Communications in Mathematics 28 (2020) 289–299
DOI: 10.2478/cm-2020-0008

Generalization of uniqueness and value sharing of meromorphic functions concerning differential polynomials

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Abstract. The motivation of this paper is to study the uniqueness problems of meromorphic functions concerning differential polynomials that share a small function. The results of the paper improve and generalize the recent results due to Fengrong Zhang and Linlin Wu [13]. We also solve an open problem as posed in the last section of [13].

1 Introduction and main results

Throughout this note, the term "meromorphic" means meromorphic in the whole complex plane, and we shall use the standard notations of Nevanlinna theory of meromorphic functions [6]. For a meromorphic function f, let $T(r,f)$ denote the Nevanlinna characteristic of f and let $S(r,f)$ be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, except possibly on a set of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$.

If for some $a \in \bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities in to account, f and g are said to share the value a IM (ignoring multiplicities).

In the recent past a number of authors worked on the uniqueness problem of meromorphic functions when differential polynomials generated by them share certain values (cf. [1], [2], [4], [5], [7], [8]). In [7] following question was asked: What

2020 MSC: Primary 30D35
Key words: Uniqueness,Meromorphic functions, Sharing value, Differential polynomials.
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can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?

Since then the progress to investigate the uniqueness of meromorphic functions which are the generating functions of different types of nonlinear differential polynomials is remarakable and continuous efforts are being put in to relax the hypothesis of the results. (see [1], [4], [5], [9], [10]). In 1997, Yang and Hua [11] proved the following result.

Theorem A. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, let $n \geq 11$ be an integer, and let $a \in \mathbb{C} \setminus \{0\}$. If $f^n f'$ and $g^n g'$ share a CM, then $f(z) \equiv dg(z)$ for some $(n+1)$-th roots of unity d, or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Without loss of generality, in Theorem A the complex number a can be replaced by 1. Noting that $\left(\frac{1}{n+2} f^{n+2} - \frac{1}{n+1} f^{n+1}\right)' = f^n (f-1) f'$, Fang and Hong [5] obtained the following result.

Theorem B. *Let f and g be two transcendental entire functions, let $n \geq 11$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2004, Lin and Yi [10] improved their result to $n \geq 7$ and also studied the case that f and g are meromorphic functions. Moreover, they discussed the other polynomial $\frac{1}{n+3} f^{n+3} - \frac{2}{n+2} f^{n+2} + \frac{1}{n+1} f^{n+1}$ of f with its derivative as $f^n(f-1)^2 f'$. In fact, Lin and Yi proved the following two theorems.

Theorem C. *Let f and g be two nonconstant meromorphic functions, and let $n \geq 12$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then*

$$f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})} \tag{1}$$

where h is a nonconstant meromorphic function.

Theorem D. *Let f and g be two nonconstant meromorphic functions, and let $n \geq 13$ be an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2011, Dyavanal [3] proved the following results, which to the knowledge of the authors probably are the first approach in which in order to consider the value sharing of two differential polynomials the multiplicities of zeros and poles of f and g are taken into account.

Theorem E. *Let f and g be two nonconstant meromorphic functions with zeros and poles of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If $f^n f'$ and $g^n g'$ share 1 CM, then $f(z) \equiv dg(z)$ for some $(n+1)$-th roots of unity d, or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

Theorem F. *Let f and g be two nonconstant meromorphic functions with zeros and poles of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s \geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then (1) holds.*

Theorem G. *Let f and g be two nonconstant meromorphic functions with zeros and poles of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f(z) \equiv g(z)$.*

Recently, Fengrong Zhang and Linlin Wu [13] discussed Theorems E–G by replacing CM with IM and reduced n for $s \geq 7$ in Theorems F–G and proved the following results.

Theorem H. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n-4)s \geq 19$ for $s = 1, 2$ and $ns \geq 28$ for $s \geq 3$. If f^nf' and g^ng' share 1 IM, then $f(z) \equiv dg(z)$ for some $(n+1)$-th root d of unity, or $g(z) = c_1e^{cz}$ and $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

Theorem I. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s. Suppose that $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 IM, where s and n are positive integers. Then we have one of the following two cases:*

1. *if $s = 1$ and $n \geq 27$, then $f(z) \equiv g(z)$, or we have (1);*
2. *if $(n-8)s \geq 19$ for $s = 2$ and $(n-4)s \geq 28$ for $s \geq 3$, then $f(z) \equiv g(z)$.*

Theorem J. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s. Suppose that $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 IM, where s and n are positive integers. If $(n-9)s \geq 19$ for $s = 1, 2$ and $(n-5)s \geq 28$ for $s \geq 3$, then $f(z) \equiv g(z)$.*

Theorem K. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, where $s\,(\geq 7)$ is a positive integer. Let n be an integer satisfying $(n-1)s \geq 13$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.*

Theorem L. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, where $s\,(\geq 7)$ is a positive integer. Let n be an integer satisfying $(n-2)s \geq 13$. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f(z) \equiv g(z)$.*

In the same paper, Fengrong Zhang and Linlin Wu posed the following open problem.

Open problem 1. *Let n, k be positive integers, and let m be a nonnegative integer. Suppose that*

$$f^n(f-1)^m f^{(k)} \text{ and } g^n(g-1)^m g^{(k)}$$

share a CM (or IM), where $a\,(\not\equiv 0, \infty)$ is a small function of f and g under what conditions can we get $f \equiv g$?

We will concentrate our attention to the above Question and provide an affirmative answer in this direction. Indeed, the following theorems which are the main results of the paper justify our claim.

Theorem 1. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, Suppose that $f^n(f-1)^m f^{(k)}$ and $g^n(g-1)^m g^{(k)}$ share 1 IM, where s, n and k are positive integers. If $(n-m-4)s \geq 4k+18$ for $s = 1, 2$ and $(n-m)s \geq 8k+23$ for $s \geq 3$, then*

1. *$f = tg$ for a constant t such that $t^d = 1$, where $d = (n+m+1, n+m, \ldots, n+1)$.*
2. *$f^n(f-1)^m f^{(k)} \equiv g^n(g-1)^m g^{(k)}$, if $\frac{f}{g}$ is not a constant.*

Remark 1. In Theorem 1 giving specific values for s, m and $k = 1$ in the condition $(n-m-4)s \geq 4k+18$, we get the following interesting cases:

1. If $s = 1$ and $m = 1$, the result reduces to Theorem I i.e., $n \geq 27$ and for $m = 2$, the result reduces to Theorem J i.e., $n \geq 28$.
2. If $s = 2$ and $m = 1$, we improve Theorem I i.e., $n \geq 16$ and for $m = 2$, we improve Theorem J i.e., $n \geq 17$.

Similarly for the condition $(n-m)s \geq 8k+23$, we get the following interesting cases:

3. If $s = 3$ and $m = 1$, we improve Theorem I i.e., $n \geq 11$ and for $m = 2$, we improve Theorem J i.e., $n \geq 12$.
4. When $s \geq 4$, $m = 1$ we obtain the value of n as $n \geq 9$ which improves Theorem I. Also when $s \geq 4$, $m = 2$ we obtain $n \geq 10$ which improves Theorem J.

We conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n.

5. We can further weaken the condition $(n-m)s \geq 8k+23$ in Theorem 1 by replacing the sharing value 1 IM by sharing a small function and ∞ IM i.e., $(n-m)s \geq 8k+22$.
6. The Statement of Theorem 1 remains same even if 1 IM is replaced with a IM in the results presented above.

Theorem 2. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, where $s\,(\geq 7)$ is a positive integer. Let n, k be an integers satisfying $(n-m)s \geq 2k+11$. If $f^n(f-1)^m f^{(k)}$ and $g^n(g-1)^m g^{(k)}$ share 1 CM, then*

1. *$f = tg$ for a constant t such that $t^d = 1$, where $d = (n+m+1, n+m, \ldots, n+1)$.*
2. *$f^n(f-1)^m f^{(k)} \equiv g^n(g-1)^m g^{(k)}$, if $\frac{f}{g}$ is not a constant.*

Remark 2.

1. If $s = 7$, $m = 1$ and $k = 1$ in Theorem 2, then Theorem 2 reduces to Theorem K i.e., $n \geq 3$.
2. If $s = 7$, $m = 2$ and $k = 1$ in Theorem 2, then Theorem 2 reduces to Theorem L i.e., $n \geq 4$.

2 Preliminary Lemmas

We denote by $\overline{N}_{(k}\big(r, \frac{1}{f-a}\big)$ the reduced counting function for zeros of $f-a$ with multiplicity no less than k. Define

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f-a}\Big) + \cdots + \overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big).$$

Lemma 1 ([14]). *Let $f(z)$ be a nonconstant meromorphic function, and let p and k be positive integers. Then*

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \le N_{p+k}\Big(r, \frac{1}{f}\Big) + k\overline{N}(r, f) + S(r, f). \tag{2}$$

Lemma 2 ([11], [12]). *Let f and g be two nonconstant meromorphic functions sharing 1 CM. Then we have one of the following three cases:*

1. $T(r, f) \le N_2\big(r, \frac{1}{f}\big) + N_2\big(r, \frac{1}{g}\big) + N_2(r, f) + N_2(r, g) + S(r, f) + S(r, g)$;
2. $f(z) \equiv g(z)$;
3. $f(z)g(z) \equiv 1$.

Lemma 3 ([13]). *Let f and g be two nonconstant meromorphic functions. If f and g share 1 IM, then we have one of the following three cases:*

1. $$T(r, f) \le N_2\Big(r, \frac{1}{f}\Big) + N_2\Big(r, \frac{1}{g}\Big) + N_2(r, f) + N_2(r, g) + 2\overline{N}(r, f) + \overline{N}(r, g) + 2\overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{g}\Big) + S(r, f) + S(r, g);$$
2. $f(z) \equiv g(z)$;
3. $f(z)g(z) \equiv 1$.

3 Proofs

Proof of Theorem 1. Let $F = f^n(f-1)^m f^{(k)}$ and $G = g^n(g-1)^m g^{(k)}$. Then F and G share 1 IM. By Lemma 3, we consider three cases.

Case 1. Suppose that

$$T(r, F) \le N_2\Big(r, \frac{1}{F}\Big) + N_2\Big(r, \frac{1}{G}\Big) + N_2(r, F) + N_2(r, G) + 2\overline{N}(r, F) + \overline{N}(r, G) + 2\overline{N}\Big(r, \frac{1}{F}\Big) + \overline{N}\Big(r, \frac{1}{G}\Big) + S(r, f) + S(r, g). \tag{3}$$

We deduce from (3) that

$$T(r, F) \le 6\overline{N}\Big(r, \frac{1}{f}\Big) + mN\Big(r, \frac{1}{f}\Big) + N_2\Big(r, \frac{1}{f^{(k)}}\Big) + 4\overline{N}\Big(r, \frac{1}{g}\Big) + mN\Big(r, \frac{1}{g}\Big) + N_2\Big(r, \frac{1}{g^{(k)}}\Big) + 4\overline{N}(r, f) + 3\overline{N}(r, g) + 2\overline{N}\Big(r, \frac{1}{f^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{g^{(k)}}\Big) + S(r, f) + S(r, g)\,. \tag{4}$$

Obviously,

$$N(r,F)=(n+m+1)N(r,f)+k\overline{N}(r,f)+S(r,f)\,. \tag{5}$$

Since

$$\begin{aligned}(n+m)m(r,f)&=m\Big(r,\frac{F}{f^{(k)}}\Big)\le m(r,F)+m\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f)\\ &=m(r,F)+T(r,f^{(k)})-N\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f)\\ &\le m(r,F)+T(r,f)+k\overline{N}(r,f)-N\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f)\,. \end{aligned}\tag{6}$$

It follows from (5), (6), and Lemma 1 that

$$\begin{aligned}(n+m-1)T(r,f)&\le T(r,F)-N(r,f)-N\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f)\\ &\le 6\overline{N}\Big(r,\frac{1}{f}\Big)+mN\Big(r,\frac{1}{f}\Big)+4\overline{N}\Big(r,\frac{1}{g}\Big)+mN\Big(r,\frac{1}{g}\Big)\\ &\quad+N_2\Big(r,\frac{1}{g^{(k)}}\Big)+4\overline{N}(r,f)+3\overline{N}(r,g)+2\overline{N}\Big(r,\frac{1}{f^{(k)}}\Big)\\ &\quad+\overline{N}\Big(r,\frac{1}{g^{(k)}}\Big)-N(r,f)+S(r,f)+S(r,g). \end{aligned}\tag{7}$$

Using Lemma 1, we get

$$\begin{aligned}N_2\Big(r,\frac{1}{g^{(k)}}\Big)+\overline{N}\Big(r,\frac{1}{g^{(k)}}\Big)&\le 2N\Big(r,\frac{1}{g}\Big)+2k\overline{N}(r,g)+S(r,g)\\ &\le 2\Big(1+\frac{k}{s}\Big)T(r,g)+S(r,g), \end{aligned}\tag{8}$$

$$\begin{aligned}2\overline{N}\Big(r,\frac{1}{f^{(k)}}\Big)&\le 2N\Big(r,\frac{1}{f}\Big)+2k\overline{N}(r,f)+S(r,f)\\ &\le 2\Big(1+\frac{k}{s}\Big)T(r,f)+S(r,f). \end{aligned}\tag{9}$$

Then substituting (8) and (9) into (7) yields

$$\begin{aligned}(n+m-1)T(r,f)&\le\Big(\frac{2k+10}{s}+m+1\Big)T(r,f)\\ &\quad+\Big(\frac{2k+7}{s}+m+2\Big)T(r,g)+S(r,f)+S(r,g). \end{aligned}\tag{10}$$

A Similar inequality for G also holds. Therefore we can conclude that

$$\begin{aligned}(n+m-1)\{T(r,f)+T(r,g)\}\le\Big(\frac{4k+17}{s}+2m+3\Big)\{T(r,f)+T(r,g)\}\\ +S(r,f)+S(r,g),\end{aligned}$$

which contradicts the condition $(n-m-4)s \geq 4k+18$ for $s=1,2$.

Again using Lemma 1, we have

$$N_2\Big(r,\frac{1}{g^{(k)}}\Big)+\overline{N}\Big(r,\frac{1}{g^{(k)}}\Big)\leq N_{k+2}\Big(r,\frac{1}{g}\Big)+N_{k+1}\Big(r,\frac{1}{g}\Big)+2k\overline{N}(r,g)+S(r,g)$$
$$\leq\Big(\frac{4k+3}{s}\Big)T(r,g)+S(r,g), \tag{11}$$
$$2\overline{N}\Big(r,\frac{1}{f^{(k)}}\Big)\leq 2N_{k+1}\Big(r,\frac{1}{f}\Big)+2k\overline{N}(r,f)+S(r,f)$$
$$\leq\Big(\frac{4k+2}{s}\Big)T(r,f)+S(r,f). \tag{12}$$

Then substituting the two inequalities in to (7) leads to

$$(n+m-1)T(r,f)\leq\Big(\frac{4k+12}{s}+m-1\Big)T(r,f)+\Big(\frac{4k+10}{s}+m\Big)T(r,g)$$
$$+S(r,f)+S(r,g). \tag{13}$$

Similarly, we can get

$$(n+m-1)\{T(r,f)+T(r,g)\}\leq\Big(\frac{8k+22}{s}+2m-1\Big)\{T(r,f)+T(r,g)\}$$
$$+S(r,f)+S(r,g),$$

which contradicts the condition $(n-m)s\geq 8k+23$ for $s\geq 3$.

Case 2. Suppose that $FG\equiv 1$, that is

$$f^n(f-1)^m f^{(k)} g^n(g-1)^m g^{(k)}\equiv 1 \tag{14}$$

Let z_0 ($\neq 0,\infty$) be a zero of f of order p. From (14) we know that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q. From (14) we obtain

$$np+p-k=nq+mq+q+k\,,$$

that is, $(n+1)(p-q)=mq+2k$, which implies that $p\geq q+1$ and $mq+2k\geq n+1$. Hence,

$$p\geq\frac{n+m-2k+1}{m}. \tag{15}$$

Let z_1 ($\neq 0,\infty$) be a zero of $(f-1)$ of order p_1, then from (14), z_1 is a pole of g of order q_1. Again by (14), we get

$$mp_1+p_1-k=nq_1+mq_1+q_1+k\,,$$

that is,

$$p_1\geq\frac{(n+m+1)s+2k}{m+1}\,. \tag{16}$$

Let $z_2\,(\neq 0,\infty)$ be a zero of f' of order p_2, that is not a zero of $f(f-1)$, then from (14), z_2 is a pole of g of order q_2. Again by (14), we get

$$p_2-(k-1)=nq_2+mq_2+q_2+k\,,$$

that is,

$$p_2\geq (n+m+1)s+2k+1. \tag{17}$$

In the same manner as above, we have the similar results for the zeros of $g^n(g-1)^m g^{(k)}$. From (14) we can write

$$\overline{N}\big(r,f^n(f-1)^m f^{(k)}\big)=\overline{N}\Big(r,\frac{1}{g^n(g-1)^m g^{(k)}}\Big),$$

i.e.,

$$\overline{N}(r,f)=\overline{N}\Big(r,\frac{1}{g^n(g-1)^m g^{(k)}}\Big)=\overline{N}\Big(r,\frac{1}{g}\Big)+\overline{N}\Big(r,\frac{1}{g-1}\Big)+\overline{N}\Big(r\frac{1}{g^{(k)}}\Big)$$

From (15) to (17), we obtain

$$\begin{aligned}\overline{N}(r,f)&\leq \frac{m}{n+m-2k+1}N\Big(r,\frac{1}{g}\Big)+\frac{m+1}{(n+m+1)s+2k}N\Big(r,\frac{1}{g-1}\Big)\\&\quad+\frac{1}{(n+m+1)s+2k+1}N\Big(r,\frac{1}{g^{(k)}}\Big)\\ \overline{N}(r,f)&\leq \Big(\frac{m}{n+m-2k+1}+\frac{m+1}{(n+m+1)s+2k}\\&\quad+\frac{k+1}{(n+m+1)s+2k+1}\Big)T(r,g)+S(r,g).\end{aligned} \tag{18}$$

By the second fundamental theorem and (18), we have

$$\begin{aligned}T(r,f)&\leq \overline{N}(r,f)+\overline{N}\Big(r,\frac{1}{f}\Big)+\overline{N}\Big(r,\frac{1}{f-1}\Big)+S(r,f)\\ T(r,f)&\leq \Big(\frac{m}{n+m-2k+1}+\frac{m+1}{(n+m+1)s+2k}+\frac{k+1}{(n+m+1)s+2k+1}\Big)T(r,g)\\&\quad+\Big(\frac{m}{n+m-2k+1}+\frac{m+1}{(n+m+1)s+2k}\Big)T(r,f)+S(r,f)+S(r,g),\end{aligned} \tag{19}$$

and a similar inequality for $T(r,g)$. Combining the two inequalities, we get

$$\begin{aligned}T(r,f)+T(r,g)\leq \Big(&\frac{2m}{n+m-2k+1}+\frac{2(m+1)}{(n+m+1)s+2k}\\&+\frac{k+1}{(n+m+1)s+2k+1}\Big)\{T(r,f)+T(r,g)\}\\&+S(r,f)+S(r,g).\end{aligned} \tag{20}$$

Giving specific values for m, s and k which satisfies $(n-m-4)s \geq 4k+18$ for $s=1,2$, we deduce that

$$T(r,f)+T(r,g) \leq (0.2655)[T(r,f)+T(r,g)]+S(r,f)+S(r,g).$$

Similarly $(n-m)s \geq 8k+23$ for $s \geq 3$, we have

$$T(r,f)+T(r,g) \leq (0.2276)[T(r,f)+T(r,g)]+S(r,f)+S(r,g).$$

Thus (20) leads to contradiction.

Case 3. $F \equiv G$, that is

$$f^n(f^m+\cdots+(-1)^i m_{C_{m-i}} f^{m-i}+\cdots+(-1)^m)f^{(k)} \equiv g^n(g^m+\cdots+(-1)^i m_{C_{m-i}} g^{m-i}+\cdots+(-1)^m)g^{(k)}. \quad (21)$$

Let $h=\frac{f}{g}$. If h is a constant, by putting $f=hg$ in (21) we get

$$g^{n+m}(h^{n+m+1}-1)+\cdots+(-1)^i m_{C_{m-i}} g^{n+m-i}(h^{n+m-i+1}-1) +\cdots+(-1)^m g^n(h^{n+1}-1)=0.$$

Which implies that $h^d=1$, where $d=(n+m+1, n+m, \ldots, n+1)$. Thus $f(z)=tg(z)$ for a constant t such that $t^d=1$.

If h is not constant then we must have

$$f^n(f-1)^m f^{(k)} \equiv g^n(g-1)^m g^{(k)}. \qquad \square$$

Proof of Theorem 2. Let $F=f^n(f-1)^m f^{(k)}$ and $G=g^n(g-1)^m g^{(k)}$. Then F and G share 1 CM. By Lemma 2, we consider three cases.

Case 1. Suppose that

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G)\,. \quad (22)$$

We deduce from (22) that

$$T(r,F) \leq 2\overline{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+mN\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g^{(k)}}\right)+2\overline{N}(r,f)+2\overline{N}(r,g)+S(r,f)+S(r,g). \quad (23)$$

It follows from (5), (6), (23) and Lemma 1 that

$$\begin{aligned}(n+m-1)T(r,f) &\leq T(r,F)-N(r,f)-N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\ &\leq 2\overline{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+mN\left(r,\frac{1}{g}\right)\\ &\quad + N_2\left(r,\frac{1}{g^{(k)}}\right)+2\overline{N}(r,f)-N(r,f)\\ &\quad + 2\overline{N}(r,g)+S(r,f)+S(r,g) \qquad (24)\end{aligned}$$

where by Lemma 1 for $N_2\big(r, \frac{1}{g^{(k)}}\big)$, we use

$$N_2\Big(r, \frac{1}{g^{(k)}}\Big) \leq N_{k+2}\Big(r, \frac{1}{g}\Big) + k\overline{N}(r, g) + S(r, g) \leq \Big(\frac{2k+2}{s}\Big)T(r, g) + S(r, g). \quad (25)$$

Then substituting (25) in to (24) yields

$$(n+m-1)T(r, f) \leq \Big(\frac{4}{s} + m - 1\Big)T(r, f) + \Big(\frac{2k+6}{s} + m\Big)T(r, g) + S(r, f) + S(r, g). \quad (26)$$

There also exists a similary inequality of $T(r, G)$. Therefore we have

$$(n+m-1)\{T(r, f) + T(r, g)\} \leq \Big(\frac{2k+10}{s} + 2m - 1\Big)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g),$$

which contradicts to $(n-m)s \geq 2k + 11$.

Case 2. Suppose $FG \equiv 1$, that is

$$f^n(f-1)^m f^{(k)} g^n(g-1)^m g^{(k)} \equiv 1\,.$$

Proceeding as in the proof of Theorem 1 (Case 2), we get a contradiction.

Case 3. $F \equiv G$, that is

$$f^n(f-1)^m f^{(k)} \equiv g^n(g-1)^m g^{(k)}\,.$$

Proceeding as in the proof of Theorem 1 (Case 3), we get a Conclusion of Theorem 2.

Hence, the proof of Theorem 2. □

Acknowledgement

The author (RM) is greatful to the University Grants Commission (UGC), New Delhi, India for Supporting her research work by providing her with a National Fellowship for Higher Education (NFHE).

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Received: 12 May, 2019
Accepted for publication: 29 July, 2019
Communicated by: Diana Barseghyan