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On a question of Schmidt and Summerer concerning 3-systems

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Abstract. Following a suggestion of W.M. Schmidt and L. Summerer, we construct a proper 3-system (P_1, P_2, P_3) with the property $\overline{\varphi}_3 = 1$. In fact, our method generalizes to provide n-systems with $\overline{\varphi}_n = 1$, for arbitrary $n \geq 3$. We visualize our constructions with graphics. We further present explicit examples of numbers $\xi_1, \ldots, \xi_{n-1}$ that induce the n-systems in question.

1 Parametric Diophantine approximation in dimension three

Let ξ_1, ξ_2 be real numbers so that the set $\{1, \xi_1, \xi_2\}$ is linearly independent over $\mathbb{Q}$. For $q > 0$ a parameter, let $K(q)$ be the box of points $(z_0, z_1, z_2) \in \mathbb{R}^3$ that satisfy

$$|z_0| \leq e^{2q}, \quad |z_1| \leq e^{-q}, \quad |z_2| \leq e^{-q}.$$

Further let Λ be the lattice consisting of the points

$$\big\{(x, \xi_1 x - y_1, \xi_2 x - y_2) : x, y_1, y_2 \in \mathbb{Z}\big\}.$$

The successive minima $\lambda_1(q), \lambda_2(q), \lambda_3(q)$ of $K(q)$ with respect to Λ as functions of q contain the essential information on the simultaneous rational approximation to ξ_1, ξ_2. It is convenient to study the logarithms of the functions $\lambda_j(q)$, denoted by $L_j(q) = \log \lambda_j(q)$ for $j = 1, 2, 3$. These functions have the nice property that their slopes are among $\{-2, 1\}$, and their sum is absolutely bounded uniformly in the parameter q. These properties motivated Schmidt and Summerer [7] to define so called 3-systems. A 3-system $P = (P_1, P_2, P_3)$ is a triple of functions $P_j : [0, \infty) \to \mathbb{R}$ with slopes among $\{-2, 1\}$ with the properties that

$$P_1(0) = P_2(0) = P_3(0) = 0,$$
$$P_1(q) \leq P_2(q) \leq P_3(q)$$

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and

$$P_1(q)+P_2(q)+P_3(q)=0 \quad \text{for every } q\geq 0.$$

Hence, locally in a neighborhood of any $q>0$, precisely one of the three functions decays while the other two rise, unless q is a switch point where some P_j are not differentiable (change slope). Moreover, for P to be a 3-system, it is additionally required that if at a switch point q some P_i changes from falling to rising and some other P_j from rising to falling, then $i<j$ unless $P_i(q)=P_j(q)$. It has been shown in [7] that every function triple (L_1,L_2,L_3) as above, associated to some (ξ_1,ξ_2), corresponds to a 3-system P up to a bounded amount, and conversely by Roy [2] that for any 3-system P there exist ξ_1,ξ_2 satisfying the $\mathbb{Q}$-linear independence condition above and so that $\sup_{q>0}\max_{j=1,2,3}|P_j(q)-L_j(q)|\ll 1$. Roy's result employs a minor technical condition on the mesh of the system P, we do not rephrase it here. Both results [2], [7] are established in more generality.

For given ξ_1,ξ_2 with induced funtions $L_j(q)$, let $\varphi_j(q)=L_j(q)/q$ and put

$$\underline{\varphi}_j=\liminf_{q\to\infty}\varphi_j(q)\,, \qquad \overline{\varphi}_j=\limsup_{q\to\infty}\varphi_j(q)\,,$$

for $j=1,2,3$. Since L_j have slopes -2 and 1 only, it is clear that

$$-2\leq\underline{\varphi}_j\leq\overline{\varphi}_j\leq 1\,, \qquad j=1,2,3\,. \tag{1}$$

By virtue of the results from [2], [7] quoted above, in the sequel we will identify the values $\underline{\varphi}_j,\overline{\varphi}_j$ with quantities derived from an associated 3-system P via

$$\underline{\varphi}_j\longleftrightarrow\liminf_{q\to\infty}\frac{P_j(q)}{q}\,, \qquad \overline{\varphi}_j\longleftrightarrow\limsup_{q\to\infty}\frac{P_j(q)}{q}\,, \qquad j=1,2,3\,, \tag{2}$$

and vice versa. M. Laurent [1] provided estimates for classical exponents of approximation related to any pair (ξ_1,ξ_2) that is $\mathbb{Q}$-linearly independent with $\{1\}$. As pointed out in [9] they translate into the language of the functions φ_j as

$$0\leq\underline{\varphi}_3\leq\overline{\varphi}_3\leq 1,$$

$$\underline{\varphi}_3+\underline{\varphi}_3\overline{\varphi}_1+\overline{\varphi}_1=0, \tag{3}$$

$$2\underline{\varphi}_1+\overline{\varphi}_3\leq-\underline{\varphi}_3(3+\underline{\varphi}_1+2\overline{\varphi}_3), \tag{4}$$

$$2\overline{\varphi}_3+\underline{\varphi}_1\geq-\overline{\varphi}_1(3+\overline{\varphi}_3+2\underline{\varphi}_1). \tag{5}$$

Schmidt and Summerer [9] recently provided additional information by including the second successive minimum in the picture.

Theorem 1 (Schmidt/Summerer, 2017). *For any ξ_1,ξ_2 with $\{1,\xi_1,\xi_2\}$ linearly independent over $\mathbb{Q}$, if $0\leq\underline{\varphi}_3<1$, additionally to the above relations we have*

$$\overline{\varphi}_2\leq\overline{\Omega}:=\frac{\overline{\varphi}_1-\underline{\varphi}_1}{2-\overline{\varphi}_1-\overline{\varphi}_1\underline{\varphi}_1}\,, \tag{6}$$

and

$$\underline{\varphi}_2 \geq \underline{\Omega} := \frac{\underline{\varphi}_3 - \overline{\varphi}_3}{2 - \underline{\varphi}_3 - \overline{\varphi}_3\underline{\varphi}_3} \,. \tag{7}$$

Moreover, these estimates are best possible in the sense that for given numbers $\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_3, \overline{\varphi}_3$ *with* $0 \leq \underline{\varphi}_3 < 1$ *and* (3), (4), (5) *there are* ξ_1, ξ_2 *with* $\{1, \xi_1, \xi_2\}$ *linearly independent over* $\mathbb{Q}$ *for whose approximation constants we have* $\underline{\varphi}_2 = \underline{\Omega}$ *and* $\overline{\varphi}_2 = \overline{\Omega}$.

Schmidt and Summerer enclose a remark to Theorem 1 pointing out that in the case $\underline{\varphi}_3 = 1$ excluded in its claim, we have $\overline{\varphi}_2 = 1$ and $\overline{\varphi}_1 = \underline{\varphi}_2 = -1/2$ (by mistake they denoted $-1/3$ instead in [9]). However, there is a gap in Theorem 1 concerning the *existence* of graphs with the property $\underline{\varphi}_3 = 1$, and related real numbers ξ_1, ξ_2. In [9] they state "But one really should prove that $\underline{\xi} = (\xi_1, \xi_2)$ with $(1, \xi_1, \xi_2)$ linearly independent over $\mathbb{Q}$ with $\overline{\varphi}_3 = 1$ exist. We invite the reader to construct a proper 3-system P with this property." The main purpose of this paper is to provide the desired construction. Before we turn to constructing the 3-system, we point out that explicit examples of $\mathbb{Q}$-linearly independent $\{1, \xi_1, \xi_2\}$ inducing $\underline{\varphi}_3 = \overline{\varphi}_3 = 1$ can be derived from previous results of the author. Concretely [4, Corollary 2.11], upon putting $k = n - 1 = 2$ and $C = \infty$, yields the following example.

Theorem 2. *Let*

$$\xi_1 = \sum_{k=1}^{\infty} 10^{-(2k-1)!} \,, \qquad \xi_2 = \sum_{k=1}^{\infty} 10^{-(2k)!} \,. \tag{8}$$

Then

$$\underline{\varphi}_1 = -2 \,, \qquad \underline{\varphi}_2 = -\frac{1}{2} \,, \qquad \underline{\varphi}_3 = 1 \,, \tag{9}$$

$$\overline{\varphi}_1 = -\frac{1}{2} \,, \qquad \overline{\varphi}_2 = 1 \,, \qquad \overline{\varphi}_3 = 1 \,. \tag{10}$$

While the results in [4] are originally formulated in the language of another type of exponents, the two types of exponents determine each other via the identities of [6, Theorem 1.4], and we derive Theorem 2. We note that for the sole purpose of $\overline{\varphi}_3 = 1$, as desired in [9] and rephrased above, in fact any numbers ξ_1, ξ_2 which are simultaneously approximable to any order by rational numbers can be chosen. In particular, one may choose the pair (ξ, ξ^2) with ξ any Liouville number, see [5, Theorem 3.1]. However, then we always have $\underline{\varphi}_3 = 0$. For Liouville's constant given as $\xi = 10^{-1!} + 10^{-2!} + 10^{-3!} + \cdots$, by [5, Theorem 3.2] in place of (9), (10) we have

$$\underline{\varphi}_1 = -2 \,, \qquad \underline{\varphi}_2 = -\frac{1}{2} \,, \qquad \underline{\varphi}_3 = 0 \,, \tag{11}$$

$$\overline{\varphi}_1 = 0 \,, \qquad \overline{\varphi}_2 = 1 \,, \qquad \overline{\varphi}_3 = 1 \,. \tag{12}$$

Alternatively to the above examples, the pure existence of pairs (ξ_1, ξ_2) inducing $\overline{\varphi}_3 = 1$ (or $\underline{\varphi}_3 = 1$) also follows from Roy's results [2] and [3, Theorem 11.5] (the latter result, already quoted in [9], provides an explicit description of the spectrum of sixtuples $\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3, \overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3$ by a system of complicated inequalities). The main concern of the question of Schmidt and Summerer appears to be the construction of a suitable 3-system, carried out in Section 2.1 below.

2 Construction of a 3-system with $\overline{\varphi}_3 = 1$

We want to present an effective construction of a 3-system with (9), (10), in particular $\overline{\varphi}_3 = 1$. It resembles the combined graph (L_1, L_2, L_3) with respect to the pair (ξ_1, ξ_2) in (8), in an idealized form. In fact the resulting 3-system can be interpreted as the idealized extremal case of the regular graph defined in [8], for the parameter $\rho = \infty$. In Section 2.3 we will briefly sketch how to modify the method to obtain a graph with (11), (12) instead, and give generalizations to n-systems.

2.1 The construction

We construct the graphs piecewise as follows. Let

$$0 < l_0 < l_1 < l_2 < l_3 < \cdots ,$$

be a fast increasing lacunary sequence of real numbers with the property

$$\lim_{i\to\infty} \frac{l_{i+1}}{l_i} = \infty . \tag{13}$$

Let $r_0 = 0$. In the interval $[r_0, l_0] = [0, l_0]$ let P_1 decay with slope -2 and P_2, P_3 rise with slope 1, so that $P_1(l_0) = -2l_0$ and $P_2(l_0) = P_3(l_0) = l_0$. Let $w_0 = l_0$ for consistency with later notation. Let l_0 be the first switch point where P_1 starts to rise and P_2 starts to decay. Then the graph of P_1 will meet the graph of P_2 at some point $(r_1, P_1(r_1))$ with $r_1 > l_0$. We may assume $l_1 > r_1$. In the interval $[r_1, l_1]$ we define P_1 as decaying with slope -2 again and the other two functions rising with slope 1. Note that $P_3(l_1) = l_1$ since it has not changed slope yet. Assume this construction of the graphs in $[0, l_1]$ was step 0 of our construction. Now we carry out how to complete the process with identical steps $1, 2, 3, \ldots$ where in step i we define the graphs of P_1, P_2, P_3 in the interval $[l_i, l_{i+1}]$. At position $q = l_1$ we let P_1 and P_3 change slopes so that P_1 rises with slope 1 and P_3 decays with slope -2. The function P_2 still rises with slope 1. We keep these slopes until P_2 meets P_3 at position $q = w_1$. Then we let P_2 decay with slope -2 and the other functions rise with slope 1 until P_2 meets P_1 at some point $(r_2, P_1(r_2))$. We may assume $l_2 > r_2$. Then we let P_1 decay with slope -2 up to $q = l_2$, and the other two functions rise with slope 1 in this interval. This completes step 1. At $q = l_2$ we let P_1 again switch from decaying to rising and conversely for P_3, and so on. When we repeat the whole process ad infinitum, we claim that P_1, P_2, P_3 represent the combined graph of a 3-system with the properties (9), (10). A sketch of such a 3-system in an initial interval is shown in Figure 1 below. For size reasons we used the slopes $-1, 1/2$ instead of $-2, 1$, thereby sketching $P_j(q)/2$ for $j = 1, 2, 3$.

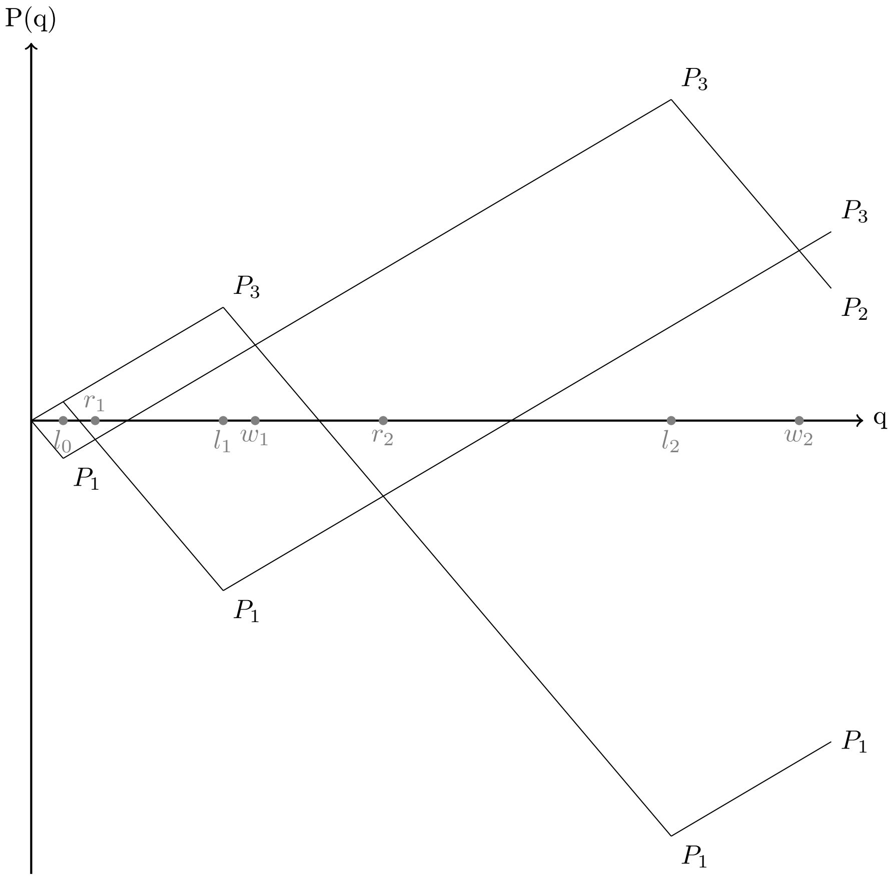

Figure 1: Visualization of case $\overline{\varphi}_3 = 1$, slopes scaled by factor 1/2.

2.2 The proof

Keep in mind for the following that the switching positions in our construction are ordered

$$0 = r_0 < l_0 = w_0 < r_1 < l_1 < w_1 < r_2 < l_2 < w_2 < \cdots ,$$

and also the identification (2). First it is clear that the process yields the combined graph of a 3-system P. Indeed, by construction there is always precisely one P_j decaying, there are infinitely many positions where $P_1 = P_2$ and $P_2 = P_3$ respectively hold, and the switches occur in a way that respects the additional 3-system condition on a local maximum having higher index than a local minimum at switch points mentioned in the introduction. To obtain (9), (10), we first look at positions $q = l_i$ and claim that

$$\lim_{i\to\infty} \frac{P_1(l_i)}{l_i} = -2 , \qquad \lim_{i\to\infty} \frac{P_2(l_i)}{l_i} = \lim_{i\to\infty} \frac{P_3(l_i)}{l_i} = 1 . \tag{14}$$

By the identification (2) and by (1) this implies $\underline{\varphi}_1 = -2$ and $\overline{\varphi}_2 = \overline{\varphi}_3 = 1$. By construction P_1 decays with slope -2 in intervals of the form $I_t := [r_t, l_t]$ for $t \geq 0$ and rises in intervals $J_t := [l_{t-1}, r_t]$ for $t \geq 1$. We next check that

$$r_t < 2l_{t-1}\,, \qquad t \geq 1. \tag{15}$$

We trivially have $P_3(l_{t-1}) - P_1(l_{t-1}) \leq l_{t-1} - (-2l_{t-1}) = 3l_{t-1}$. On the other hand, since P_1 decays in J_t with slope -2 whereas P_3 rises with slope 1, the function $P_3 - P_1$ has slope 3 in J_t so that they must meet within distance $3l_{t-1}/3 = l_{t-1}$ in the first coordinate on the right from l_{t-1}. This intersection point has first coordinate r_t, and we deduce (15).

The estimate (15) and the assumption (13) clearly imply that the sums of the lengths of the intervals I_t over $t = 1, 2, \ldots, i$ exceeds the according sums of the intervals J_t by any given factor $\rho > 0$ for large enough i, i.e.

$$\sum_{t=0}^{i} |I_t| > \rho \sum_{t=1}^{i} |J_t|\,, \qquad i \geq i_0(\rho).$$

Thus since

$$l_i = \sum_{t=0}^{i} |I_t| + \sum_{t=1}^{i} |J_t|$$

and

$$P_1(l_i) = -2\sum_{t=0}^{i} |I_t| + \sum_{t=1}^{i} |J_t|\,,$$

indeed for sufficiently large i we have

$$\frac{P_1(l_i)}{l_i} < -\frac{2+\rho^{-1}}{1+\rho^{-1}}\,.$$

As we can choose ρ arbitrarily large indeed $\lim_{i\to\infty} P_1(l_i)/l_i = -2$, hence $\underline{\varphi}_1 = -2$ by (1), (2). Since P_2 and P_3 rise with slope 1 in any I_t we infer the remaining claims of (14) by a very similar argument, or directly by using the bounded sum property at $q = l_i$.

Next we show

$$\lim_{q\to\infty} \frac{P_3(q)}{q} = 1\,. \tag{16}$$

By construction P_3 has local minima precisely at positions w_i and it rises with slope 1 everywhere outside of the intervals $[l_i, w_i]$, in which it decays with slope -2. In view of (1) it suffices to check that

$$\liminf_{i\to\infty} \frac{P_3(w_i)}{w_i} \geq 1\,. \tag{17}$$

By construction

$$P_3(w_i) = l_0 - 2\sum_{j=0}^{i} (w_j - l_j) + \sum_{j=0}^{i-1} (l_{j+1} - w_j)\,.$$

Hence, in view of (13), to verify (17) it suffices to check

$$\lim_{i\to\infty} \frac{w_i}{l_i} = 1\,. \tag{18}$$

Now by construction in the interval $[l_i, w_i]$ the function P_2 rises with slope 1 whereas P_3 decays with slope -2, hence $w_i = l_i + u_i$ with u_i defined implicitly by the identity $P_3(l_i) - 2u_i = P_2(l_i) + u_i$, that is $w_i = l_i + (P_3(l_i) - P_2(l_i))/3$. On the other hand, by (14) we have $P_2(l_i) = l_i(1 + o(1))$ and $P_3(l_i) = l_i(1 + o(1))$, hence inserting we derive $w_i = l_i(1 + o(1))$ as $i \to \infty$, as desired. Thus (16) is shown.

Finally we show that

$$\lim_{i\to\infty} \frac{P_1(r_i)}{r_i} = \lim_{i\to\infty} \frac{P_2(r_i)}{r_i} = -\frac{1}{2}\,. \tag{19}$$

Since by construction the local maxima of P_1 and the local minima of P_2 both are attained precisely at the positions r_i, the remaining identities from (9) and (10) are implied. Let $K_t = [w_{t-1}, r_t]$, so that $K_t \subseteq J_t$ and by (13), (18) the complement $J_t \setminus K_t$ is small compared to J_t. In K_t, the function P_1 rises with slope 1 whereas P_2 decays with slope -2. Moreover, by (14) and (18) and since the slopes are bounded

$$\lim_{i\to\infty} \frac{P_1(w_i)}{w_i} = -2\,, \qquad \lim_{i\to\infty} \frac{P_2(w_i)}{w_i} = 1\,.$$

Combining these two facts and by definition of r_i, for large i we readily conclude $r_i = w_{i-1}(2 - o(1))$ and thus the asymptotic value at r_i is

$$P_1(r_i) = P_1(w_{i-1}) + r_i - w_{i-1} = w_{i-1}(-1 + o(1))\,,$$

hence indeed $P_1(r_i)/r_i = -1/2 + o(1)$ for large i. Thus (19) holds and the proof is finished.

2.3 Generalizations and variations

A similar construction as in Section 2.1 can be done in arbitrary dimension n, where the slopes of the P_j are among $\{-n+1, 1\}$. Instead of one sequence $(w_i)_{i\geq 0}$ with $l_i < w_i < r_{i+1}$, we obtain $n-2$ sequences $(w_i^h)_{i\geq 0}, 1 \leq h \leq n-2$, induced by positions where P_{n-h+1} meets P_{n-h}, ordered

$$l_i < w_i^1 < w_i^2 < \cdots < w_i^{n-2} < r_{i+1}\,.$$

We derive n-systems $P = (P_1, \ldots, P_n)$ whose approximation constants (via identification (2)) satisfy

$$\underline{\varphi}_1 = -n+1\,, \qquad \underline{\varphi}_2 = \frac{2-n}{2}\,, \qquad \underline{\varphi}_j = 1\,, \quad 3 \leq j \leq n,$$

and

$$\overline{\varphi}_1 = \frac{2-n}{2}\,, \qquad \overline{\varphi}_j = 1\,, \quad 2 \leq j \leq n.$$

Again this resembles the special case $\rho = \infty$ of the regular graph [8] in dimension n, and suitable numbers $(\xi_1, \ldots, \xi_{n-1})$ inducing these approximation constants arise from [4, Corollary 2.11] upon taking $k = n-1$, $C = \infty$, a particular choice is

$$\xi_j = \sum_{k=0}^{\infty} 10^{-(k(n-1)+j)!}, \qquad 1 \le j \le n-1.$$

Finally, we sketch the construction of a 3-system P with the properties (11), (12) in place of (9), (10). We have to alternate between the construction of Section 2.1 and another type of intermediate construction. Take i a large integer and follow the construction from Section 2.1 up to $q = q_0 =: l_i$. Recall $P_1(l_i) \approx -2l_i$ and $P_j(l_i) \approx l_i$ for $j = 2, 3$ by (14). Then we make the first intermediate construction. Starting from q_0, let P_1 rise with slope 1 and P_2, P_3 decay with slope roughly $-1/2$ in not too short intervals. The latter can be easily realized by changing the slopes of P_2, P_3 rapidly so that there are many positions q with equality $P_2(q) = P_3(q)$. One may take these equality positions an arithmetic sequence $b_0, b_1 = b_0 + D, b_2 = b_0 + 2D, \ldots, b_h = b_0 + hD$ with some $b_0 \ge q_0, h \ge 0$ and some small increment $D > 0$, in the following way. Fix $D > 0$ small. Let $(b_0, P_2(b_0))$ be the intersection point of the line passing through $(q_0, P_2(q_0))$ with slope 1 (graph of P_2) and the line passing through $(q_0, P_3(q_0))$ with slope -2 (graph of P_3), corresponding to w_i in Section 2.1. In $[b_0, b_0 + D/2]$, let P_2 decay with slope -2 and P_3 rise with slope 1. Then at $q_0 + D/2$ interchange the slopes, such that at $b_1 = b_0 + D$ we have $P_2(b_1) = P_3(b_1) = P_2(b_0) - D/2$. We repeat this procedure and stop at the largest index h so that the resulting graphs of P_2, P_3 remain positive on $[0, b_h]$. For simplicity let $\tilde{q} := b_h$. Notice that $P_j(b_l) - P_j(b_0) = -(b_l - b_0)/2 = -lD/2$ for $l = 0, 1, \ldots, h$. Therefore, by (14) and since D is small, it is easy to see that $|P_j(\tilde{q})|$ are all small for $j = 1, 2, 3$. Now starting at $\tilde{q}$, let P_1, P_3 rise with slope 1 and P_2 decay with slope -2 until the graphs of P_1 and P_2 meet at some position q_1. Since $|P_j(\tilde{q})|$ are all small, the expressions $q_1 - \tilde{q}$ and $|P_j(q_1)|$ for $j = 1, 2, 3$, are small (like $o(q_1)$) as well. This ends the first intermediate construction, illustrated in Figure 2 below (again slopes are scaled with factor 1/2). Now we essentially apply the initial construction (step 0) from Section 2.1 from the interval $[0, l_1]$ again, starting from $q = q_1$ instead of $q = 0$. Let us denote by q_2 the right endpoint in this construction, that is the value corresponding to l_1 from Section 2.1. Notice that P_1 has a local minimum inside the interval $[q_1, q_2]$, corresponding to l_0 from Section 2.1, and another one at the right endpoint q_2. Since $|P_j(q_1)|$ are small for $j = 1, 2, 3$, the P_j indeed behave in $[q_1, q_2]$ essentially like they do in the construction of Section 2.1 in the interval $[0, l_1]$ (see Figure 1). In particular, as for $q = q_0$, at $q = q_2$ again we have $P_1(q_2) \approx -2q_2$ and $P_j(q_2) \approx q_2$ for $j = 2, 3$. Hence at this point we again switch to the intermediate construction to define the P_j in some interval $[q_2, q_3]$. We repeat this iterative process of constructing P in $[q_{2k}, q_{2k+1}]$ and then in $[q_{2k+1}, q_{2k+2}]$, for all $k \ge 1$. It can be checked that the resulting combined graph satisfies (11), (12). Notice hereby that the condition $\underline{\varphi}_2 = -1/2$ forced us to copy the behavior of the P_j on $[0, l_1]$, and not only on $[0, l_0]$, in intervals $[q_{2k+1}, q_{2k+2}]$. The procedure can again be generalized to dimension n to provide n-systems with the properties

$$\underline{\varphi}_1 = -n+1\,, \qquad \underline{\varphi}_2 = \frac{2-n}{2}\,, \qquad \underline{\varphi}_j = 0\,, \quad 3 \leq j \leq n,$$

and

$$\overline{\varphi}_1 = 0\,, \qquad \overline{\varphi}_j = 1\,, \quad 2 \leq j \leq n.$$

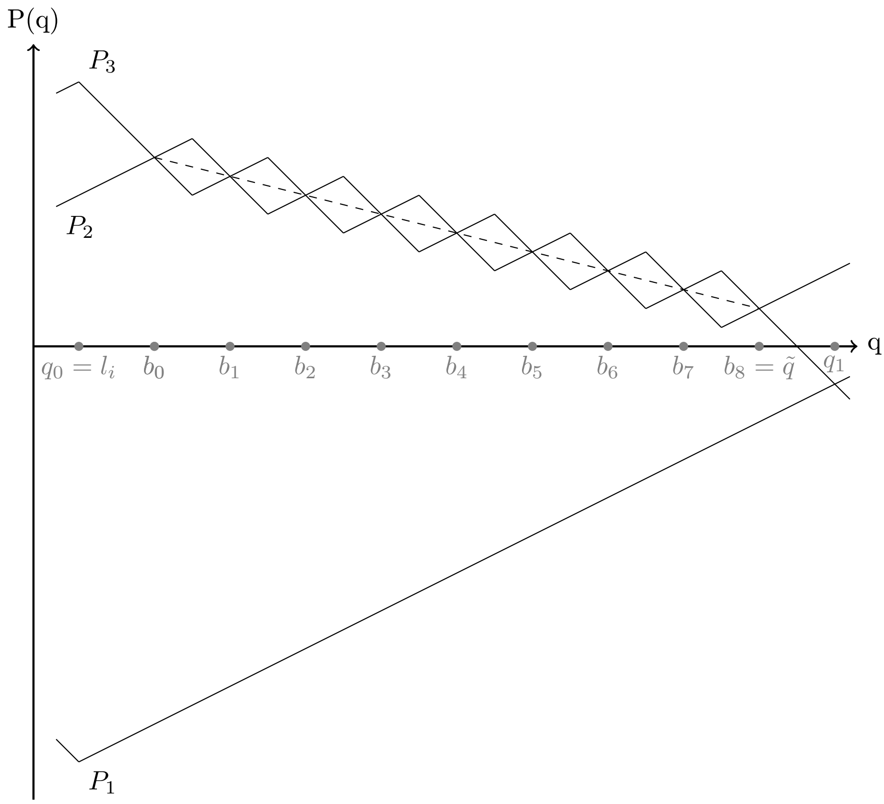

Figure 2: Intermediate construction in $[q_0, q_1]$, slopes scaled by factor 1/2.

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References

[1] M. Laurent: Exponents of Diophantine approximation in dimension two. *Canadian Journal of Mathematics* 61 (1) (2009) 165–189.

[2] D. Roy: On Schmidt and Summerer parametric geometry of numbers. *Annals of Mathematics* (2015) 739–786.

[3] D. Roy: On the topology of Diophantine approximation spectra. *Compositio Mathematica* 153 (7) (2017) 1512–1546.

[4] J. Schleischitz: Diophantine approximation and special Liouville numbers. *Communications in Mathematics* 21 (1) (2013) 39–76.

[5] J. Schleischitz: On approximation constants for Liouville numbers. *Glasnik matematički* 50 (2) (2015) 349–361.

[6] W.M. Schmidt, L. Summerer: Parametric geometry of numbers and applications. *Acta Arithmetica* 140 (2009) 67–91.

[7] W.M. Schmidt, L. Summerer: Diophantine approximation and parametric geometry of numbers. *Monatshefte für Mathematik* 169 (1) (2013) 51–104.

[8] W.M. Schmidt, L. Summerer: Simultaneous approximation to three numbers. *Moscow Journal of Combinatorics and Number Theory* 3 (1) (2013) 84–107.

[9] W.M. Schmidt, L. Summerer: Simultaneous approximation to two reals: bounds for the second successive minimum. *Mathematika* 63 (3) (2017) 1136–1151.

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