mathbb{C}$ be a root of ϕ. We denote by $\mu(\phi)(\zeta)$ the multiplicity of ζ. Now let $c \in \mathcal{A}$ be a root of f and let $m = \mu(\bar{f}f)(\alpha_c + \mathbf{i}\beta_c)$. Then,

(1) if c is isolated, we define its multiplicity $\mu(f)(c)$, as a root of f, to be m;

(2) if c is spherical, its multiplicity is set to be $2m$.

Note that:

Remark 3. Let t_0 be a root of f and write $f(t) = g(t)(t - t_0)$. Then, t_0 is simple (multiple) if and only if $g(t_1) \neq 0$, $(g(t_1) = 0)$ for any $t_1 \sim t_0$ respectively. Moreover, a *primitive* polynomial f has simple roots if and only if $\bar{f}f$ does.

2.1 $|J(f)| \geq 0$ over $\mathcal{A}$

In this paragraph we will show that $|J(f)|$ of a polynomial f is non negative over $\mathbb{O}$; the case of $\mathcal{A} = \mathbb{H}$ is similar. In particular, we will prove that if t_0 is a root of f, t_0 is simple if and only if $|J(f)(t_0)| > 0$. Thus, at a multiple root $|J(f)|$ vanishes.

A first indication of $|J(f)(t)|$ being non negative is when $t \in \mathbb{R}$. Indeed, if $t = r \in \mathbb{R}$, divide $f(t) - f(r)$ by $(t - r)$ to get

$$f(t) - f(r) = g(t)(t - r)\,.$$

Since r commutes with every element of $\mathcal{A}$, we see that

$$f(t) - f(r) = g(t) * (t - r)\,.$$

Thus,

$$|J(f)(r)| = |g(r)|^8 |I| \geq 0 .$$

Now let

$$t_0 = \tau_0 + \sum_{k=1}^{7} \mathbf{e}_k \tau_k = \tau_0 + \tau \in \mathcal{A} .$$

Since

$$J(f)(t_0) = J(f - f(t_0))(t_0)$$

we may assume that $f(t_0) = 0$. Further, by replacing f with $f(t + \tau_0)$ we see that $f(\tau) = 0$. Now let $\eta \in \mathcal{A}$, $|\eta| = 1$ so that $\eta * \tau * \eta^{-1} = \mathbf{i}s$, $s = |\tau|$. If $u = \eta^{-1} t \eta$ and $F(t) = (f \circ u)(t)$, then $F(\mathbf{i}s) = 0$ and

$$|J(F)| = |J(f(u))||J(u)| .$$

But $|J(u)| = 1$ since $J(u)$ is nothing but an orthogonal matrix. Finally, by replacing t with t/s in $F(t)$ we may assume that $f(\mathbf{i}) = 0$. Therefore, it is enough to show that $|J(f)(\mathbf{i})| \geq 0$.

Divide $f(t)$ by $t - \mathbf{i}$ to get $f(t) = g(t)(t - \mathbf{i})$. Let

$$g(t) = b_m t^m + \cdots + b_1 t + b_0 ,$$

$b_k \in \mathcal{A}$. We write

$$f(t) = b_0(t - \mathbf{i}) + b_1 t(t - \mathbf{i}) + \cdots + b_m t^m (t - \mathbf{i}) .$$

Let A be the matrix

$$A = \begin{bmatrix} N & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \\ 0 & 0 & -N & 0 \\ 0 & 0 & 0 & -N \end{bmatrix}, \quad \text{where } N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Notice that $A^2 = -I$. Furthermore, we have

Lemma 2. *$J(t^k(t - \mathbf{i}))(\mathbf{i}) = A^k$ for $k \geq 0$.*

Proof. We use induction on k. When $k = 0$ we have $J(t - \mathbf{i}) = I = A^0$. Now we have

$$t^{k+1}(t - \mathbf{i}) = (t^{k+1} - \mathbf{i}t^k)t .$$

Set

$$t^{k+1} - \mathbf{i}t^k = a^0 + \sum_{m=1}^{7} \mathbf{e}_m a^m ,$$

$$t = x_0 + \sum_{m=1}^{7} \mathbf{e}_m x_m .$$

Then, a calculation shows that

$$J(t^{k+1}(t-\mathbf{i}))(\mathbf{i}) = \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_4 \end{bmatrix},$$

where

$$M_1 = \begin{bmatrix} -a^1_{x_0} & -a^1_{x_1} \\ a^0_{x_0} & a^0_{x_1} \end{bmatrix}, \qquad M_2 = \begin{bmatrix} a^3_{x_2} & a^3_{x_3} \\ -a^2_{x_2} & -a^2_{x_3} \end{bmatrix},$$

$$M_3 = \begin{bmatrix} a^5_{x_4} & a^5_{x_5} \\ -a^4_{x_4} & -a^4_{x_5} \end{bmatrix}, \qquad M_4 = \begin{bmatrix} a^7_{x_6} & a^7_{x_7} \\ -a^6_{x_6} & -a^6_{x_7} \end{bmatrix}.$$

But then,

$$J(t^{k+1}(t-\mathbf{i}))(\mathbf{i}) = A\left(J(t^k(t-\mathbf{i}))(\mathbf{i})\right) = A \cdot A^k .$$

This finishes induction and the proof. □

In view of the above Lemma we get

$$J(f)(\mathbf{i}) = b_0 I + b_1 A + b_2 A^2 + \cdots + b_m A^m = \sum_{k=0}^{m} (-1)^k b_{2k} I + \sum_{l=0}^{m} (-1)^l b_{2l+1} A .$$

Set

$$\sum_{k=0}^{m} (-1)^k b_{2k} = B_e , \quad \sum_{l=0}^{m} (-1)^l b_{2l+1} = B_o .$$

We claim that $|B_e I + B_o A| \geq 0$. Indeed, if either of B_e, B_o is zero there is nothing to prove. Suppose then $B_e * B_o \neq 0$. Then it is enough to show $|I + CA| \geq 0$ for $C = B_o/B_e$. If

$$C = \gamma_0 + \sum_{i=1}^{7} \mathbf{e}_i \gamma_i = \gamma_0 + \gamma ,$$

a calculation – via Maple – shows that

$$|I + CA| = \left[(1 - |\gamma|^2)^2 + 2\gamma_0^2(1 + |C|^2)\right]\left[(\gamma_1 - 1)^2 + |C|^2 - \gamma_1^2\right]^2$$

The above proves the claim. Moreover, $|I + CA|$ vanishes precisely when $C = \gamma$ or $C = \mathbf{i}$; that is C is an imaginary unit. In short, $|B_e I + B_o A| = 0$ if and only $B_e + B_o \delta = 0$, for a suitable imaginary unit δ.

Let $\delta \in \mathcal{A}$ be an imaginary unit. Recall that $\delta \sim \mathbf{i}$. Then, $g(\delta) = B_e + B_o \delta$, since $\delta^2 = -1$. Thus, if $g(\delta) \neq 0$, which in turn says that $\mu(f)(\mathbf{i}) = 1$,

$$|B_e I + B_o A| > 0 .$$

On the other hand, if $g(\gamma) = 0$, which means that $\mu(f)(\mathbf{i}) \geq 2$, then $|J(f)(\mathbf{i})|$ vanishes, as required. We summarize the above into the following:

Theorem 2. *Let $f(t)$ be a polynomial of positive degree. Then, $|J(f)(t)| \geq 0$ for all $t \in \mathcal{A}$. Moreover, if t_0 is a root of f, t_0 is* simple *if and only if $|J(f)(t_0)| > 0$.*

3 Degrees of maps in $\mathcal{A}$

In this section we will prove that any regular polynomial of positive degree over $\mathcal{A}$ has a root in $\mathcal{A}$. In addition, we will show that the topological degree of products of maps is additive.

Let $f\colon \mathcal{A} \to \mathcal{A}$ be a smooth map with the property that $\lim_{|t|\to\infty}|f(t)| = \infty$. Then, if Σ is the spherical compactification of $\mathcal{A}$ [Σ is $S^{\mathbf{m}+1}$], f can be smoothly extended to give $\hat{f}\colon \Sigma \to \Sigma$; i.e. $\hat{f}(t) = f(t)$, $t \in \mathcal{A}$ and $\hat{f}(\infty) = \infty$. In that case we define $\deg(f) := \deg(\hat{f})$. Now suppose that $g\colon \mathcal{A} \to \mathcal{A}$ is also smooth with the same property as f. Consider the map $F(t) := f(t) * g(t)\colon \mathcal{A} \to \mathcal{A}$. Obviously $\lim_{|t|\to\infty}|F(t)| = \infty$. Here is our first result:

Proposition 2. *Let f, g, F be as above. Then,* $\deg(F) = \deg(f) + \deg(g)$.

Proof. According to Sard's Theorem, there exist $\mathbf{h}, \mathbf{z} \in \mathcal{A}$ so that:

(1) $\mathbf{h}, \mathbf{z}$ are regular values of f, g and

(2) the equations $f(t) - \mathbf{h} = 0$, $g(t) - \mathbf{z} = 0$ have no common roots.

Since Σ is compact, the sets

$$A = \{t \in \mathcal{A} : f(t) - \mathbf{h} = 0\}$$

and

$$B = \{t \in \mathcal{A} : g(t) - \mathbf{z} = 0\}$$

are finite. Then,

$$\deg(f) = \sum_{a\in A} \operatorname{sign}|J(f)(a)|$$

and

$$\deg(g) = \sum_{b\in b} \operatorname{sign}|J(g)(b)|\,.$$

Let

$$\Phi(t) = (f(t) - \mathbf{h}) * (g(t) - \mathbf{z})\,.$$

Obviously, $\lim_{|t|\to\infty}|\Phi(t)| = \infty$. In that case for $a \in A$, $b \in B$ we have

$$\operatorname{sign}|J(\Phi)(a)| = \operatorname{sign}|J(f)(a)| \cdot |g(a) - \mathbf{z}|^{\mathbf{m}+1}\,,$$
$$\operatorname{sign}|J(\Phi)(b)| = \operatorname{sign}|J(g)(b)| \cdot |f(b) - \mathbf{h}|^{\mathbf{m}+1}\,.$$

The above calculation shows that 0 is a regular value of Φ and thus $\deg(\Phi) = \deg(f) + \deg(g)$. Finally, since

$$\Phi = f * g - f * \mathbf{z} - \mathbf{h} * g + \mathbf{h} * \mathbf{z}\,,$$

Φ is homotopic to F via the homotopy

$$\phi_r(t) = (f * g)(t) + (1 - r)((-f * \mathbf{z} - \mathbf{h} * g + \mathbf{h} * \mathbf{z})(t))$$

for $t \in \mathcal{A}$ and $\phi_r(\infty) = \infty$, $0 \le r \le 1$. $\square$

Using the result above and simple induction, we get:

Corollary 1. *The degree of* $t^n\colon \mathcal{A} \to \mathcal{A}$, $n \in \mathbb{N}$ *is equal to* n.

Note that Corollary 1 is the same as Lemma 4 of [8] which was proved using K-theoretical tools.

Now, we are ready to prove the Fundamental Theorem of Algebra over $\mathcal{A}$.

Theorem 3. *Any* regular *polynomial of positive degree over* $\mathcal{A}$ *has a root in* $\mathcal{A}$.

Proof. Let $f(t) = \sum_{k=0}^{n} \phi^k(t)$ be regular with $n \geq 1$. Then, since

$$\lim_{|t|\to\infty} |\phi^n(t)| = \infty,$$

we see that $\lim_{|t|\to\infty}|f(t)| = \infty$ as well. A slight modification of the proof of Lemma 1 of [3] shows that f is homotopic to the map $t^n\colon \mathcal{A} \to \mathcal{A}$. Corollary 1 shows that $\deg(f) = n$ and therefore f is onto. $\square$

Regularity is a necessary condition for a polynomial to have roots as the following example indicates:

Example 2. The polynomial $f(t) = \mathbf{i}t^2\mathbf{j} + \mathbf{j}t^2\mathbf{i} - 1$ has no roots over $\mathbb{H}$.

In the above, observe that $\lim_{|t|\to\infty}|\mathbf{i}t^2\mathbf{j} + \mathbf{j}t^2\mathbf{i}|$ does not exist.

Let now S denote the unit sphere in $\mathcal{A}$; i.e. $S = S^{\mathbf{m}}$. Obviously S is equipped with the multiplication $(*)$ in $\mathcal{A}$. Let $k \in \mathbb{Z}$ and consider the map $h(t)\colon S \to S$, $h(t) = t^k$. We then have:

Lemma 3. *The degree of* h *is equal to* k.

Proof. If $k = 0$, h is constant and thus has degree 0. Suppose first that $k \geq 1$. For a fixed $0 < r < 1$, consider the polynomial $p(t) = t^k - \mathbf{i}r$. Then, p has simple roots (Remark 3) $\rho_i \in \mathcal{A}$, $i = 1, \ldots, k$ with $\rho_i \in \mathrm{Int}(S)$ since $|\rho_i| = r < 1$. Further, from Theorem 2 we get $|J(p)(\rho_i)| > 0$ for each i. Let $g(t)\colon S \to S$ be defined by $g(t) = \frac{p(t)}{|p(t)|}$. In that case,

$$\deg(g) = \sum_{i=1}^{k} \mathrm{sign}|J(p)(\rho_i)| = k\,,$$

[7, Lemma 3, p. 36]. We now claim that $|g(t) - h(t)| < 2$ for $t \in S$. Indeed, note first that $|g(t) - h(t)| \leq 2$. Furthermore, if for some $t_0 \in S$,

$$|g(t_0) - h(t_0)| = 2\,,$$

we must have $g(t_0) = -h(t_0)$; that is

$$t_0^k(1 + |t_0^k - \mathbf{i}r|) = \mathbf{i}r\,,$$

a contradiction to $0 < r < 1$. Thus, $|g(t) - h(t)| < 2$ and this shows that h and g are homotopic, [7, p. 52]. Therefore, $\deg(g) = \deg(h) = k$.

Let now $k = -1$. Then, $h(t) = t^{-1}$ is nothing but a composition of m reflections. Indeed, if $S = S^3$ and

$$\begin{aligned} f_1 &= (x, -y, z, w)\,, \\ f_2 &= (x, y, -z, w)\,, \\ f_3 &= (x, y, z, -w)\,, \end{aligned}$$

then

$$h(x, y, z, w) = (x, -y, -z, -w) = (f_1 \circ f_2 \circ f_3)(x, y, z, w)\,.$$

The case of $S = S^7$ is similar. Therefore, $\deg(h) = (-1)^m$.

Finally, let $k \leq -2$. If $f(t) = t^{-1}$ we observe that $(h \circ f)(t) = t^{-k}$ and thus $\deg(h)\deg(f) = -k$, or $\deg(h) = k$. This finishes the proof. $\square$

Let now $f, g\colon S \to S$ be smooth and $F\colon S \to S$ be defined by $F(t) = f(t) * g(t)$. Then,

Theorem 4. $\deg(F) = \deg(f) + \deg(g)$.

Proof. Let $\deg(f) = n$, $\deg(g) = k$. According to Hopf's Theorem, [7], f, g are smoothly homotopic to $f_1, g_1\colon S \to S$ where $f_1(t) = t^n$ and $g_1(t) = t^k$. Let

$$\phi_r(t), \psi_r(t)\colon [0,1] \times S \to S$$

be smooth maps so that

$$\phi_r(0) = f(t)\,, \quad \phi_r(1) = f_1$$

and

$$\psi_r(0) = g(t)\,, \quad \psi_r(1) = g_1\,.$$

Define

$$\Phi_r(t) = \phi_r(t) * \psi_r(t)\colon [0,1] \times S \to S\,.$$

Notice that $\Phi_r(t)$ is continuous and

$$\begin{aligned} \Phi_r(1) &= f(t) * g(t)\,, \\ \Phi_r(0) &= f_1(t) * g_1(t)\,. \end{aligned}$$

Thus, $f * g$ is homotopic to the map $t \to t^{n+k}$. The latter implies that

$$\deg(F) = n + k = \deg(f) + \deg(g)$$

as required. $\square$

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