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Solutions of the Diophantine Equation $7X^2 + Y^7 = Z^2$ from Recurrence Sequences

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Abstract. Consider the system $x^2 - ay^2 = b$, $P(x, y) = z^2$, where P is a given integer polynomial. Historically, the integer solutions of such systems have been investigated by many authors using the congruence arguments and the quadratic reciprocity. In this paper, we use Kedlaya's procedure and the techniques of using congruence arguments with the quadratic reciprocity to investigate the solutions of the Diophantine equation $7X^2 + Y^7 = Z^2$ if $(X, Y) = (L_n, F_n)$ (or $(X, Y) = (F_n, L_n)$) where $\{F_n\}$ and $\{L_n\}$ represent the sequences of Fibonacci numbers and Lucas numbers respectively.

1 Introduction

The Lucas sequences $\{U_n(P, Q)\}$, with the parameters P and Q, are defined by

$$U_0(P, Q) = 0\,, \quad U_1(P, Q) = 1 \quad \text{and} \quad U_n(P, Q) = PU_{n-1} - QU_{n-2}\,,$$

for $n \geq 2$, and the associated Lucas sequences $\{V_n(P, Q)\}$ are defined similarly with the initial terms

$$V_0(P, Q) = 2 \quad \text{and} \quad V_1(P, Q) = P\,.$$

Terms of Lucas sequences and associated Lucas sequences satisfy the identity

$$V_n(P, Q)^2 - DU_n(P, Q)^2 = 4Q^n,$$

where $D = P^2 - 4Q$. It is easy to see that the sequences of Fibonacci numbers and Lucas numbers are $\{F_n\} = \{U_n(1, -1)\}$ and $\{L_n\} = \{V_n(1, -1)\}$ respectively. On the other hand, the Diophantine equation of the form $AX^2 + BY^r = CZ^2$,

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where A, B, C, and r are nonzero integers such that $r > 1$, has either no integer solutions or infinitely many nontrivial solutions in integers X, Y, and Z (see, e.g., [7] or [11]). In this paper, we investigate the integer solutions (X, Y, Z) of the Diophantine equation

$$7X^2 + Y^7 = Z^2. \tag{1}$$

According to [11, page 111] the solutions of the above equation can be parametrized as

$$\begin{aligned} X &= 7a_1^6 b_1 + 245a_1^4 b_1^3 + 1029a_1^2 b_1^5 + 343b_1^7, \\ Y &= a_1^2 - 7b_1^2, \\ Z &= a_1^7 + 147a_1^5 b_1^2 + 1715a_1^3 b_1^4 + 2401a_1 b_1^6, \end{aligned}$$

where a_1 and b_1 are arbitrary integers, which provide infinitely many solutions of equation (1). In this paper, we deal with special solutions of this equation, namely where $(X, Y) = (L_n, F_n)$ (or $(X, Y) = (F_n, L_n)$), which are clearly equivalent to the solutions of the systems

$$L_n^2 - 5F_n^2 = \pm 4\,, \quad 7L_n^2 + F_n^7 = Z^2$$

and

$$L_n^2 - 5F_n^2 = \pm 4\,, \quad L_n^7 + 7F_n^2 = Z^2\,.$$

In other words, we examine the solutions to the following systems of Diophantine equations

$$x^2 - 5y^2 = \pm 4, \quad 7x^2 + y^7 = z^2, \tag{2}$$

$$x^2 - 5y^2 = \pm 4, \quad x^7 + 7y^2 = z^2, \tag{3}$$

where $x = L_n$, $y = F_n$, and $z = Z$. A solution (x, y, z) of any system in (2) or (3) represents a solution (x, y) of one of its special Pell equations with the restriction given by the corresponding equation.

Historically, several authors investigated the existence and nonexistence of the integer solutions of certain systems of Diophantine equations of the form

$$x^2 - ay^2 = b, \quad P(x, y) = z^2, \tag{4}$$

where a is a positive integer that is not a perfect square, b is a nonzero integer, and $P(x, y)$ is a polynomial with integer coefficients. Many of the studies of systems of the form (4) use Baker's results on linear forms in logarithms of algebraic numbers [2] to give an upper bound on the size of the solutions. Using this bound with some techniques of Diophantine approximation, Baker and Davenport [3] proved that there is no solution in nonnegative integers other than $(x, y, z) = (1, 1, 1)$ or $(19, 11, 31)$ for the system

$$x^2 - 3y^2 = -2, \quad z^2 - 8y^2 = -7.$$

Brown [4] proved that the equations

$$y^2 - 8t^2 = 1, \quad u^2 - 5t^2 = 1,$$

have no common solution other than $(y, t, u) = (1, 0, 1)$ using Grinstead's technique [8]. Szalay [14] presented an alternative procedure for solving systems of simultaneous Pell equations

$$a_1x^2 + b_1y^2 = c_1, \quad a_2x^2 + b_2z^2 = c_2$$

in nonnegative integers x, y and z, with relatively small coefficients. He implemented the algorithm of this procedure in Magma to verify famous examples and give a new theorem related to such systems. In general, one can guarantee the finiteness of the number of solutions of (4) by the work of Thue [15] or Siegel [13]. On the other hand, many authors have given elementary solutions to systems of the form (4) such as Cohn [5], who considered the case where P is a linear polynomial. Cohn's method uses congruence arguments to eliminate some cases, and a clever invocation of quadratic reciprocity to handle the remaining cases. The congruence arguments are very sufficient if there exists no solution in such a system, however they fail in the presence of a solution. Mohanty and Ramasamy [10] adapted this method to show that the system of equations

$$x^2 - 5y^2 = -20\,, \quad z^2 - 2y^2 = 1,$$

has no solution other than $(x, y, z) = (0, 2, 3)$. Muriefah and Al Rashed [1] showed that the system

$$x^2 - 5y^2 = 4\,, \quad z^2 - 442x^2 = 441$$

has no integer solutions using a similar method to that presented by Mohanty and Ramasamy.

Additionally, Peker and Cenberci [12] proved that the system

$$y^2 - 10x^2 = 9\,, \quad z^2 - 17x^2 = 16$$

cannot be solved simultaneously in nonzero integers x, y, z using the same method with Muriefah and Rashed. Kedlaya [9] gave a general procedure, based on the methods of Cohn and the theory of Pell equations, that solves many systems of the form (4). In fact, he applied this approach on several examples, in which P is univariate with degree at most two. Moreover, in some cases this procedure fails to solve a system completely. To investigate the solutions of the Diophantine equation (1) from the sequences of Fibonacci numbers and Lucas numbers, we use Kedlaya's procedure and similar techniques adapted by the methods of Mohanty and Ramasamy, Muriefah and Rashed, and Peker and Cenberci to determine and prove whether each of the four systems of equations in (2) and (3) has a solution. We employ Kedlaya's procedure and the techniques of using the congruence arguments and the quadratic reciprocity to prove that the system

$$x^2 - 5y^2 = 4\,, \quad 7x^2 + y^7 = z^2$$

has no more solutions other than $(x, y, z) = (3, 1, \pm 8)$, and each of the other three systems can not be solved simultaneously.

2 Auxiliary results

For the proofs of our theorems, we need the following Lemma 1 presented by Copley [6], Lemmas 2, 3 and 4 and a procedure presented by Kedlaya [9] for checking if a given list of solutions to a system of the form (4) is complete, and a remark shows the general forms of nonnegative solutions for the Pell type equations

$$x^2 - 5y^2 = \pm 4\,.$$

Lemma 1. *Let $(x_k + y_k\sqrt{a})$, where $k = 0, 1, 2, 3, \ldots$, be the solution of $x^2 - ay^2 = b$ in a fixed class C, where b is a given nonzero integer and a is a positive integer which is not a square, then*

$$x_{-k} = x_k, \quad y_{-k} = -y_k, \tag{5}$$

$$x_{k+r} = u_r x_k + av_r y_k, \tag{6}$$

$$y_{k+r} = u_r y_k + v_r x_k, \tag{7}$$

where $(u_r + v_r\sqrt{a}) = (u_1 + v_1\sqrt{a})^r$ such that (u_1, v_1) is the fundamental solution of the Pell equation $u^2 - av^2 = 1$.

Lemma 2. *For all k, ω, r we have $y_{k+2\omega r} \equiv (-1)^\omega y_k \pmod{u_r}$ and $y_{k+2\omega r} \equiv y_k \pmod{v_r}$.*

(Of course, the same result holds for u_k, v_k, or x_k as well).

Lemma 3. *For all k, ω we have $v_k \mid v_{\omega k}$; if ω is odd, we also have $u_k \mid u_{\omega k}$.*

Lemma 4. *If the sequence $\{f_k\}$ satisfies the recurrence relation*

$$f_{k+1} = 2f_k u_1 - f_{k-1}$$

then for any positive integer χ, $\{f_k \pmod{\chi}\}$ is completely periodic.

(Of course, the same result holds for $f_k = u_k, v_k, x_k$, or y_k as well).

The Procedure: Denote (u_k, v_k) be the k-th solution of the Pell equation

$$u^2 - av^2 = 1.$$

For each base solution (x_0, y_0) of the equation $x^2 - ay^2 = b$, let S be the set of integers m such that (x_m, y_m) is in the given list of solutions. One can prove that $P(x_m, y_m)$ is a prefect square if and only if $m \in S$ as follows (without having to give up):

- For each $m \in S$, let $\alpha = P(-x_m, -y_m)$.
- If $|\alpha|$ is a perfect square, we give up; otherwise, let β be the product of all the primes that divide α an odd number of times.
- Let l be the period of $\{u_k \pmod{\beta}\}$ (the period is guaranteed by Lemma 4) and d be the largest odd divisor of l.

- Let q be the largest integer such that $2^q \mid l$, unless 4 does not divide l, in which case let $q = 2$.
- Let s be the order of 2 in the group $(\mathbb{Z}/d\mathbb{Z})^\times$.
- Define the set, $U = \left\{t \in \{0, \dots, d-1\} : \left(\frac{u_{2^q t}}{\beta}\right) = -1\right\}$.
- If U is empty, we give up; otherwise find an odd number j such that for each $\varepsilon = q, \dots, q+s-1$, there exist $t \in U$ and $g \mid j$ with $2^{\varepsilon - q} g \equiv t \pmod{\beta}$.
- Let $\gamma_m = 2^q j$ and γ be twice the least common multiple of γ_m for all $m \in S$.
- Find an integer δ with the following property: for every $k \in \{0, \dots, \delta\gamma - 1\}$, either $k \equiv m \pmod{2\gamma_m}$ for some $m \in S$; or there exists a prime number p such that $P(x_k, y_k)$ is a nonresidue mod p, with $\{x_i \pmod{p}\}$ and $\{y_i \pmod{p}\}$ have periods dividing $\delta\gamma$. Using Lemmas 2 and 3, we note that the period condition can be guaranteed by having $p \mid v_\kappa$ for some κ where $2\kappa \mid \delta\gamma$.
- Suppose that δ can be found satisfying the specified properties. To show that $P(x_m, y_m)$ is a prefect square if and only if $m \in S$, assume that there exists $k \notin S$ such that $P(x_k, y_k)$ is a perfect square. By the construction of δ, there exists m such that $k \equiv m \pmod{2\gamma_m}$, or else there exists a prime number p such that $P(x_k, y_k)$ is a nonresidue $\pmod{p}$. Since $k \notin S$, so $k \neq m$ and $k = m + 2^{\varepsilon+1} jh$ for some h, ε with h odd and $\varepsilon \geq q$. Using Lemma 2, we get $x_k \equiv -x_m \pmod{u_{j2^\varepsilon}}$ and $y_k \equiv -y_m \pmod{u_{j2^\varepsilon}}$. Therefore,

$$P(x_k, y_k) \equiv P(-x_m, -y_m) = \alpha \pmod{u_{j2^\varepsilon}}.$$

  The construction gives that for some $t \in U$ and some $g \mid j$ with $2^{\varepsilon - q} g \equiv t \pmod{\beta}$. It is clear that $\varepsilon \geq q \geq 2$ and $\{u_k \pmod{8}\}$ has period dividing 4. Thus, the Jacobi symbols $\left(\frac{-1}{u_{2^\varepsilon g}}\right)$ and $\left(\frac{2}{u_{2^\varepsilon g}}\right)$ both equal 1. Since $|\alpha|/\beta$ is a perfect square and $u_{g2^\varepsilon} \mid u_{j2^\varepsilon}$ by Lemma 3, we have by quadratic reciprocity

$$\left(\frac{P(x_k, y_k)}{u_{2^\varepsilon g}}\right) = \left(\frac{\alpha}{u_{2^\varepsilon g}}\right) = \left(\frac{\beta}{u_{2^\varepsilon g}}\right) = \left(\frac{u_{2^\varepsilon g}}{\beta}\right) = \left(\frac{u_{2^q t}}{\beta}\right) = -1,$$

  which contradicts the assumption that $P(x_k, y_k)$ is a perfect square.

Remark 1. The Pell equation $u^2 - 5v^2 = 1$ has the fundamental solution $(u_1, v_1) = (9, 4)$, and the Pell type equation $x^2 - 5y^2 = 4$ has three non associated classes of solutions with the fundamental solutions $3 + \sqrt{5}$, $3 - \sqrt{5}$, and 2. Therefore, its general solutions are given by

$$x_k + y_k\sqrt{5} = (3 + \sqrt{5})(9 + 4\sqrt{5})^k, \tag{8}$$

$$x_k + y_k\sqrt{5} = (3 - \sqrt{5})(9 + 4\sqrt{5})^k, \tag{9}$$

$$x_k + y_k\sqrt{5} = (2)(9 + 4\sqrt{5})^k, \tag{10}$$

respectively. Similarly, the general solutions of the Pell type equation $x^2-5y^2=-4$ are given by

$$x_k + y_k\sqrt{5} = (1+\sqrt{5})(9+4\sqrt{5})^k, \tag{11}$$

$$x_k + y_k\sqrt{5} = (-1+\sqrt{5})(9+4\sqrt{5})^k, \tag{12}$$

$$x_k + y_k\sqrt{5} = (4+2\sqrt{5})(9+4\sqrt{5})^k, \tag{13}$$

respectively.

3 Main results

Theorem 1. *Suppose that* $X = L_n$ *and* $Y = F_n$*, then the Diophantine equation* (1) *has no more solutions other than* $(X,Y,Z) = (3,1,\pm 8)$.

Proof. To prove this theorem, we have to show that $(3,1,8)$ and $(3,1,-8)$ are the only solutions to the systems of the simultaneous Diophantine equations in (2). In fact, they are the only solutions to the system

$$x^2 - 5y^2 = 4\,, \tag{14}$$

$$7x^2 + y^7 = z^2\,, \tag{15}$$

where $x = L_n$, $y = F_n$ and $z = Z$. Now, let $P(x,y) = 7x^2 + y^7$. Considering equation (8) and using Kedlaya's procedure, it is possible to show that $P(x_m, y_m)$ is a perfect square if and only if $m \in S = \{0\}$ and the set

$$\{(x_0, y_0, z)\} = \{(3,1,-8), (3,1,8)\}$$

is a complete list of solutions to the system of the Diophantine equations (14) and (15) with the procedure's output: $\alpha = \beta = 62$, $l = d = 5$, $q = 2$, $s = 3$, $U = \{2,3\}$, $\gamma_m = 60$, $\gamma = 120$, and $\delta = 1$ such that for $k = 0$, $k \equiv m \equiv 0 \pmod{120}$. Following the last step in the procedure, one can easily show that there exists no k other than $k = 0$ such that $k \equiv 0 \pmod{120}$ and $P(x_k, y_k)$ is a perfect square. Assume, for the sake of contradiction, that there exists $k \notin S$ such that $k \equiv 0 \pmod{120}$ and $P(x_k, y_k)$ is a perfect square. Therefore, $k = 2^{\varepsilon+1}jh = 2^{\varepsilon+1}15h$ for some h, ε with h odd and $\varepsilon \geq q = 2$. Using Lemma 2, we obtain

$$x_k \equiv -x_0 = -3 \pmod{u_{2^\varepsilon 15}} \quad \text{and} \quad y_k \equiv -y_0 = -1 \pmod{u_{2^\varepsilon 15}}\,,$$

which imply $P(x_k, y_k) \equiv P(-3,-1) = 62 = \alpha \pmod{u_{2^\varepsilon 15}}$. Since

$$2^{\varepsilon - q} g \equiv t \pmod{\beta}$$

for some $t \in U = \{2,3\}$ and some $g \mid 15$ and $|\alpha|/\beta$ is equal 1 which is a perfect square, we get $u_{2^\varepsilon g} \mid u_{2^\varepsilon 15}$ by Lemma 3. Moreover, we have the Jacobi symbol $\left(\frac{2}{u_{2^\varepsilon g}}\right)$ is equal 1. Therefore, we obtain by the quadratic reciprocity that

$$\left(\frac{P(x_k,y_k)}{u_{2^\varepsilon g}}\right) = \left(\frac{62}{u_{2^\varepsilon g}}\right) = \left(\frac{u_{2^\varepsilon g}}{62}\right) = \left(\frac{u_{2^2 t}}{62}\right) = \left(\frac{37}{62}\right) = -1$$

for all t, contradicting the assumption that $P(x_k, y_k)$ is a perfect square. Next, we consider $k \neq 0$. From equations (6) and (7) in Lemma 1, we can write

$$x_{k+15} = (3220013013190122249)x_k + 5(1440033597185408060)y_k, \tag{16}$$
$$y_{k+15} = (3220013013190122249)y_k + (1440033597185408060)x_k, \tag{17}$$

which imply

$$x_{k+15} \equiv x_k \pmod{11} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{11}, \tag{18}$$
$$x_{k+15} \equiv -x_k \pmod{17} \quad \text{and} \quad y_{k+15} \equiv -y_k \pmod{17}, \tag{19}$$
$$x_{k+15} \equiv x_k \pmod{19} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{19}, \tag{20}$$
$$x_{k+15} \equiv 35y_k \pmod{41} \quad \text{and} \quad y_{k+15} \equiv 7x_k \pmod{41}, \tag{21}$$
$$x_{k+15} \equiv -x_k \pmod{61} \quad \text{and} \quad y_{k+15} \equiv -y_k \pmod{61}, \tag{22}$$
$$x_{k+15} \equiv 40y_k \pmod{107} \quad \text{and} \quad y_{k+15} \equiv 8x_k \pmod{107}, \tag{23}$$
$$x_{k+15} \equiv x_k \pmod{181} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{181}, \tag{24}$$
$$x_{k+15} \equiv x_k \pmod{541} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{541}, \tag{25}$$
$$x_{k+15} \equiv -x_k \pmod{109441} \quad \text{and} \quad y_{k+15} \equiv -y_k \pmod{109441}, \tag{26}$$
$$x_{k+15} \equiv 4160200y_k \pmod{\xi} \quad \text{and} \quad y_{k+15} \equiv 832040x_k \pmod{\xi}, \tag{27}$$

where $\xi = 10783342081$. From (18), equation (15) becomes $z^2 \equiv 7x_k^2 + y_k^7 \pmod{11}$. If $k \equiv 1 \pmod{15}$, then $x_k \equiv x_1 \equiv 3 \pmod{11}$ and $y_k \equiv y_1 \equiv 10 \pmod{11}$ which imply $z^2 \equiv 7 \pmod{11}$, but the Legendre symbol $\left(\frac{7}{11}\right) = -1$. So $k \not\equiv 1 \pmod{15}$. Next, if $k \equiv 3 \pmod{15}$, then $z^2 \equiv 6 \pmod{11}$ which is impossible since $\left(\frac{6}{11}\right) = -1$. Hence, $k \not\equiv 3 \pmod{15}$. From (19), equation (15) implies $z^2 \equiv 7x_k^2 - y_k^7 \pmod{17}$. If $k \equiv 4 \pmod{15}$, we get $x_k \equiv x_4 \equiv 4 \pmod{17}$ and $y_k \equiv y_4 \equiv 13 \pmod{17}$. Thus, $z^2 \equiv 6 \pmod{17}$, but $\left(\frac{6}{17}\right) = -1$. Therefore, $k \not\equiv 4 \pmod{15}$. If $k \equiv 5 \pmod{15}$ leads to $z^2 \equiv 14 \pmod{17}$ and this gives a contradiction again. Thus, $k \not\equiv 5 \pmod{15}$. Using (21) and from equation (15), we get $z^2 \equiv 17x_k^7 + 6y_k^2 \pmod{41}$. If $k \equiv 12, 14 \pmod{15}$, we obtain $z^2 \equiv 29 \pmod{41}$. This is impossible since 29 is a quadratic nonresidue modulo 41, hence $k \not\equiv 12, 14 \pmod{15}$. From (22), equation (15) gives $z^2 \equiv 7x_k^2 - y_k^7 \pmod{61}$. Similarly, if $k \equiv 7 \pmod{15}$ or $k \equiv 9 \pmod{15}$ then $z^2 \equiv 43 \pmod{61}$ or $z^2 \equiv 29 \pmod{61}$ respectively. But, these yield a contradiction since $\left(\frac{43}{61}\right) = -1 = \left(\frac{29}{61}\right)$. So, $k \not\equiv 7, 9 \pmod{15}$. Finally, using (25), equation (15) implies $z^2 \equiv 7x_k^2 + y_k^7 \pmod{541}$ which is impossible if $k \equiv 2, 6, 8, 10, 11, 13 \pmod{15}$. Therefore, $k \not\equiv 2, 6, 8, 10, 11, 13 \pmod{15}$. Here, we have proved the completeness of the given list of solutions related to equation (8). Then, it remains to show that the equations (14) and (15) have no common solution at the equations (9) and (10) using the above techniques of congruence arguments and the quadratic reciprocity.

Now, we consider equation (9). By using (18), we get $z^2 \equiv 7 \pmod{11}$ if $k \equiv 0 \pmod{15}$. However $\left(\frac{7}{11}\right) = -1$. Also, if $k \equiv 2 \pmod{15}$, we get a contradiction since $z^2 \equiv 6 \pmod{11}$ is impossible. Therefore, $k \not\equiv 0, 2 \pmod{15}$. Next, from (20), (15) leads to $z^2 \equiv 7x_k^2 + y_k^7 \pmod{19}$. If $k \equiv 1 \pmod{15}$, then $x_k \equiv x_1 \equiv 7 \pmod{19}$ and $y_k \equiv y_1 \equiv 3 \pmod{19}$. So, $z^2 \equiv 3 \pmod{19}$, but this is impossible since 3 is a quadratic nonresidue modulo 19, hence $k \not\equiv 1 \pmod{15}$. From (21), if $k \equiv 10 \pmod{15}$, then $z^2 \equiv 14 \pmod{41}$. This again leads to a contradiction since $\left(\frac{14}{41}\right) = -1$, thus $k \not\equiv 10 \pmod{15}$. Using (22), the equation $z^2 \equiv 7x_k^2 - y_k^7 \pmod{61}$ leads to $z^2 \equiv 51 \pmod{61}$, $z^2 \equiv 29 \pmod{61}$, or $z^2 \equiv 43 \pmod{61}$ if $k \equiv 4 \pmod{15}$, $k \equiv 6 \pmod{15}$, or $k \equiv 8 \pmod{15}$ respectively. But, $29, 43$ and 51 are quadratic nonresidues modulo 61, which implies $k \not\equiv 4, 6, 8 \pmod{15}$. If we use equation (24), (15) implies $z^2 \equiv 7x_k^2 + y_k^7 \pmod{181}$. Here, we face a contradiction if $k \equiv 3, 7, 9 \pmod{15}$. Therefore, $k \not\equiv 3, 7, 9 \pmod{15}$. From (25), the equation $z^2 \equiv 7x_k^2 + y_k^7 \pmod{541}$ and $k \equiv 5, 11, 12, 13, 14 \pmod{15}$ yield a contradiction. So $k \not\equiv 5, 11, 12, 13, 14 \pmod{15}$.

Finally, we consider (10). If $k \equiv 0 \pmod{15}$, then $z^2 \equiv 11 \pmod{17}$, again giving a contradiction since $\left(\frac{11}{17}\right) = -1$. Moreover, if $k \equiv 5 \pmod{15}$, then $z^2 \equiv 5 \pmod{17}$. This is again impossible, so $k \not\equiv 0, 5 \pmod{15}$. Now, we use equation (20). If $k \equiv 1 \pmod{15}$, then $x_k \equiv x_1 \equiv 18 \pmod{19}$ and $y_k \equiv y_1 \equiv 8 \pmod{19}$.This implies $z^2 \equiv 15 \pmod{19}$, but 15 is a quadratic nonresidue modulo 19. Hence, $k \not\equiv 1 \pmod{15}$. From (24), we get $z^2 \equiv 155 \pmod{181}$ if $k \equiv 2 \pmod{15}$. But, $\left(\frac{155}{181}\right) = -1$. Furthermore, if $k \equiv 3 \pmod{15}$, we obtain $z^2 \equiv 22 \pmod{181}$, again yielding a contradiction. Similarly,

$$k \equiv 4, 6, 9, 10, 11, 12, 13, 14 \pmod{15}$$

again leads to a contradiction. So

$$k \not\equiv 2, 3, 4, 6, 9, 10, 11, 12, 13, 14 \pmod{15}.$$

Using equation (25) with $k \equiv 7 \pmod{15}$, we get $z^2 \equiv 502 \pmod{541}$. This is impossible since $\left(\frac{502}{541}\right) = -1$. Therefore, $k \not\equiv 7 \pmod{15}$. If we use equation (27) for $k \equiv 8 \pmod{15}$, then (15) implies

$$z^2 \equiv 3401662621 \pmod{10783342081}.$$

This is impossible and hence $k \not\equiv 8 \pmod{15}$. We have thus proved that the equations (14) and (15) have no common solutions other than

$$(x, y, z) = (3, 1, \pm 8) = (L_2, \{F_1, F_2\}, z) = (X, Y, Z).$$

To complete the proof of the theorem, we must show that the other system of the simultaneous Diophantine equations in (2)

$$x^2 - 5y^2 = -4, \tag{28}$$

$$7x^2 + y^7 = z^2, \tag{29}$$

has no integer solution (x, y, z) such that $x = L_n, y = F_n$ and $z = Z$. Again, we use the same techniques of congruence arguments and the quadratic reciprocity to exhaust all the possibilities of $k \equiv \rho \pmod{r}$ for a proper r and $\rho = 0, 1, 2, \ldots, r-1$. From equations (6) and (7), we can write

$$x_{k+10} = (1730726404001)x_k + 5(774004377960)y_k, \tag{30}$$

$$y_{k+10} = (1730726404001)y_k + (774004377960)x_k, \tag{31}$$

which lead to

$$x_{k+10} \equiv x_k \pmod{11} \quad \text{and} \quad y_{k+10} \equiv y_k \pmod{11}, \tag{32}$$
$$x_{k+10} \equiv 15y_k \pmod{23} \quad \text{and} \quad y_{k+10} \equiv 3x_k \pmod{23}, \tag{33}$$
$$x_{k+10} \equiv x_k \pmod{31} \quad \text{and} \quad y_{k+10} \equiv y_k \pmod{31}, \tag{34}$$
$$x_{k+10} \equiv -x_k \pmod{41} \quad \text{and} \quad y_{k+10} \equiv -y_k \pmod{41}, \tag{35}$$
$$x_{k+10} \equiv x_k \pmod{61} \quad \text{and} \quad y_{k+10} \equiv y_k \pmod{61}, \tag{36}$$
$$x_{k+10} \equiv 85y_k \pmod{241} \quad \text{and} \quad y_{k+10} \equiv 17x_k \pmod{241}, \tag{37}$$
$$x_{k+10} \equiv -x_k \pmod{2521} \quad \text{and} \quad y_{k+10} \equiv -y_k \pmod{2521}. \tag{38}$$

First, we consider (11). From (32), equation (29) gives $z^2 \equiv 7x_k^2 + y_k^7 \pmod{11}$. If $k \equiv 0, 3 \pmod{10}$, then $z^2 \equiv 8 \pmod{11}$. But, 8 is a quadratic nonresidue modulo 11. So $k \not\equiv 0, 3 \pmod{10}$. Using (34), equation (29) implies $z^2 \equiv 7x_k^2 + y_k^7 \pmod{31}$. If $k \equiv 2 \pmod{10}$, then $x_k \equiv x_2 \equiv 25 \pmod{31}$ and $y_k \equiv y_2 \equiv 16 \pmod{31}$ which yield $z^2 \equiv 12 \pmod{31}$. This is impossible, hence $k \not\equiv 2 \pmod{10}$. Moreover, if $k \equiv 4 \pmod{10}$, then $z^2 \equiv 15 \pmod{31}$, again leading to a contradiction. So $k \not\equiv 4 \pmod{10}$. From (36), we get $z^2 \equiv 7x_k^2 + y_k^7 \pmod{61}$. If $k \equiv 1 \pmod{10}$, then $z^2 \equiv 44 \pmod{61}$. However, $\left(\frac{44}{61}\right) = -1$, thus $k \not\equiv 1 \pmod{10}$. In a similar way, if $k \equiv 5, 6, 9 \pmod{10}$, we obtain a contradiction again. Therefore, $k \not\equiv 5, 6, 9 \pmod{10}$. From (37), equation (29) leads to $z^2 \equiv 23x_k^7 + 206y_k^2 \pmod{241}$. If $k \equiv 7 \pmod{10}$ or $k \equiv 8 \pmod{10}$, then $z^2 \equiv 153 \pmod{241}$ or $z^2 \equiv 68 \pmod{241}$ respectively, again giving a contradiction. Hence, $k \not\equiv 7, 8 \pmod{10}$.

Next, we consider (12). From (35), equation (29) gives

$$z^2 \equiv 7x_k^2 - y_k^7 \pmod{41},$$

which is impossible if $k \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}$. This requires

$$k \not\equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}.$$

Using (36), we get $z^2 \equiv 44 \pmod{61}$ if $k \equiv 9 \pmod{10}$. This gives a contradiction again, so $k \not\equiv 9 \pmod{10}$. From (38), we obtain $z^2 \equiv 7x_k^2 - y_k^7 \pmod{2521}$. Then $z^2 \equiv 1129 \pmod{2521}$ if $k \equiv 8 \pmod{10}$. But, $\left(\frac{1129}{2521}\right) = -1$. Hence, $k \not\equiv 8 \pmod{10}$.

Finally, we consider (13). Using (33), equation (29) implies $z^2 \equiv 2x_k^7 + 11y_k^2 \pmod{23}$, which is impossible if $k \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}$. This forces

$$k \not\equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}.$$

It remains to consider $k \equiv 8, 9 \pmod{10}$. Here we use equation (35). If $k \equiv 8 \pmod{10}$ or $k \equiv 9 \pmod{10}$, then $z^2 \equiv 30 \pmod{41}$ or $z^2 \equiv 35 \pmod{41}$ respectively. But, 30 and 35 are quadratic nonresidues modulo 41. So $k \not\equiv 8, 9 \pmod{10}$. Thus, the simultaneous Diophantine equations (28) and (29) can not be solved simultaneously. Hence, Theorem 1 is proved. □

Theorem 2. *The Diophantine equation* (1) *has no solutions in integers* X, Y, *and* Z *if* $X = F_n$ *and* $Y = L_n$.

Proof. We prove this theorem by showing the simultaneous Diophantine equations in (3) have no common solutions. Firstly, we consider the system of Diophantine equations

$$x^2 - 5y^2 = 4, \tag{39}$$
$$x^7 + 7y^2 = z^2, \tag{40}$$

where $x = L_n$, $y = F_n$ and $z = Z$. To prove this system has no solution, we follow the same approach used in the proof of Theorem 1 to exhaust all the possibilities of $k \equiv \rho \pmod{15}$ for $\rho = 0, 1, 2, \dots, 14$, with using some equations of (16)–(27). Firstly, we consider (8). From (18), equation (40) gives $z^2 \equiv x_k^7 + 7y_k^2 \pmod{11}$. If $k \equiv 3 \pmod{15}$, then $z^2 \equiv 7 \pmod{11}$. This is impossible, so $k \not\equiv 3 \pmod{15}$. Using (23), we get $z^2 \equiv 51 \pmod{107}$ if $k \equiv 7 \pmod{15}$. But, $\left(\frac{51}{107}\right) = -1$. So $k \not\equiv 7 \pmod{15}$. From (24), we get a contradiction if $k \equiv 0, 1, 2, 5, 6, 8, 10, 12 \pmod{15}$. To exclude the rest possibilities, we use (25) which leads to $z^2 \equiv x_k^7 + 7y_k^2 \pmod{541}$. If $k \equiv 4 \pmod{15}$, then $z^2 \equiv 206 \pmod{541}$. This is a contradiction since 206 is a quadratic nonresidue modulo 541. Similarly, $k \equiv 9, 11, 13, 14 \pmod{15}$ leads to a contradiction again. Therefore, $k \not\equiv 4, 9, 11, 13, 14 \pmod{15}$.

Now, we consider (9). From (22), equation (40) implies $z^2 \equiv 7y_k^2 - x_k^7 \pmod{61}$. Starting with $k \equiv 8 \pmod{15}$, we get $z^2 \equiv 43 \pmod{61}$. Again, we get a contradiction, thus $k \not\equiv 8 \pmod{15}$. Using (24), we get $z^2 \equiv x_k^7 + 7y_k^2 \pmod{181}$. If $k \equiv 0 \pmod{15}$, then $x_k \equiv x_0 \equiv 3 \pmod{181}$ and $y_k \equiv y_0 \equiv 180 \pmod{181}$ which give $z^2 \equiv 22 \pmod{181}$. This is impossible, since $\left(\frac{22}{181}\right) = -1$. Furthermore, $k \equiv 3, 5, 7, 9, 10, 13, 14 \pmod{15}$ yields a contradiction again. Therefore,

$$k \not\equiv 0, 3, 5, 7, 9, 10, 13, 14 \pmod{15}.$$

From (25), the equation $z^2 \equiv x_k^7 + 7y_k^2 \pmod{541}$ is impossible if $k \equiv 1, 2, 4, 6, 11 \pmod{15}$. So $k \not\equiv 1, 2, 4, 6, 11 \pmod{15}$. Using (26), we get $z^2 \equiv 7y_k^2 - x_k^7 \pmod{109441}$. If $k \equiv 12 \pmod{15}$, then $z^2 \equiv 98563 \pmod{109441}$. This is impossible since 98563 is a quadratic nonresidue modulo 109441. Hence, $k \not\equiv 12 \pmod{15}$.

Finally, we consider (10). Equation (24) leads to $z^2 \equiv x_k^7 + 7y_k^2 \pmod{181}$, which is impossible if

$$k \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 \pmod{15}.$$

Therefore, they are all excluded. Hence, the simultaneous Diophantine equations (39) and (40) have no common solution. We finish the proof of the theorem by proving the system of the Diophantine equations

$$x^2 - 5y^2 = -4, \tag{41}$$
$$x^7 + 7y^2 = z^2, \tag{42}$$

where $x = L_n$, $y = F_n$ and $z = Z$, can not be solved simultaneously. Again, we use some appropriate equations of (30)–(38) to exclude all the possibilities of $k \equiv \rho \pmod{10}$ for $\rho = 0, 1, 2, \dots, 9$. First of all, we consider equation (11). From (32), equation (42) leads to $z^2 \equiv 8 \pmod{11}$, $z^2 \equiv 6 \pmod{11}$, or $z^2 \equiv 10 \pmod{11}$ if $k \equiv 0 \pmod{10}$, $k \equiv 3 \pmod{10}$, or $k \equiv 4 \pmod{10}$ respectively. However, 6, 8, and 10 are quadratic nonresidues modulo 11. So $k \not\equiv 0, 3, 4 \pmod{10}$. Using (33), we get $z^2 \equiv 17x_k^2 + 11y_k^7 \pmod{23}$. If $k \equiv 1 \pmod{10}$, then $z^2 \equiv 21 \pmod{23}$. This yields a contradiction, hence $k \not\equiv 1 \pmod{10}$. If we use (35), we obtain $z^2 \equiv 7y_k^2 - x_k^7 \pmod{41}$ which can not be held for $k \equiv 5, 6, 7, 8, 9 \pmod{10}$. Then $k \not\equiv 5, 6, 7, 8, 9 \pmod{10}$. From (36), we get $z^2 \equiv 31 \pmod{61}$ for $k \equiv 2 \pmod{10}$. But, $\left(\frac{31}{61}\right) = -1$. Therefore, $k \not\equiv 2 \pmod{10}$.

Next, we consider (12). Equation (32) and $k \equiv 0 \pmod{10}$ give $z^2 \equiv 6 \pmod{11}$, again yielding a contradiction. So $k \not\equiv 0 \pmod{10}$. Similarly, we get a contradiction again if we use (35) for $k \equiv 1, 2, 3, 4, 5, 7 \pmod{10}$. Hence,

$$k \not\equiv 1, 2, 3, 4, 5, 7 \pmod{10}.$$

From (37), we obtain $z^2 \equiv 95x_k^2 + 220y_k^7 \pmod{241}$. If $k \equiv 6 \pmod{10}$, then $z^2 \equiv 7 \pmod{241}$. Moreover, if $k \equiv 8 \pmod{10}$ or $k \equiv 9 \pmod{10}$, then $z^2 \equiv 21 \pmod{241}$ or $z^2 \equiv 37 \pmod{241}$ respectively. But, 7, 21, and 37 are quadratic nonresidues modulo 241. So $k \not\equiv 6, 8, 9 \pmod{10}$.

Lastly, we consider (13). From (32), we have $z^2 \equiv 8 \pmod{11}$ if $k \equiv 3 \pmod{10}$. This is impossible. Therefore, $k \not\equiv 3 \pmod{10}$. Equation (36) leads to a contradiction again if $k \equiv 6 \pmod{10}$. So $k \not\equiv 6 \pmod{10}$. In a similar way, we can use (35) to eliminate all the remaining possibilities of $k \equiv \rho \pmod{10}$ such that $\rho = 0, 1, 2, 4, 5, 7, 8, 9$. Hence, the simultaneous Diophantine equations (41) and (42) have no common solution. Therefore, Theorem 2 is completely proved. □

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