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Existence of entropy solutions to nonlinear degenerate parabolic problems with variable exponent and L^1-data

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Abstract. In the present paper, we prove existence results of entropy solutions to a class of nonlinear degenerate parabolic $p(\cdot)$-Laplacian problem with Dirichlet-type boundary conditions and L^1 data. The main tool used here is the Rothe method combined with the theory of variable exponent Sobolev spaces.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $(d \geq 2)$ be an open bounded domain with a connected Lipschitz boundary $\partial\Omega$ and T be a fixed positive real number. Our aim of this paper is to prove existence results of entropy solutions for the nonlinear degenerate parabolic problem with variable exponent

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta^{\omega}_{p(x)} u = f & \text{in } Q_T := (0,T) \times \Omega, \\ u = 0 & \text{on } \Sigma_T := (0,T) \times \partial\Omega, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1}$$

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where $p(\cdot)$ is a continuous function defined on $\overline{\Omega}$ with $p(x) > 1$ for all $x \in \overline{\Omega}$, ω is a measurable positive and a.e. finite function defined in $\mathbb{R}^d$, the datum f is a non-regular function.

For $u = u(t,x)$, with $(t,x) \in (0,T) \times \Omega$, we denote by

$$\nabla u = \nabla_x u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d} \right)$$

the gradient of function u with respect to the space variables only ($\Omega \subset \mathbb{R}^d$). The operator

$$\Delta^{\omega}_{p(\cdot)} u = \operatorname{div}\big(\omega |\nabla u|^{p(\cdot)-2} \nabla u\big)$$

is called ω-$p(\cdot)$-Laplacian, which becomes the ω-p-Laplacian when $p(\cdot) \equiv p$ (i.e. when $p(\cdot)$ is a constant function).

In recent years, the study of partial differential equations and variational problems with variable exponent has received considerable attention in many models coming from various branches of mathematical physics, such as elastic mechanics, electrorheological fluid dynamics and image processing, etc. Degenerate phenomena appear in the area of oceanography, turbulent fluid flows, induction heating and electrochemical problems (see for example [7], [12], [17]). The notion of entropy solutions was introduced by Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, J.L. Vazquez in [6], this notion was then adapted by many authors to study some nonlinear elliptic and parabolic problems with a constant or variable exponent and Dirichlet or Neumann boundary conditions (see for example [1], [4], [14], [16], [19], and [20]).

Recently, when $\omega(x) \equiv 1$, the existence and uniqueness of entropy solutions of $p(x)$-Laplace equation with L^1 data were proved in [18] by Sanchón and Urbano, this notion was adapted to study the entropy solutions for nonlinear elliptic equations with variable exponents by Chao Zhang in [19] and the existence of solutions for some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui et al. in [3] in terms of the penalization method. A. Jamea et al. are proved in [14] the existence of entropy solutions to a class of nonlinear parabolic problem by using Rothe time-discretization method combined with the theory of variable exponent Sobolev spaces. Fortunately, Kim, Wang and Zhang in [15] have shown good properties of a function space and the so-called weighted variable exponent Lebesgue-Sobolev spaces and the existence and some properties of solutions for degenerate elliptic equations with exponent variable have been proved by Ky Ho, Inbo Sim in [11]. Other work in direction of degenerate parabolic equations can be found in [2].

In this paper, by using the Rothe's method, we study the existence question of entropy solutions to nonlinear parabolic problem (1), we apply here a time discretization of this continuous problem (1) by Euler forward scheme and we show existence, uniqueness and stability of entropy solutions to the discretized problem. After, we will construct from the entropy solution of discretized problem a sequence that we show converging to an entropy solution of the nonlinear parabolic problem (1). We recall that the Euler forward scheme has been used by several authors while studying time discretization of nonlinear parabolic problems, we refer to the works [8], [9], [13], [14] for some details. The advantage of our method is that we cannot only obtain the existence of entropy solutions to the problem (1), but also to compute the numerical approximations.

2 Preliminaries and notations

In this section we give some notations and definitions and state some results which will be used in this work.

Let Ω be a bounded open domain in $\mathbb{R}^d$ $(d \geq 2)$, we consider the following set

$$C^+(\overline{\Omega}) = \{p\colon \overline{\Omega} \to \mathbb{R}^+ : p \text{ is continuous and such that } 1 < p_- < p_+ < \infty\}\,,$$

where

$$p_- = \min_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p_+ = \max_{x\in\overline{\Omega}} p(x)\,.$$

Let ω be a measurable positive and a.e. finite function defined in $\mathbb{R}^d$ and satisfied the following integrability conditions.

(H1) $\omega \in L^1_{\text{loc}}(\Omega)$ and $\omega^{\frac{-1}{p(x)-1}} \in L^1_{\text{loc}}(\Omega)$,

(H2) $\omega^{-s(x)} \in L^1_{\text{loc}}(\Omega)$, where $s(x) \in \big(\frac{N}{p(x)}, \infty\big) \cap \big(\frac{1}{p(x)-1}, \infty\big]$.

For $p(\cdot) \in C^+(\overline{\Omega})$, we define the weighted Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega,\omega)$ by

$$L^{p(\cdot)}(\Omega,\omega) = \left\{u\colon \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^{p(x)}\omega(x)\,\mathrm{d}x < \infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot),\omega} = \|u\|_{L^{p(\cdot)}(\Omega,\omega)} = \inf\left\{\lambda > 0,\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \omega(x)\,\mathrm{d}x \leq 1\right\}.$$

We denote by $L^{p'(\cdot)}(\Omega,\omega^*)$ the conjugate space of $L^{p(\cdot)}(\Omega,\omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

and where

$$\omega^*(x) = \omega(x)^{1-p'(x)} \quad \text{for all } x \in \Omega\,.$$

On the space $L^{p(\cdot)}(\Omega,\omega)$, we consider the function $\rho_{p(\cdot),\omega}\colon L^{p(\cdot)}(\Omega,\omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot),\omega}(u) = \rho_{L^{p(\cdot)}(\Omega,\omega)}(u) = \int_\Omega |u(x)|^{p(x)}\omega(x)\,\mathrm{d}x\,.$$

The connection between $\rho_{p(\cdot),\omega}$ and $\|\cdot\|_{p(\cdot),\omega}$ is established by the next result.

Proposition 1. *Let u be an element of $L^{p(\cdot)}(\Omega,\omega)$, the following assertions hold:*

i) $\|u\|_{p(\cdot),\omega} < 1$ *(respectively* $>, = 1$*)* $\Leftrightarrow \rho_{p(\cdot),\omega}(u) < 1$ *(respectively* $>, = 1$*).*

ii) If $\|u\|_{p(\cdot),\omega} < 1$ *then* $\|u\|^{p+}_{p(\cdot),\omega} \leq \rho_{p(\cdot),\omega}(u) \leq \|u\|^{p-}_{p(\cdot),\omega}$.

iii) If $\|u\|_{p(\cdot),\omega} > 1$ *then* $\|u\|^{p-}_{p(\cdot),\omega} \leq \rho_{p(\cdot),\omega}(u) \leq \|u\|^{p+}_{p(\cdot),\omega}$.

iv) $\|u\|_{p(\cdot),\omega} \to 0 \Leftrightarrow \rho_{p(\cdot),\omega}(u) \to 0$ *and* $\|u\|_{p(\cdot),\omega} \to \infty \Leftrightarrow \rho_{p(\cdot),\omega}(u) \to \infty$.

Proof. See Proposition 2.1 in [4] or Theorem 1.3 in [10]. □

The weighted Sobolev space with variable exponent is defined by

$$W^{1,p(\cdot)}(\Omega,\omega) = \left\{u \in L^{p(\cdot)} \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega,\omega)\right\},$$

with the norm

$$\|u\|_{1,p(\cdot),\omega} = \|u\|_{p(\cdot)} + \|u\|_{p(\cdot),\omega}\,, \quad \forall u \in W^{1,p(\cdot)}(\Omega,\omega)\,.$$

In the following of this paper, the space $W_0^{1,p(\cdot)}(\Omega,\omega)$ denote the closure of C_0^∞ in $W^{1,p(\cdot)}(\Omega,\omega)$.

Let $p(\cdot), s(\cdot)$ are two elements of space $C^+(\overline{\Omega})$ where the function $s(\cdot)$ satisfies the hypothesis (H2), we define the following functions

$$p^*(x) = \frac{Np(x)}{N-p(x)} \quad \text{for } p(x) < N\,,$$

$$p_s(x) = \frac{p(x)s(x)}{1+s(x)} < p(x)\,,$$

$$p_s^*(x) = \begin{cases} \dfrac{p(x)s(x)}{(1+s(x))N - p(x)s(x)} & \text{if } N > p_s(x), \\ +\infty & \text{if } N \leq p_s(x), \end{cases}$$

for almost all $x \in \Omega$.

Proposition 2. *Let* $\Omega \in \mathbb{R}^d$ *be an open set of* $\mathbb{R}$, $p(\cdot) \in C^+(\overline{\Omega})$ *and let hypothesis (H1) be satisfied, we have*

$$L^{p(\cdot)}(\Omega,\omega) \hookrightarrow L^1_{\text{loc}}(\Omega)\,.$$

Proof. See Proposition 2.8 in [15]. □

Proposition 3. *Let hypothesis (H1) be satisfied, the space* $(W^{1,p(\cdot)}(\Omega,\omega), \|u\|_{1,p(\cdot),\omega})$ *is a separable and reflexive Banach space.*

Proof. See Theorem 2.10 in [15]. □

Proposition 4. *Assume that hypotheses (H1) and (H2) hold and* $p(\cdot), s(\cdot) \in C^+(\overline{\Omega})$, *then we have the continuous embedding*

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow W^{1,p_s(x)}(\Omega,\omega)\,.$$

Moreover, we have the compact embedding

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$$

provided that $r \in C^+(\overline{\Omega})$ *and* $1 \leq r(x) < p_s^*(x)$ *for all* $x \in \Omega$.

Proof. See Proposition 2.11 in [15]. □

Given a constant $k > 0$, we define the cut function $T_k\colon \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \min(k, \max(s, -k)) = \begin{cases} s & \text{if } |s| \le k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k u$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.

Let the function $\Theta_k\colon \mathbb{R} \to \mathbb{R}^+$ (is the primitive function of T_k) defined by

$$\Theta_k(s) = \int_0^s T_k(t)\,\mathrm{d}t = \begin{cases} \frac{s^2}{2} & \text{if } |s| \le k \\ k|s| - \frac{k^2}{2} & \text{if } |s| > k \end{cases}$$

Throughout this paper, $\langle \cdot, \cdot \rangle$ denotes the duality between X and X^* (dual space of X, the set of continuous linear functional on X).

We have as in the paper [5]

$$\int_0^t \Big\langle \frac{\partial v}{\partial s}, T_k(v) \Big\rangle \,\mathrm{d}s = \int_\Omega \Theta_k(v(t))\,\mathrm{d}x - \int_\Omega \Theta_k(v(0))\,\mathrm{d}x\,,$$

where $v \in W_0^{1,x} L_M(Q_T) \cap L^\infty(Q_T)$.

We define also the space

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega) = \big\{ u\colon \Omega \to \mathbb{R} : \\ u \text{ is measurable and } T_k(u) \in W_0^{1,p(\cdot)}(\Omega, \omega) \text{ for all } k > 0 \big\}\,.$$

By Proposition 1.1 in [19], we have the following result.

Proposition 5. *For every function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega)$, there exists a unique measurable function $v\colon \Omega \to \mathbb{R}^d$, which we call the very weak gradient of u and denote $v = \nabla u$ such that*

$$\nabla T_k(u) = v \chi_{\{|u| \le k\}} \quad \text{for a.e. } x \in \Omega \text{ and for all } k > 0\,,$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^d$. Moreover, if u belongs to $W_0^{1,p(\cdot)}(\Omega, \omega)$, then the function v coincides with the weak gradient of u.

Lemma 1. *For $\xi, \eta \in \mathbb{R}^d$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \le |\xi|^{p-2}\xi \cdot (\xi - \eta)\,,$$

where a dot denote the Euclidian scalar product in $\mathbb{R}^d$.

Proof. See Lemma 3.2 in [1]. □

Theorem 1. *Assume that (H1) and (H2) hold, then the following problem*

$$\begin{cases} H(x,u,\nabla u) - \operatorname{div}(a(x,\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

has a least one entropy solution, i.e., there exists $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega,\omega)$ *such that* $u \geq \psi$ *a.e. in* Ω *and*

$$\int_\Omega H(x,u,\nabla u) T_k(u-\varphi)\,\mathrm{d}x + \int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-\varphi)\,\mathrm{d}x \leq \int_\Omega f T_k(u-\varphi)\,\mathrm{d}x\,,$$

for all $\varphi \in K_\psi \cap L^\infty(\Omega)$*, where* ψ *is a measurable function such that*

$$\psi^+ \in W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega) \quad (\psi^+ = \max\{\psi, 0\})$$

and

$$K_\psi = \left\{u \in W_0^{1,p(\cdot)}(\Omega,\omega) : u \geq \psi \text{ a.e. in } \Omega\right\}.$$

The function $a\colon \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying the following assumptions:

$$\begin{gathered} |a(x,\xi)| \leq \beta\omega(x)^{\frac{1}{p(x)}}\left(k(x) + \omega(x)^{\frac{1}{p'(x)}}|\xi|^{p(x)-1}\right), \\ [a(x,\xi) - a(x,\eta)]\cdot(\xi-\eta) > 0\,, \quad \forall \xi \neq \eta\,, \\ a(x,\xi)\cdot\xi \geq \alpha\omega(x)|\xi|^{p(x)}\,, \end{gathered}$$

for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$, where $k(x)$ is positive function lying in $L^{p'(x)}(\Omega)$ and $\alpha, \beta > 0$.

The nonlinear term $H\colon \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function satisfying

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)\omega(x)|\xi|^{p(x)}\,,$$

where $g\colon \mathbb{R}\to\mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$ and $\gamma(x)$ belongs to $L^1(\Omega)$. The datum f is in $L^1(\Omega)$.

Proof. See Theorem 4.2 in [4]. □

Remark 1. Hereinafter k, τ, T are strictly positive real numbers, N is a strictly positive natural number and $C(X), C_i(X)$ ($i \in \mathbb{N}$) are positive constants depending only on X.

3 Main result

In this section we give the notion of entropy solution to nonlinear degenerate parabolic problem (1) and we state the main result of this paper, firstly and in addition to the hypotheses (H1) and (H2) listed earlier, we suppose the following assumption:

(H3) $f \in L^1(Q_T)$ and $u_0 \in W_0^{1,p(\cdot)}(\Omega, \omega)$.

Definition 1. A measurable function $u\colon Q_T \to \mathbb{R}$ is an entropy solution of parabolic problem (1) in Q_T if $u(\cdot, 0) = u_0$ in Ω,

$$u \in C(0,T; L^1(\Omega)), \quad T_k(u) \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega,\omega))$$

for all $k > 0$ and

$$\begin{aligned} &\int_0^t \Big\langle \frac{\partial \varphi}{\partial s},\ T_k(u-\varphi) \Big\rangle \,\mathrm{d}s + \int_0^t \int_\Omega \omega(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla T_k(u-\varphi) \,\mathrm{d}x \,\mathrm{d}s \\ \le &\int_\Omega \Theta_k(u(0)-\varphi(0)) \,\mathrm{d}x - \int_\Omega \Theta_k(u(t)-\varphi(t)) \,\mathrm{d}x + \int_0^t \int_\Omega f T_k(u-\varphi) \,\mathrm{d}x \,\mathrm{d}s \end{aligned} \tag{3}$$

for all $\varphi \in L^\infty(Q_T) \cap L^{p_-}(0,T; W^{1,p(\cdot)}(\Omega,\omega)) \cap W^{1,1}(0,T; L^1(\Omega))$ and $t \in [0,T]$.

Our main result is the following theorem

Theorem 2. *Assume that hypotheses (H1), (H2) and (H3) hold, then the nonlinear degenerate parabolic problem* (1) *has a least one entropy solution.*

4 Proof of main result

The proof of our main result is divided into three steps, in the first one, using Euler forward scheme, we discretize the continuous problem (1) and study the existence and uniqueness questions of entropy solutions to the discretized problems. In the second step, we give some stability results for the discrete entropy solutions. Finally and by Rothe function, we construct a sequence of functions that we show that this sequence converges to an entropy solution of nonlinear degenerate parabolic problem (1).

Step 1. The semi-discrete problem

By Euler forward scheme, we discretize the problem (1), we obtain the following problems

$$\begin{cases} U_n - \tau \Delta^\omega_{p(x)} U_n = \tau f_n + U_{n-1} & \text{in } \Omega, \\ U_n = 0 & \text{on } \partial\Omega, \\ U_0 = u_0 & \text{in } \Omega, \end{cases} \tag{4}$$

where $N\tau = T$, $0 < \tau < 1$, $1 \le n \le N$, $t_n = n\tau$ and

$$f_n(\cdot) = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(s, \cdot) \,\mathrm{d}s \quad \text{in } \Omega\,.$$

Definition 2. An entropy solution to the discretized problem (4) is a sequence $(U_n)_{0\leq n\leq N}$ such that $U_0 = u_0$ and for $n = 1, 2, \dots, N$, U_n is defined by induction as an entropy solution to the problem

$$\begin{cases} u - \tau\Delta^{\omega}_{p(x)} u = \tau f_n + U_{n-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

i.e. for all $n \in \{1, 2, \dots, N\}$, $U_n \in \mathcal{T}_0^{1,p(\cdot)}(\Omega,\omega)$ and for all $\varphi \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$, $k > 0, \tau > 0$, we have

$$\int_\Omega U_n T_k(U_n - \varphi)\,\mathrm{d}x + \tau \int_\Omega \omega(x)|\nabla U_n|^{p(x)-2}\nabla U_n \cdot \nabla T_k(U_n - \varphi)\,\mathrm{d}x \leq \int_\Omega (\tau f_n + U_{n-1}) T_k(U_n - \varphi)\,\mathrm{d}x\,. \tag{5}$$

Lemma 2. *Let hypotheses (H1), (H2) and (H3) be satisfied. If $(U_n)_{0\leq n\leq N}$ is an entropy solution of discretized problem (4), then for all $n = 1, \dots, N$, we have $U_n \in L^1(\Omega)$.*

Proof. For $n = 1$, we take $\varphi = 0$ in inequality (5), we get

$$\int_\Omega U_1 T_k(U_1)\,\mathrm{d}x \leq \int_\Omega \tau f_1 T_k(U_1)\,\mathrm{d}x + \int_\Omega u_0 T_k(U_1)\,\mathrm{d}x\,.$$

This implies

$$0 \leq \int_\Omega U_1 \frac{T_k(U_1)}{k}\,\mathrm{d}x \leq \left(\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}\right).$$

On the other hand, for each $x \in \Omega$, we have

$$\lim_{k\to 0} U_1(x)\frac{T_k(U_1(x))}{k} = |U_1(x)|\,.$$

Then by Fatou's lemma, we deduce that $U_1 \in L^1(\Omega)$ and

$$\|U_1\|_{L^1(\Omega)} \leq \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}\,.$$

By induction, we deduce in the same manner that $U_n \in L^1(\Omega), \forall n = 1, \dots, N$. □

Theorem 3. *Assume that hypotheses (H1), (H2) and (H3) hold, then the discretized problem (4) has a unique entropy solution $(U_n)_{0\leq n\leq N}$ and $U_n \in L^1(\Omega) \cap \mathcal{T}_0^{1,p(\cdot)}(\Omega,\omega)$ for all $n = 1, \dots, N$.*

Proof. Existence. For $n = 1$, we rewrite the discretized problem (4) as

$$\begin{aligned} u - \tau\Delta^{\omega}_{p(x)} u = g \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{6}$$

where $u = U_1$ and $g = \tau f_1 + u_0$, by using hypothesis (H3), the function g is an element of $L^1(\Omega)$, then, by Theorem 1, the problem (6) has an entropy solution U_1 in $L^1(\Omega) \cap \mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega)$.

By induction, using the same argument above, we prove that the problem (4) has an entropy solution $(U_n)_{0 \leq n \leq N}$ and $U_n \in L^1(\Omega) \cap \mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega)$ for all $n = 1, \ldots, N$.

Uniqueness. We firstly need the following two lemmas.

Lemma 3. *Let hypotheses (H1), (H2) and (H3) be satisfied, if $(U_n)_{0 \leq n \leq N}$, $N \in \mathbb{N}$ is an entropy solution of problem* (4)*, then for all $n = 1, \ldots, N$ and $k > 0$, there exists a positive constant C such that*

$$\operatorname{meas}\{|U_n| > k\} \leq \frac{C(M+1)^{\frac{(p_s^*)_-}{p_-}}}{k^{((p_s^*)_-) \times (1 - \frac{1}{p_-})}},$$

where $M = \|\tau f_n + U_{n-1}\|_{L^1(\Omega)}$ and

$$(p_s^*)_- := \frac{(p_-) \times (s_-) \times (N)}{((s_-) + 1) \times (N) - (p_-) \times (s_-)}.$$

Proof. See Proposition 3.4 in [19]. □

Lemma 4. *Let hypotheses (H1), (H2) and (H3) be satisfied, if $(U_n)_{0 \leq n \leq N}$, $N \in \mathbb{N}$ is an entropy solution of problem* (4)*, then for all $k > 0$, $n = 1, \ldots, N$ and $h > 0$, we have*

$$\lim_{h \to \infty} \int_{\{h < |U_n| < k+h\}} \omega(x) |\nabla U_n|^{p(x)} \, \mathrm{d}x = 0.$$

Proof. We take in entropy inequality (5), $\varphi = T_h(U_n)$, we get

$$\int_\Omega U_n T_k(U_n - T_h(U_n)) \, \mathrm{d}x + \tau \int_\Omega \omega(x) |\nabla U_n|^{p(x)-2} \nabla U_n \cdot \nabla T_k(U_n - T_h(U_n)) \, \mathrm{d}x \leq \int_\Omega (\tau f_n + U_{n-1}) T_k(U_n - T_h(U_n)) \, \mathrm{d}x. \quad (7)$$

Firstly, we have

$$\int_\Omega U_n T_k(U_n - T_h(U_n)) \, \mathrm{d}x = \int_{\{|U_n| > h\}} U_n T_k(U_n - h \operatorname{sign}(U_n)) \, \mathrm{d}x,$$

and

$$\begin{aligned} \operatorname{sign}(U_n) \chi_{\{|U_n| > h\}} &= \operatorname{sign}(U_n - h \operatorname{sign}(U_n)) \chi_{\{|U_n| > h\}} \\ &= \operatorname{sign}(T_k(U_n - h \operatorname{sign}(U_n))) \chi_{\{|U_n| > h\}}, \end{aligned}$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^d$.

Then

$$\int_\Omega U_n T_k(U_n - T_h(U_n)) \, \mathrm{d}x \geq 0.$$

Or, we have

$$\nabla T_k(U_n - T_h(U_n))\chi_{\{h<|U_n|<k+h\}} = \nabla U_n \chi_{\{h<|U_n|<k+h\}} \,.$$

Therefore, inequality (7) becomes

$$\tau \int_{\{h<|U_n|<k+h\}} \omega(x)|\nabla U_n|^{p(x)} \,\mathrm{d}x \leq k \int_{\{|U_n|>h\}} |\tau f_n + U_{n-1}| \,\mathrm{d}x \,.$$

By Lemma 3, we deduce that meas$\{|U_n| > h\}$ tends to zero as h go to infinity, then, we conclude that

$$\lim_{h\to\infty} \int_{\{|U_n|>h\}} |\tau f_n + U_{n-1}| \,\mathrm{d}x = 0 \,.$$

Hence

$$\lim_{h\to\infty} \int_{\{h<|U_n|<k+h\}} \omega(x)|\nabla U_n|^{p(x)} \,\mathrm{d}x = 0 \,. \qquad \square$$

Now, let $(U_n)_{0\leq n\leq N}$ and $(V_n)_{0\leq n\leq N}$, $N \in \mathbb{N}$ be two entropy solutions of problem (4). For $n = 1$, we write $U_1 = u$ and $V_1 = v$ (for simplification), and let h, k be two positive real numbers. For the solution u, we take $\varphi = T_h(v)$ as test function and for the solution v we take $\varphi = T_h(u)$ as test function in entropy inequality (5), then we have

$$\int_\Omega u T_k(u - T_h(v)) \,\mathrm{d}x + \int_\Omega \omega(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(u - T_h(v)) \,\mathrm{d}x \leq \int_\Omega f T_k(u - T_h(v)) \,\mathrm{d}x$$

and

$$\int_\Omega v T_k(v - T_h(u)) \,\mathrm{d}x + \int_\Omega \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla T_k(v - T_h(u)) \,\mathrm{d}x \leq \int_\Omega f T_k(v - T_h(u)) \,\mathrm{d}x \,.$$

By summing up the two above inequalities, and letting h go to infinity, we find by applying Dominated Convergence Theorem that

$$\int_\Omega (u - v)T_k(u - v) \,\mathrm{d}x + \lim_{h\to\infty} \mathcal{I}(k; h) \leq 0 \,, \tag{8}$$

where

$$\mathcal{I}(k; h) = \int_\Omega \omega(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(u - T_h(v)) \,\mathrm{d}x + \int_\Omega \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla T_k(v - T_h(u)) \,\mathrm{d}x \,.$$

We consider the following decomposition

$$\Omega_h^1 = \{|u| \le h; |v| \le h\}\,, \qquad \Omega_h^2 = \{|u| \le h; |v| > h\}\,,$$
$$\Omega_h^3 = \{|u| > h; |v| \le h\}\,, \qquad \Omega_h^4 = \{|u| > h; |v| > h\}\,,$$

and for $i = 1, \dots 4$, we take

$$\begin{aligned}\mathcal{I}_i(k;h) = &\int_{\Omega_h^i} \omega(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(u - T_h(v))\,\mathrm{d}x \\ &+ \int_{\Omega_h^i} \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla T_k(v - T_h(u))\,\mathrm{d}x\,.\end{aligned}$$

Firstly, we have

$$\begin{aligned}\mathcal{I}_1(k;h) &= \int_{\Omega_h^1} \omega(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(u - T_h(v))\,\mathrm{d}x \\ &\quad + \int_{\Omega_h^1} \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla T_k(v - T_h(u))\,\mathrm{d}x \\ &= \int_{\Omega_h^k(1)} \omega(x)\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla(u - v)\,\mathrm{d}x\,,\end{aligned}$$

where

$$\Omega_h^k(1) = \{|u - v| \le k; |u| \le h; |v| \le h\}\,.$$

The function ω is positive, then we have

$$\mathcal{I}_1(k;h) \ge 0\,.$$

On the other hand, we have

$$\begin{aligned}\mathcal{I}_2(k;h) &= \int_{\Omega_h^2} \omega(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(u - T_h(v))\,\mathrm{d}x \\ &\quad + \int_{\Omega_h^2} \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla T_k(v - T_h(u))\,\mathrm{d}x \\ &= \int_{\Omega_h^{2,1}} \omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x + \int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla(v - u)\,\mathrm{d}x\,,\end{aligned}$$

where

$$\begin{aligned}\Omega_h^{2,1} &= \{|u| \le h; |v| > h; |u - h\,\mathrm{sign}(v)| \le k\} \\ \Omega_h^{2,2} &= \{|u| \le h; |v| > h; |u - v| \le k\}\,.\end{aligned}$$

Which implies by positivity of ω that

$$\begin{aligned}\mathcal{I}_2(k;h) = &\int_{\Omega_h^{2,1}} \omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x + \int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)}\,\mathrm{d}x \\ &- \int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla u\,\mathrm{d}x \\ \ge &- \int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla u\,\mathrm{d}x\,.\end{aligned}$$

Now, let

$$\mathcal{I}_2^1(k;h) = -\int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla u\,\mathrm{d}x\,,$$

and

$$\mathcal{I}_2^{1;1}(k;h) = \int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla(v-u)\,\mathrm{d}x\,,$$

$$\mathcal{I}_2^{1;2}(k;h) = \int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)}\,\mathrm{d}x\,.$$

On the one hand, we have

$$\mathcal{I}_2^1(k;h) = \mathcal{I}_2^{1;1}(k;h) - \mathcal{I}_2^{1;2}(k;h)\,.$$

We apply Lemma 1, we deduce that

$$\mathcal{I}_2^{1;1}(k;h) \geq \int_{\Omega_h^{2,2}} \frac{1}{p(x)}\omega(x)|\nabla v|^{p(x)}\,\mathrm{d}x - \int_{\Omega_h^{2,2}} \frac{1}{p(x)}\omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x\,.$$

Observing that

$$\begin{aligned}\int_{\Omega_h^{2,2}} \frac{1}{p(x)}\omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x &\leq \frac{1}{p_-}\int_{\Omega_h^{2,2}} \omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x\\ &\leq \frac{1}{p_-}\int_{\{h\leq|u|\leq h+k\}} \omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x\,.\end{aligned}$$

Then, by applying the Lemma 4, we get that

$$\lim_{h\to\infty}\int_{\Omega_h^{2,2}} \frac{1}{p(x)}\omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x = 0\,.$$

This implies that

$$\lim_{h\to\infty}\mathcal{I}_2^{1;1}(k;h) \geq 0\,.$$

On the other hand, we have

$$\begin{aligned}\mathcal{I}_2^{1;2}(k;h) &= \int_{\Omega_h^{2,2}} \omega(x)|\nabla v|^{p(x)}\,\mathrm{d}x\\ &\leq \int_{\{h\leq|v|\leq h+k\}} \omega(x)|\nabla v|^{p(x)}\,\mathrm{d}x\,.\end{aligned}$$

Therefore, the Lemma 4 implies that

$$\lim_{h\to\infty}\mathcal{I}_2^{1;2}(k;h) = 0\,.$$

This implies that

$$\lim_{h\to\infty}\mathcal{I}_2(k;h) = 0\,.$$

We use the same manner used above, we show that

$$\lim_{h\to\infty} \mathcal{I}_3(k;h) \geq 0\,.$$

Finally, for $\mathcal{I}_4(k;h)$, we have

$$\begin{aligned}\mathcal{I}_4(k;h) &= \int_{\Omega_h^4} \omega(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla T_k(u-T_h(v))\,\mathrm{d}x\\ &\quad+\int_{\Omega_h^4} \omega(x)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla T_k(v-T_h(u))\,\mathrm{d}x\\ &= \int_{\Omega_h^{4,1}} \omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x + \int_{\Omega_h^{4,2}} \omega(x)|\nabla v|^{p(x)}\,\mathrm{d}x \geq 0\,,\end{aligned}$$

where

$$\begin{aligned}\Omega_h^{4,1} &= \{|u|>h; |v|>h; |u-h\,\mathrm{sign}(v)|\leq k\}\,,\\ \Omega_h^{4,2} &= \{|u|>h; |v|>h; |v-h\,\mathrm{sign}(u)|\leq k\}\,.\end{aligned}$$

Therefore, the above results we allow to conclude that

$$\lim_{h\to\infty} \mathcal{I}(k;h) \geq 0\,.$$

Then, inequality (8) becomes

$$\int_\Omega (u-v)T_k(u-v)\,\mathrm{d}x \leq 0\,,$$

i.e.

$$\int_\Omega (u-v)\frac{T_k(u-v)}{k}\,\mathrm{d}x \leq 0\,.$$

Taking the limit as $k\to 0$, by Dominated Convergence Theorem, we get

$$\|u-v\|_{L^1(\Omega)} \leq 0\,.$$

This implies that

$$u = v \quad \text{a.e. in } \Omega\,.$$

By induction, we prove that

$$\forall n = 1,\dots,N,\quad \|U_n - V_n\|_{L^1(\Omega)} = 0\,.$$

□

Step 2. Stability results

Theorem 4. *Assume that hypotheses (H1), (H2) and (H3) hold. If $(U_n)_{1\leq n\leq N}$ is an entropy solution of the discretized problem (4), then for all $n = 1,\dots,N$, we have*

i) $\|U_n\|_{L^1(\Omega)} \leq C(u_0, f)$, *ii)* $\sum_{i=1}^{n} \|U_i - U_{i-1}\|_{L^1(\Omega)} \leq C(u_0, f)$,

iii) $\tau \sum_{i=1}^{n} \|T_k(U_i)\|^{p_-}_{W_0^{1,p(\cdot)}(\Omega,\omega)} \leq C(u_0, f, T, k)$,

where $C(u_0, f)$ and $C(u_0, f, T, k)$ are positive constants independents of N.

Proof. For i). Let $i \in \{1, 2, \ldots, N\}$, we take $\varphi = 0$ as a test function in entropy formulation of the discretized problem (4), we get

$$\int_\Omega U_i T_k(U_i)\,\mathrm{d}x \leq \tau \int_\Omega f_i T_k(U_i)\,\mathrm{d}x + \int_\Omega U_{i-1} T_k(U_i)\,\mathrm{d}x\,.$$

This inequality implies

$$\int_\Omega U_i \frac{T_k(U_i)}{k}\,\mathrm{d}x \leq \tau \|f_i\|_{L^1(\Omega)} + \|U_{i-1}\|_{L^1(\Omega)}\,. \tag{9}$$

Noting that

$$\lim_{k\to 0} \frac{T_k(s)}{k} = \operatorname{sign}(s)\,. \tag{10}$$

Therefore, passing to limit in (9) and using Fatou's lemma, we deduce that

$$\|U_i\|_{L^1(\Omega)} \leq \tau \|f_i\|_{L^1(\Omega)} + \|U_{i-1}\|_{L^1(\Omega)}\,.$$

Summing the above inequality from $i = 1$ to n, we deduce that

$$\|U_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}\,.$$

Hence, the stability result i) is then proved.

For ii). Let $i \in \{1, 2, \ldots, N\}$, taking $\varphi = T_k(U_{i-1})$ as a test function in entropy formulation of the discretized problem (4), we get

$$\begin{aligned}\int_\Omega (U_i - U_{i-1}) T_\tau(U_i - T_k(U_{i-1}))\,\mathrm{d}x \\ + \tau \int_\Omega \omega(x) |\nabla U_i|^{p(x)-2} \nabla U_i \cdot \nabla T_\tau(U_i - T_k(U_{i-1}))\,\mathrm{d}x \\ \leq \tau \int_\Omega f_i T_\tau(U_i - T_k(U_{i-1}))\,\mathrm{d}x\,.\end{aligned}$$

This implies that

$$\begin{aligned}\int_\Omega (U_i - U_{i-1}) \frac{T_\tau(U_i - T_k(U_{i-1}))}{\tau}\,\mathrm{d}x \\ + \int_\Omega \omega(x) |\nabla U_i|^{p(x)-2} \nabla U_i \cdot \nabla T_\tau(U_i - T_k(U_{i-1}))\,\mathrm{d}x \\ \leq \tau \|f_i\|_{L^1(\Omega)}\,.\end{aligned} \tag{11}$$

However, we have

$$\int_\Omega \omega(x)|\nabla U_i|^{p(x)-2}\nabla U_i \cdot \nabla T_\tau(U_i - T_k(U_{i-1}))\,\mathrm{d}x = \int_{\Omega_1^i(\tau,k)} \omega(x)|\nabla U_i|^{p(x)-2}\nabla U_i \cdot (\nabla U_i - \nabla U_{i-1})\,\mathrm{d}x + \int_{\Omega_2^i(\tau,k)} \omega(x)|\nabla U_i|^{p(x)}\,\mathrm{d}x\,,$$

where

$$\Omega_1^i(\tau,k) = \{|U_i - U_{i-1}| \le \tau; |U_{i-1}| \le k\}\,,$$
$$\Omega_2^i(\tau,k) = \{|U_i - k\,\mathrm{sign}(U_{i-1})| \le \tau; |U_{i-1}| > k\}.$$

Then, by using Lemma 1, we deduce from the inequality (11) that

$$\int_\Omega (U_i - U_{i-1})\frac{T_\tau(U_i - T_k(U_{i-1}))}{\tau}\,\mathrm{d}x + \int_{\Omega_1^i(\tau,k)} \omega(x)\left(\frac{1}{p(x)}|\nabla U_i|^{p(x)} - \frac{1}{p(x)}|\nabla U_{i-1}|^{p(x)}\right)\mathrm{d}x \le \tau\|f_i\|_{L^1(\Omega)}\,.$$

Summing the above inequality from $i = 1$ to n, we deduce that

$$\sum_{i=1}^{n}\int_\Omega (U_i - U_{i-1})\frac{T_\tau(U_i - T_k(U_{i-1}))}{\tau}\,\mathrm{d}x \le \frac{1}{p_-}\int_\Omega \omega(x)|\nabla u_0|^{p(x)}\,\mathrm{d}x + \|f\|_{L^1(Q_T)}\,. \quad (12)$$

Then, we pass to limit in (12) as $k \to \infty$, $\tau \to 0$, we apply (10), Fatou's lemma and hypothesis (H3) to conclude the stability result ii).

For iii). Let $i \in \{1, 2, \ldots, N\}$, taking $\varphi = 0$ as a test function in entropy formulation of the discretized problem (4), we get

$$\tau\int_\Omega \omega(x)|\nabla T_k(U_i)|^{p(x)}\,\mathrm{d}x \le \tau k\|f_i\|_{L^1(\Omega)} + k\|U_i - U_{i-1}\|_{L^1(\Omega)}\,\mathrm{d}x\,.$$

This inequality implies that

$$\tau\sum_{i=1}^{n}\rho_{p(\cdot),\omega}(\nabla T_k(U_i))\,\mathrm{d}x \le \tau k\sum_{i=1}^{n}\|f_i\|_{L^1(\Omega)} + k\sum_{i=1}^{n}\|U_i - U_{i-1}\|_{L^1(\Omega)}\,\mathrm{d}x\,.$$

Then by hypothesis (H3) and the stability result ii), we conclude that

$$\tau\sum_{i=1}^{n}\rho_{p(\cdot),\omega}(\nabla T_k(U_i))\,\mathrm{d}x \le C_1(f, u_0, k)\,. \quad (13)$$

On the other hand, let $s_0 = \{i \in 1, 2, \dots, n : \|\nabla T_k(U_i)\|_{p(x),\omega(x)} \leq 1\}$, we have

$$\begin{aligned}\tau \sum_{i=1}^{n} \|\nabla T_k(U_i)\|^{p_-}_{p(x),\omega(x)} &= \tau \sum_{i \in s_0} \|\nabla T_k(U_i)\|^{p_-}_{p(x),\omega(x)} + \tau \sum_{i \notin s_0} \|\nabla T_k(U_i)\|^{p_-}_{p(x),\omega(x)} \\ &\leq T + \tau \sum_{i \notin s_0} \rho_{p(\cdot),\omega}(\nabla T_k(U_i)) .\end{aligned}$$

We deduce by inequality (13) that

$$\tau \sum_{i=1}^{n} \|\nabla T_k(U_i)\|^{p_-}_{p(x),\omega(x)} \leq C_2(f, u_0, T, k) .$$

Hence the stability result iii) is established. □

Step 3. Entropy solution of continuous problem

Let us introduce the following piecewise linear extension (called Rothe function)

$$\begin{cases} u_N(0) := u_0, \\ u_N(t) := U_{n-1} + (U_n - U_{n-1}) \frac{(t - t_{n-1})}{\tau}, \quad \forall t \in (t_{n-1}, t_n], n = 1, \dots, N \text{ in } \Omega . \end{cases} \tag{14}$$

And the following piecewise constant function

$$\begin{cases} \overline{u}_N(0) := u_0 , \\ \overline{u}_N(t) := U_n , \quad \forall t \in (t_{n-1}, t_n], n = 1, \dots, N \text{ in } \Omega . \end{cases} \tag{15}$$

We have by Theorem 3 that for any $N \in \mathbb{N}$, the entropy solution $(U_n)_{1 \leq n \leq N}$ of problems (4) is unique, thus, the two sequences $(u_N)_{N \in \mathbb{N}}$ and $(\overline{u}_N)_{N \in \mathbb{N}}$ are uniquely defined.

Lemma 5. *Let hypotheses (H1), (H2) and (H3) be satisfied, then for all $N \in \mathbb{N}$, we have*

i) $\|\overline{u}_N - u_N\|_{L^1(Q_T)} \leq \frac{1}{N} C(T, u_0, f)$,

ii) $\|\frac{\partial u_N}{\partial t}\|_{L^1(Q_T)} \leq C(T, u_0, f)$,

iii) $\|u_N\|_{L^1(Q_T)} \leq C(T, u_0, f)$,

iv) $\|\overline{u}_N\|_{L^1(Q_T)} \leq C(T, u_0, f)$,

v) $\|T_k(\overline{u}_N)\|_{L^{p_-}(0,T,W_0^{1,p(\cdot)}(\Omega,\omega))} \leq C(T, u_0, f, k)$,

where $C(T, u_0, f)$ and $C(T, u_0, f, k)$ are positive constants independent of N.

Proof. For i). For $N \in \mathbb{N}$, we have

$$\begin{aligned}
\|\overline{u}_N - u_N\|_{L^1(Q_T)} &= \int_0^T \int_\Omega |\overline{u}_N - u_N| \,\mathrm{d}x \,\mathrm{d}t \\
&= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|U_n - U_{n-1}\|_{L^1(\Omega)} \frac{(t_n - t)}{\tau} \,\mathrm{d}t \\
&= \frac{\tau}{2} \sum_{n=1}^N \|U_n - U_{n-1}\|_{L^1(\Omega)} \\
&\leq \frac{T}{2N} \sum_{n=1}^N \|U_n - U_{n-1}\|_{L^1(\Omega)} \,.
\end{aligned}$$

Then, we use Theorem 4, we conclude the result i).

For ii). We have for $n = 1, \dots, N$ and $t \in (t_{n-1}, t_n]$

$$\frac{\partial u_N(t)}{\partial t} = \frac{(U_n - U_{n-1})}{\tau} \,.$$

This implies that

$$\begin{aligned}
\left\| \frac{\partial u_N}{\partial t} \right\|_{L^1(Q_T)} &= \int_0^T \int_\Omega \left| \frac{\partial u_N}{\partial t} \right| \mathrm{d}x \,\mathrm{d}t \\
&= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{\tau} \|U_n - U_{n-1}\|_{L^1(\Omega)} \,\mathrm{d}t \\
&= \sum_{n=1}^N \|U_n - U_{n-1}\|_{L^1(\Omega)} \,.
\end{aligned}$$

Using theorem 4, we conclude the result ii).

We follow the same techniques used above to show the estimates iii), iv) and v). □

Lemma 6. *Let hypotheses (H1), (H2) and (H3) be satisfied, the sequence $(\overline{u}_N)_{N \in \mathbb{N}}$ converges in measure and a.e. in Q_T.*

Proof. Let ε, r, k be positive real numbers and let $N, M \in \mathbb{N}$, we have the following inclusion

$$\begin{aligned}
&\{|\overline{u}_N - \overline{u}_M| > r\} \\
&\quad \subset \{|\overline{u}_N| > k\} \cup \{|\overline{u}_M| > k\} \cup \{|\overline{u}_N| \leq k, |\overline{u}_M| \leq k, |\overline{u}_N - \overline{u}_M| > r\} \,.
\end{aligned}$$

By Markov inequality and Lemma 5, we deduce

$$\begin{aligned}
\operatorname{meas}\{|\overline{u}_N| > k\} &\leq \frac{1}{k} \|\overline{u}_N\|_{L^1(Q_T)} \\
&\leq \frac{1}{k} C(T, u_0, f) \,,
\end{aligned}$$

and

$$\text{meas}\{|\overline{u}_M| > k\} \leq \frac{1}{k} C(T, u_0, f)\,.$$

This implies for sufficiently large k that

$$\text{meas}(\{|\overline{u}_N| > k\} \cup \{|\overline{u}_M| > k\}) \leq \frac{\varepsilon}{2}\,. \tag{16}$$

On the other hand, by Lemma 5, the sequence $(T_k(\overline{u}_N)_{N\in\mathbb{N}})$ is bounded in space $L^{p(\cdot)}(Q_T, \omega)$. Then, there exists a subsequence, still denoted by $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ such that $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^{p(\cdot)}(Q_T, \omega)$ and in measure. Therefore, there exists $N_0 \in \mathbb{N}$ such that for all $N, M \geq N_0$, we have

$$\text{meas}(\{|\overline{u}_N| \leq k, |\overline{u}_M| \leq k, |\overline{u}_N - \overline{u}_M| > r\}) < \frac{\varepsilon}{2}\,. \tag{17}$$

Consequently, by (16) and (17), $(\overline{u}_N)_{N\in\mathbb{N}}$ converges in measure and there exists a measurable function on Q_T, u such that

$$\overline{u}_N \to u \quad \text{a.e. in } Q_T$$

This finish the proof of Lemma 6. □

Lemma 7. *There exists a function u in $L^1(Q_T)$ such that*

$$T_k(u) \in L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega, \omega))$$

for all $k > 0$ and:

i) u_N converges to u in $L^1(Q_T)$,

ii) $\overline{u}_N$ converges to u in $L^1(Q_T)$,

iii) $\nabla T_k(\overline{u}_N)$ converges to $\nabla T_k(u)$ weakly in $L^{p(\cdot)}(Q_T, \omega)$,

iv) $T_k(\overline{u}_N)$ converges to $T_k(u)$ weakly in $L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega, \omega))$.

Proof. For iii) and iv). By v) of Lemma 5, we have

$$(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}} \text{ is bounded in } L^{p(\cdot)}(Q_T, \omega)\,.$$

Then, there exists a subsequence, still denoted $(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}}$ such that

$$(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}} \text{ converges weakly to an element v in } L^{p(\cdot)}(Q_T, \omega)\,.$$

However

$$T_k(\overline{u}_N) \text{ converges to } T_k(u) \text{ in } L^{p(\cdot)}(Q_T, \omega)\,.$$

Hence, it follows that

$$\nabla T_k(\overline{u}_N) \text{ converges to } \nabla T_k(u) \text{ weakly in } L^{p(\cdot)}(Q_T, \omega)\,,$$

and by v) of Lemma 5, we conclude that

$$T_k(\overline{u}_N) \text{ converges to } T_k(u) \text{ weakly in } L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega, \omega))\,.$$ □

In order to show that the limit function u is an entropy solution of problem (1), we need the following result.

Lemma 8. *The sequence $(u_N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$.*

Proof. For $\varphi \in L^\infty(Q_T)\cap L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))\cap W^{1,1}(0,T;L^1(\Omega))$, the inequality (5) implies that

$$\int_0^t \left\langle \frac{\partial u_N}{\partial s}, T_k(\overline{u}_N-\varphi)\right\rangle \mathrm{d}s + \int_0^t\int_\Omega \omega(x)|\nabla \overline{u}_N|^{p(x)-2}\nabla\overline{u}_N\cdot\nabla T_k(\overline{u}_N-\varphi)\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t\int_\Omega f_N T_k(\overline{u}_N-\varphi)\,\mathrm{d}x\,\mathrm{d}s\,, \quad (18)$$

where $f_N(t,x)=f_n(x), \forall t\in(t_{n-1},t_n]$, $n=1,\dots,N$.

We consider the two partitions $(t_n=n\tau_N)_{n=1}^N$ and $(t_m=m\tau_M)_{m=1}^M$ of interval $[0,T]$ and the corresponding semi-discrete solutions $(u_N(t),\overline{u}_N(t))$, $(u_M(t),\overline{u}_M(t))$ defined by (14) and (15).

Let $h>0$, for the semi-discrete solution $(u_N(t),\overline{u}_N(t))$ we take $\varphi=T_h(\overline{u}_M)$ and for the semi-discrete solution $(u_M(t),\overline{u}_M(t))$ we take $\varphi=T_h(\overline{u}_N)$. Summing the two inequalities and letting h go to infinity, we have for $k=1$

$$\int_0^t\left\langle \frac{\partial(u_N-u_M)}{\partial s}, T_1(\overline{u}_N-\overline{u}_M)\right\rangle \mathrm{d}s + \lim_{h\to\infty} I_{N,M}(h) \le \|f_N-f_M\|_{L^1(Q_T)}\,, \quad (19)$$

where

$$I_{N,M}(h)=\int_0^t\int_\Omega \omega(x)\Big(|\nabla\overline{u}_N|^{p(x)-2}\nabla\overline{u}_N\cdot\nabla T_1(\overline{u}_N-T_h(\overline{u}_M)) + |\nabla\overline{u}_M|^{p(x)-2}\nabla\overline{u}_M\cdot\nabla T_1(\overline{u}_M-T_h(\overline{u}_N))\Big)\,\mathrm{d}x\,\mathrm{d}s\,.$$

The above inequality (19) becomes

$$\int_\Omega \Theta_1(u_N(t)-u_M(t))\,\mathrm{d}x + \lim_{h\to\infty} I_{N,M}(h) \le \|f_N-f_M\|_{L^1(Q_T)} + \left|\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s}, T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\right\rangle \mathrm{d}s\right|. \quad (20)$$

By Hölder inequality

$$\left|\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s}, T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\right\rangle \mathrm{d}s\right| \le \left\|\frac{\partial(u_N-u_M)}{\partial s}\right\|_{L^1(Q_T)} \|T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\|_{L^\infty(Q_T)}\,.$$

This implies by Lemma 5 that

$$\left|\int_0^t \left\langle \frac{\partial(u_N - u_M)}{\partial s} T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M) \right\rangle \mathrm{d}s\right| \\ \leq 2C(T, u_0, f)\|T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\|_{L^\infty(Q_T)}\,.$$

We have

$$\lim_{N,M\to\infty} \|T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\|_{L^\infty(Q_T)} = 0\,.$$

Then

$$\lim_{N,M\to\infty} \left|\int_0^t \left\langle \frac{\partial(u_N - u_M)}{\partial s}, T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M) \right\rangle \mathrm{d}s\right| = 0\,. \tag{21}$$

We also have

$$\lim_{N,M\to\infty} \|f_N - f_M\|_{L^1(Q_T)} = 0\,.$$

Then, the inequality (20) becomes

$$\lim_{N,M\to\infty} \int_\Omega \Theta_1\big(u_N(t) - u_M(t)\big)\,\mathrm{d}x + \lim_{N,M\to\infty}\lim_{h\to\infty} I_{N,M}(h) \leq 0\,. \tag{22}$$

Using the same technique used in proof of Theorem 3, we prove that

$$\lim_{N,M\to\infty}\lim_{h\to\infty} I_{N,M}(h) \geq 0\,.$$

Thus, by inequality (22), we get

$$\lim_{N,M\to\infty} \int_\Omega \Theta_1\big(u_N(t) - u_M(t)\big)\,\mathrm{d}x = 0\,. \tag{23}$$

By definition of Θ_1, we have

$$\int_{\{|u_N - u_M|<1\}} |u_N(t) - u_M(t)|^2\,\mathrm{d}x \\ + \frac{1}{2}\int_{\{|u_N - u_M|\geq 1\}} |u_N(t) - u_M(t)|\,\mathrm{d}x \\ \leq \int_\Omega \Theta_1\big(u_N(t) - u_M(t)\big)\,\mathrm{d}x\,.$$

This implies that

$$\begin{aligned}\int_\Omega |u_N(t)-u_M(t)|\,\mathrm{d}x &= \int_{\{|u_N-u_M|<1\}} |u_N(t)-u_M(t)|\,\mathrm{d}x \\ &\quad + \int_{\{|u_N-u_M|\geq 1\}} |u_N(t)-u_M(t)|\,\mathrm{d}x \\ &\leq C(\Omega)\Big(\int_{\{|u_N-u_M|<1\}} |u_N(t)-u_M(t)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\quad + \int_{\{|u_N-u_M|\geq 1\}} |u_N(t)-u_M(t)|\,\mathrm{d}x \\ &\leq C(\Omega)\Big(\int_\Omega \Theta_1\big(u_N(t)-u_M(t)\big)\,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\quad + 2\int_\Omega \Theta_1\big(u_N(t)-u_M(t)\big)\,\mathrm{d}x\,.\end{aligned}$$

Therefore, by the result (23), we conclude that $(u_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,T;L^1(\Omega))$ and

$$(u_N)_{N\in\mathbb{N}} \quad \text{converges to } u \text{ in } C(0,T;L^1(\Omega))\,. \qquad \square$$

To finish this step, we verify that the limit function u is an entropy solution of the problem (1). Let $N\in\mathbb{N}$, since $u_N(0)=U_0=u_0$, then $u(0,\cdot)=u_0$. Let $N\to\infty$ in inequality (18) and using the above results, we get that the limit function u is an entropy solution of the problem (1).

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