sciendo

Communications in Mathematics 28 (2020) 199–229
DOI: 10.2478/cm-2020-0014

Reductive homogeneous spaces and nonassociative algebras

Alberto Elduque

Abstract. The purpose of these survey notes is to give a presentation of a classical theorem of Nomizu [21] that relates the invariant affine connections on reductive homogeneous spaces and nonassociative algebras.

These notes have been around for about 20 years. They were polished on the occasion of a CIMPA research school, held in Marrakech (April 13–24, 2015), where an initial version became the notes of a course with the same title in this school. The title of the school was 'Géométrie différentielle et algèbres non associatives'. The author is indebted to the participants in the course for their questions and comments, that improved the initial exposition.

The purpose of the notes is to present a classical result by Nomizu [21] that relates the invariant affine connections on reductive homogeneous spaces to nonassociative algebras defined on the tangent space of a point. This allows us to give precise algebraic descriptions of these connections, of their torsion and curvature tensors, ...

Nomizu's result constitutes a very nice bridge between these two areas mentioned in the title of the school.

Modulo basic results, the presentation is self contained. The first three sections review the results on smooth manifolds, on affine connections and on Lie groups and Lie algebras that will be needed for our purpose. The fourth section is devoted to study the invariant affine connections on homogeneous spaces, following the presentation in [1, Chapter IV].

Nomizu's Theorem is presented and proved in Section 5, based on the previous work. The proof given by Nomizu in his seminal paper is quite different.

2020 MSC: Primary 17B60; Secondary 53B05, 53C30, 22F30, 17A99
Key words: Reductive homogeneous space, invariant affine connection, Lie-Yamaguti algebra
Supported by grants MTM2017-83506-C2-1-P (AEI/FEDER, UE) and E22_17R (Gobierno de Aragón, Grupo de referencia "Álgebra y Geometría", cofunded by Feder 2014-2020 "Construyendo Europa desde Aragón")
Affiliation: Departamento de Matemáticas e Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail: elduque@unizar.es

Also, Lie-Yamaguti algebras will be introduced in Section 5, Exercise 9. Finally, in Section 6 left and bi-invariant affine connections in Lie groups will be studied. Several classes of algebras, like Lie-admissible, flexible, associative or left-symmetric, appear naturally.

Several exercises, complementing the theory, are scattered through the text.

The material presented is well known (see [1], [14], [15]). It is expected that this presentation will be useful to the readers, and will whet their appetite to learn more about the subject.

1 Manifolds

1.1 Manifolds

In these notes an n-dimensional *manifold* will indicate a second countable Hausdorff topological space M covered by a family of open sets $\{\mathcal{U}_\lambda : \lambda \in \Lambda\}$, together with coordinate maps

$$\begin{aligned} \phi_\lambda \colon \mathcal{U}_\lambda &\to \mathbb{R}^n \\ p &\mapsto \big(x_1(p), x_2(p), \dots, x_n(p)\big), \end{aligned}$$

such that ϕ_λ is a homeomorphism of $\mathcal{U}_\lambda$ into its range, an open set in $\mathbb{R}^n$, satisfying that

$$\phi_\lambda \circ \phi_\mu^{-1} \colon \phi_\mu(\mathcal{U}_\lambda \cap \mathcal{U}_\mu) \to \phi_\lambda(\mathcal{U}_\lambda \cap \mathcal{U}_\mu)$$

is a C^∞-map (i.e., all the partial derivatives of any order exist and are continuous). The family of "charts" $\{(\mathcal{U}_\lambda, \phi_\lambda) : \lambda \in \Lambda\}$ is called an *atlas* of the manifold.

The reader may consult [7], [25] or [1] for the basic facts in this section.

Example 1. The euclidean space $\mathbb{R}^n$ with the trivial atlas $\{(\mathbb{R}^n, \mathrm{id})\}$. (Or any open subset of $\mathbb{R}^n$.)

Example 2. The unit circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with the atlas consisting of the following four charts:

$$\begin{aligned} \mathcal{U}_{\mathrm{top}} &= \{(x,y) \in \mathbb{S}^1 : y > 0\} & \phi_{\mathrm{top}} &\colon \mathcal{U}_{\mathrm{top}} \to \mathbb{R}, (x,y) \mapsto \arccos x\,, \\ \mathcal{U}_{\mathrm{bottom}} &= \{(x,y) \in \mathbb{S}^1 : y < 0\} & \phi_{\mathrm{bottom}} &\colon \mathcal{U}_{\mathrm{bottom}} \to \mathbb{R}, (x,y) \mapsto \arccos x\,, \\ \mathcal{U}_{\mathrm{right}} &= \{(x,y) \in \mathbb{S}^1 : x > 0\} & \phi_{\mathrm{right}} &\colon \mathcal{U}_{\mathrm{right}} \to \mathbb{R}, (x,y) \mapsto \arcsin y\,, \\ \mathcal{U}_{\mathrm{left}} &= \{(x,y) \in \mathbb{S}^1 : x < 0\} & \phi_{\mathrm{left}} &\colon \mathcal{U}_{\mathrm{left}} \to \mathbb{R}, (x,y) \mapsto \arcsin y\,. \end{aligned}$$

Here we assume $\arccos \colon (-1,1) \to (0,\pi)$ and $\arcsin \colon (-1,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Note that, for example, $\mathcal{U}_{\mathrm{top}} \cap \mathcal{U}_{\mathrm{left}} = \{(x,y) \in \mathbb{S}^1 : y > 0, x < 0\}$ and the change of coordinates is the map:

$$\begin{aligned} \phi_{\mathrm{top}} \circ \phi_{\mathrm{left}}^{-1} \colon \left(0, \frac{\pi}{2}\right) &\to \left(\frac{\pi}{2}, \pi\right) \\ t &\mapsto \pi - t\,, \end{aligned}$$

which is clearly C^∞.

Example 3. The cartesian product $M \times N$ of an m-dimensional manifold M and an n-dimensional manifold N is naturally an $(m+n)$-dimensional manifold.

A continuous map $\Psi\colon M \to N$ between two manifolds M and N of respective dimensions m and n is *smooth* if for any charts $(\mathcal{U}_\lambda, \phi_\lambda)$ and $(\mathcal{V}_\mu, \varphi_\mu)$ of M and N respectively, the map (between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$):

$$\varphi_\mu \circ \Psi \circ \phi_\lambda^{-1}\colon \phi_\lambda\big(\mathcal{U}_\lambda \cap \Psi^{-1}(\mathcal{V}_\mu)\big) \to \varphi_\mu(\mathcal{V}_\mu)$$

is of class C^∞.

In particular, we will consider the set $\mathcal{C}^\infty(M)$ of smooth maps from the manifold M into $\mathbb{R}$. This is a unital, commutative, associative ring with the pointwise addition and multiplication of maps. Actually, it is an algebra over $\mathbb{R}$, as we can multiply maps by constants.

1.2 Vector fields

A *nonassociative real algebra* is a real vector space A endowed with a bilinear map (multiplication) $m\colon A \times A \to A$. We will usually (but not always!) write xy for $m(x,y)$. The notions of homomorphism, subalgebra, ideal, ..., are the natural ones. Note that the word *nonassociative* simply means 'not necessarily associative'.

For example, a *Lie algebra* is a nonassociative algebra L with multiplication $(x,y) \mapsto [x,y]$ such that

- $[x,x] = 0$ for any $x \in L$ (the product is *anticommutative*),
- $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for any $x,y,z \in L$ (*Jacobi identity*).

Given a nonassociative algebra A and a homomorphism $\varphi\colon A \to \mathbb{R}$, a *$\varphi$-derivation* is a linear map $d : A \to \mathbb{R}$ such that $d(xy) = d(x)\varphi(y) + \varphi(x)d(y)$ for any $x, y \in A$. The set of φ-derivations forms a vector space.

On the other hand, a *derivation* of A is a linear map $D\colon A \to A$ such that $D(xy) = D(x)y + xD(y)$ for any $x, y \in A$. The set of derivations $\mathrm{Der}(A)$ is a Lie algebra with the usual 'bracket': $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Exercise 1. Check that indeed, if D_1 and D_2 are derivations of A, so is $[D_1, D_2]$.

Given a manifold M and a point $p \in M$, the map $\varphi_p\colon \mathcal{C}^\infty(M) \to \mathbb{R}$ given by evaluation at p: $\varphi_p(f) := f(p)$, is a homomorphism. The *tangent space* T_pM at p is the vector space of φ_p-derivations of $\mathcal{C}^\infty(M)$[1]. That is, the elements of T_pM (called *tangent vectors* at p) are linear maps $v_p\colon \mathcal{C}^\infty(M) \to \mathbb{R}$, such that

$$v_p(fg) = v_p(f)g(p) + f(p)v_p(g)$$

for any $f, g \in \mathcal{C}^\infty(M)$.

Given a chart $(\mathcal{U}, \phi)$ of M with $p \in \mathcal{U}$ and $\phi(q) = \big(x_1(q), \ldots, x_n(q)\big)$ for any $q \in \mathcal{U}$, we have the natural tangent vectors $\frac{\partial}{\partial x_i}\big|_p$ given by

$$f \mapsto \frac{\partial(f \circ \phi^{-1})}{\partial x_i}\big(\phi(p)\big).$$

[1] There are other possible definitions, but this is suitable for our purposes.

Actually, the elements $\frac{\partial}{\partial x_i}\big|_p$, $i = 1, \dots, n$, form a basis of T_pM.

The disjoint union

$$TM := \bigcup_{p \in M} T_pM$$

is called the *tangent bundle* of M. It has a natural structure of manifold with atlas $\{(\tilde{\mathcal{U}}_\lambda, \tilde{\phi}_\lambda) : \lambda \in \Lambda\}$, where $\tilde{\mathcal{U}}_\lambda = \bigcup_{p \in \mathcal{U}_\lambda} T_pM$ and

$$\begin{aligned} \tilde{\phi}_\lambda \colon \tilde{\mathcal{U}}_\lambda &\to \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \\ \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}\Big|_p \in T_pM &\mapsto \big(\phi_\lambda(p), (\alpha_1, \dots, \alpha_n)\big)\,. \end{aligned}$$

A *vector field* of a manifold M is a smooth section of the natural projection $\pi \colon TM \to M$, $v_p \in T_pM \mapsto p$. That is, a vector field is a smooth map

$$\begin{aligned} X \colon M &\to TM \\ p &\mapsto X_p\,, \end{aligned}$$

such that $X_p \in T_pM$ for any $p \in M$. Any vector field X induces a derivation of the algebra $\mathcal{C}^\infty(M)$ as follows: for any $f \in \mathcal{C}^\infty(M)$, the image of f under this derivation, also denoted by X, is the smooth map

$$Xf \colon p \mapsto X_p(f)\,.$$

Any derivation of $\mathcal{C}^\infty(M)$ is obtained in this way from a vector field. Therefore, we will identify the set of vector fields, denoted by $\mathfrak{X}(M)$, with the Lie algebra $\mathrm{Der}\big(\mathcal{C}^\infty(M)\big)$.

Exercise 2. Let R be a commutative associative algebra and let $\mathcal{L} = \mathrm{Der}(R)$ be its Lie algebra of derivations (keep in mind the case $R = \mathcal{C}^\infty(M)$!).

- Prove that $\mathcal{L}$ is a module over R by means of $(aD)(b) = aD(b)$ for any $a, b \in R$.
- Prove that $[aD, bE] = ab[D, E] + \big(aD(b)\big)E - \big(bE(a)\big)D$ for any $a, b \in R$ and $D, E \in \mathcal{L}$.

Let $\Phi \colon M \to N$ be a smooth map between two manifolds, and let $v_p \in T_pM$ be a tangent vector at a point $p \in M$. Then we can push forward v_p to a tangent vector $\Phi_*(v_p)$ defined by $\Phi_*(v_p)(f) = v_p(f \circ \Phi)$ for any $f \in \mathcal{C}^\infty(N)$. The 'tangent map' $\Phi_* \colon TM \to TN$, $v_p \in T_pM \mapsto \Phi_*(v_p) \in T_{\Phi(p)}N$ is smooth.

In particular, if $\gamma \colon (-\epsilon, \epsilon) \to M$ is a smooth map with $\gamma(0) = p$ (a curve centered at p), then

$$\dot{\gamma}(0) := \gamma_* \left(\frac{d}{dt}\Big|_{t=0} \right) : f \mapsto \frac{d(f \circ \gamma)}{dt}(0)$$

is a tangent vector at $p \in M$. Any tangent vector is obtained in this way.

Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be Φ-*related* if $\Phi_*(X_p) = Y_{\Phi(p)}$ for any $p \in M$. In other words, X and Y are Φ-related if

$$(Yf) \circ \Phi = X(f \circ \Phi)$$

for any $f \in \mathcal{C}^\infty(N)$.

Exercise 3. Consider vector fields $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ such that X_i and Y_i are Φ-related, for $i = 1, 2$. Prove that $[X_1, X_2]$ and $[Y_1, Y_2]$ are also Φ-related.

If the smooth map Φ above is a diffeomorphism (i.e.; it is smooth, bijective, and the inverse is also smooth), then Φ induces an isomorphism of Lie algebras (denoted by Φ_* too):

$$\begin{aligned} \Phi_* \colon \mathfrak{X}(M) &\to \mathfrak{X}(N) \\ X \mapsto \Phi_*(X) : f \in \mathcal{C}^\infty(N) &\mapsto X(f \circ \Phi) \circ \Phi^{-1} \in \mathcal{C}^\infty(N). \end{aligned}$$

In this case, the vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)$ are Φ-related if and only if $Y = \Phi_*(X)$.

1.3 Flows

Let $X \in \mathfrak{X}(M)$ be a vector field on a manifold M. For any $p \in M$ there exists a positive real number $\epsilon > 0$, a neighbourhood U of p in M, and a smooth map

$$\begin{aligned} \Phi \colon (-\epsilon, \epsilon) \times U &\to M \\ (t, m) &\mapsto \Phi(t, m)\,, \end{aligned}$$

such that for any $f \in \mathcal{C}^\infty(M)$ and any $m \in U$ we have $\Phi(0, m) = m$, and

$$X_{\Phi(t,m)} f = \frac{d}{dt} f\big(\Phi(t, m)\big) \left(= \lim_{s \to 0} \frac{f\big(\Phi(t+s, m)\big) - f\big(\Phi(t, m)\big)}{s} \right). \tag{1}$$

The map Φ is a *local flow* of X at p.

Local flows always exist by the fundamental existence and uniqueness theorem for first-order differential equations, which also implies that

$$\Phi(t_1 + t_2, m) = \Phi(t_1, \Phi(t_2, m)) \tag{2}$$

when this makes sense.

The vector field X is said to be *complete* if there exists a *global flow*

$$\Phi \colon \mathbb{R} \times M \to M\,,$$

so that (1) holds for any $m \in M$ and $t \in \mathbb{R}$. (Note that, conversely, any smooth map $\Phi \colon (-\epsilon, \epsilon) \times M \to M$ satisfying (2) determines a vector field by equation (1).)

If X is a complete vector field with global flow Φ, for any $t \in \mathbb{R}$ the map $\Phi_t \colon M \to M$ given by $\Phi_t(m) = \Phi(t, m)$ is a diffeomorphism and, for any vector

field $Y \in \mathfrak{X}(M)$, the Lie bracket $[X,Y]$ can be computed as follows (see [7, Theorem 4.3.2]):

$$[X,Y] = \left.\frac{d}{dt}\right|_{t=0} (\Phi_{-t})_*(Y) = \lim_{t\to 0} \frac{(\Phi_{-t})_*(Y) - Y}{t}\,. \tag{3}$$

Thus, the value at a point p: $[X,Y]_p$, is given by

$$[X,Y]_p = \lim_{t\to 0} \frac{(\Phi_{-t})_*(Y_{\Phi_t(p)}) - Y_p}{t}$$

(a limit in the vector space T_pM). Actually, for $f \in \mathcal{C}^\infty(M)$, consider the smooth map

$$\begin{aligned} H\colon \mathbb{R}\times\mathbb{R} &\to \mathbb{R}\\ (t,s) &\mapsto H(t,s) = Y_{\Phi_t(p)}(f\circ\Phi_s)\,. \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial H}{\partial t}(0,0) &= \left.\frac{d}{dt}\right|_{t=0} Y_{\Phi_t(p)}(f) = \left.\frac{d}{dt}\right|_{t=0} (Yf)(\Phi_t(p)) = X_p(Yf),\\ \frac{\partial H}{\partial s}(0,0) &= \left.\frac{d}{ds}\right|_{s=0} Y_p(f\circ\Phi_s) = Y_p\left(\left.\frac{d}{ds}\right|_{s=0} (f\circ\Phi_s)\right)\\ &\qquad \text{(by the equality of mixed partial derivatives)}\\ &= Y_p(Xf)\,. \end{aligned}$$

Hence we obtain

$$\left.\frac{d}{dt}\right|_{t=0} Y_{\Phi_t(p)}(f\circ\Phi_{-t}) = \left.\frac{d}{dt}\right|_{t=0} H(t,-t) = X_p(Yf) - Y_p(Xf) = [X,Y]_p(f),$$

as indicated in (3).

Two vector fields $X, Y \in \mathfrak{X}(M)$ commute (i.e.; $[X,Y]=0$) if and only if the corresponding flows commute: for any $m \in M$ there is a $\delta_m > 0$ such that

$$\Phi_t \circ \Psi_s(m) = \Psi_s \circ \Phi_t(m)$$

for $-\delta_m < t, s < \delta_m$ [7, Theorem 2.8.20].

2 Affine connections

Affine connections constitute the extension to "non flat" manifolds of the idea of directional derivative of vector fields. It provides the idea of "parallel transport".

Definition 1. An *affine connection* on a manifold M is an $\mathbb{R}$-bilinear map

$$\begin{aligned} \nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (X,Y) &\mapsto \nabla_X Y\,, \end{aligned}$$

with the following properties:

1. ∇ is $\mathcal{C}^\infty(M)$-linear in the first component:

$$\nabla_{fX} Y = f \nabla_X Y$$

for any $X, Y \in \mathfrak{X}(M)$ and any $f \in \mathcal{C}^\infty(M)$. (Recall from Exercise 2 that $\mathfrak{X}(M)$ is a module for $\mathcal{C}^\infty(M)$.)

2. For any $X, Y \in \mathfrak{X}(M)$ and any $f \in \mathcal{C}^\infty(M)$, the equation

$$\nabla_X (fY) = (Xf)Y + f \nabla_X Y$$

holds.

The first property implies that for any $X, Y \in \mathfrak{X}(M)$ and any point $p \in M$, $\big(\nabla_X Y\big)_p$ depends only on X_p and Y, so we may define $\nabla_v Y \in T_p M$ for any $v \in T_p M$ and $Y \in \mathfrak{X}(M)$, which we should think as the derivative of the vector field Y in the direction v at the point p.

Also, the second property shows that $\big(\nabla_X Y\big)_p$ depends on the values of Y in a neighbourhood of p. Thus affine connections can be restricted to open subsets of M, in particular to domains of charts. If (U, ϕ) is a chart of M with $\phi(q) = \big(x_1(q), \dots, x_n(q)\big)$, we obtain the $\mathcal{C}^\infty(U)$-basis of the vector fields in $\mathfrak{X}(U)$: $\big\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big\}$. By properties (1) and (2), the restriction of the affine connection ∇ on M to U is determined by the values

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \,,$$

for some maps $\Gamma_{ij}^k \in \mathcal{C}^\infty(U)$. These maps Γ_{ij}^k are called the *Christoffel symbols* of ∇ in the chart (U, ϕ).

Then, for $X, Y \in \mathfrak{X}(M)$, their restrictions to U are of the form $X|_U = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ and $Y|_U = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$ ($f_i, g_i \in \mathcal{C}^\infty(U)$ for $i = 1, \dots, n$), and the vector field $\big(\nabla_X Y\big)|_U$ is given by:

$$\big(\nabla_X Y\big)|_U = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} f_i \frac{\partial g_k}{\partial x_i} + \sum_{i,j=1}^{n} f_i g_j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k} \,. \tag{4}$$

Definition 2. Given an affine connection ∇ on a manifold M, for any vector field $X \in \mathfrak{X}(M)$ the *Nomizu operator*

$$\boldsymbol{L}_X \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is the $\mathcal{C}^\infty(M)$-linear form such that

$$\boldsymbol{L}_X Y = \nabla_X Y - [X, Y]$$

for any $Y \in \mathfrak{X}(M)$.

Exercise 4. Check that $\boldsymbol{L}_X$ is indeed $\mathcal{C}^\infty(M)$-linear.

More generally, let ϕ be a unital commutative and associative ring, R a commutative associative ϕ-algebra and $D = \mathrm{Der}(R)$ its Lie algebra of derivations (this is a module over R by Exercise 2). Define an *affine connection* ∇ on (R, D) as a ϕ-bilinear map

$$\begin{aligned} \nabla\colon D \times D &\to D \\ (\delta_1, \delta_2) &\mapsto \nabla_{\delta_1}\delta_2\,, \end{aligned}$$

subject to:

- $\nabla_{r\delta_1}\delta_2 = r\nabla_{\delta_1}\delta_2$,
- $\nabla_{\delta_1}(r\delta_2) = \delta_1(r)\delta_2 + r\nabla_{\delta_1}\delta_2$,

for any $r \in R$ and $\delta_1, \delta_2 \in D$. Prove that for $\delta \in D$, the *Nomizu operator* $\boldsymbol{L}_\delta\colon D \to D$, $\delta' \mapsto \nabla_\delta\delta' - [\delta, \delta']$, is R-linear.

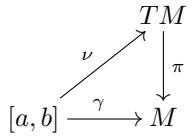
The $\mathcal{C}^\infty(M)$-linearity of $\boldsymbol{L}_X$ implies that for any $q \in M$, the tangent vector $\big(\boldsymbol{L}_X Y\big)_q$ depends only on Y_q, so we have a well defined endomorphism

$$(\boldsymbol{L}_X)|_q\colon T_qM \to T_qM\,.$$

Also, $(\boldsymbol{L}_X)|_q$ depends only on the values of X in a neighbourhood of q.

In order to define what we understand by "parallel transport" we need some preliminaries.

Definition 3. Let $a, b \in \mathbb{R}$, $a < b$, and let $\gamma\colon [a, b] \to M$ be a smooth map into a manifold M (this means that there is $\epsilon > 0$ such that γ extends to a smooth map $(a-\epsilon, b+\epsilon) \to M$). A *vector field* along the curve γ is a smooth map $\nu\colon [a, b] \to TM$ such that the diagram



commutes.

We will denote by $\mathfrak{X}(\gamma)$ the vector space of vector fields along γ.

For example, the derivative $\dot\gamma$ given by:

$$\dot\gamma(t) := \gamma_*\left(\frac{d}{dt}\right) \in T_{\gamma(t)}M$$

is a vector field along γ. If (U, ϕ) is a chart such that the image of γ is contained in U and $\phi(\gamma(t)) = (\gamma_1(t), \dots, \gamma_n(t))$, then

$$\dot\gamma(t) = \sum_{i=1}^{n} \gamma_i'(t)\left(\frac{\partial}{\partial x_i}\right)_{\gamma(t)}.$$

Given an affine connection ∇ on the manifold M, a curve $\gamma\colon [a,b] \to M$ and a vector field $Y \in \mathfrak{X}(M)$, in local coordinates we can express Y as

$$Y = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}\,,$$

for smooth maps g_i. Then we can compute

$$\begin{aligned}
\nabla_{\dot\gamma(t)} Y &= \sum_{i=1}^{n} \nabla_{\dot\gamma(t)} \left(g_i \frac{\partial}{\partial x_i} \right) \\
&= \sum_{i=1}^{n} \left(\dot\gamma(t)(g_i) \frac{\partial}{\partial x_i}\Big|_{\gamma(t)} + g_i(\gamma(t)) \sum_{k=1}^{n} \gamma_j'(t) \Gamma_{ji}^k(\gamma(t)) \frac{\partial}{\partial x_k}\Big|_{\gamma(t)} \right) \\
&= \sum_{k=1}^{n} \left(\frac{dg_k(\gamma(t))}{dt} + \sum_{i,j=1}^{n} \gamma_j'(t) g_i(\gamma(t)) \Gamma_{ji}^k(\gamma(t)) \right) \frac{\partial}{\partial x_k}\Big|_{\gamma(t)}\,,
\end{aligned}$$

and this depends only on the values of Y in the points $\gamma(t)$. This allows us, given a vector field along γ: $\nu \in \mathfrak{X}(\gamma)$, which in local coordinates appears as $\nu(t) = \sum_{i=1}^{n} \nu_i(t) \frac{\partial}{\partial x_i}\big|_{\gamma(t)}$, to define

$$\nabla_{\dot\gamma(t)} \nu \in T_{\gamma(t)} M\,,$$

by means of

$$\nabla_{\dot\gamma(t)} \nu = \sum_{k=1}^{n} \left(\frac{d\nu_k(t)}{dt} + \sum_{i,j=1}^{n} \gamma_j'(t) \nu_i(t) \Gamma_{ji}^k(\gamma(t)) \right) \frac{\partial}{\partial x_k}\Big|_{\gamma(t)} \in T_{\gamma(t)} M\,.$$

In this way we obtain an operator

$$\begin{aligned}
\mathfrak{X}(\gamma) &\to \mathfrak{X}(\gamma) \\
\nu &\mapsto \Big(t \mapsto \nabla_{\dot\gamma(t)} \nu \Big).
\end{aligned}$$

Definition 4. With M and γ as above, a vector field along γ: $\nu \in \mathfrak{X}(\gamma)$, is said to be *parallel* if $\nabla_{\dot\gamma(t)} \nu = 0$ for any $a \le t \le b$.

The existence and uniqueness of solutions for ordinary differential equations prove that for any $a \le c \le b$, if $v_0 \in T_{\gamma(c)} M$, then there is a unique parallel vector field ν along γ such that $\nu_{\gamma(c)} = v_0$. This field ν is called the *parallel transport of* v_0 *along* γ.

Definition 5. The curve γ is said to be a *geodesic* if $\dot\gamma$ is parallel along γ.

There are two important tensors attached to any affine connection on a manifold.

Definition 6. Let ∇ be an affine connection on a manifold M.

- The *torsion tensor* of ∇ is the $\mathcal{C}^\infty(M)$-bilinear map:

$$\begin{aligned} T\colon \mathfrak{X}(M)\times\mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (X,Y) &\mapsto T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]\,. \end{aligned}$$

  If the torsion tensor is identically 0, then ∇ is said to be *symmetric* (or *torsion-free*).

- The *curvature tensor* of ∇ is the $\mathcal{C}^\infty(M)$-trilinear map:

$$\begin{aligned} R\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (X,Y,Z) &\mapsto R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \end{aligned}$$

  If the curvature tensor is identically 0, then ∇ is said to be *flat*.

Exercise 5. Check that the torsion and curvature tensors are indeed $\mathcal{C}^\infty(M)$-linear in each component. Define these concepts in the general setting of Exercise 2.

The $\mathcal{C}^\infty(M)$-linearity implies that the values of $T(X,Y)$ or $R(X,Y)Z$ at a given point $p\in M$ depend only on the values of the vector fields X, Y and Z at this point. Hence it makes sense to consider the torsion $T(u,v)$ or curvature $R(u,v)w$ for $u,v,w\in T_pM$.

Exercise 6. Prove that an affine connection ∇ is symmetric if its Christoffel symbols in any chart satisfy $\Gamma_{ij}^k = \Gamma_{ji}^k$ for any $1\le i,j,k\le n$.

Exercise 7. A *pseudo-Riemannian* manifold is a manifold M endowed with a non-degenerate symmetric $\mathcal{C}^\infty(M)$-bilinear form (the *pseudo-metric*)

$$g\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathcal{C}^\infty(M).$$

Again the value $g(X,Y)$ at a point p depends only on X_p and Y_p, so g restricts to an $\mathbb{R}$-bilinear map $g_p\colon T_pM\times T_pM\to\mathbb{R}$. The nondegeneracy of g means that g_p is nondegenerate for any $p\in M$.

Prove that for any pseudo-Riemannian manifold (M,g) there is a unique affine connection ∇ such that:

(i) ∇ is symmetric,

(ii) $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y,\nabla_X Z)$ for any $X,Y,Z\in\mathfrak{X}(M)$.

This connection is called the *Levi-Civita connection*. For this connection, the parallel transport defines isometries among the tangent spaces at different points. [Hint: Permute cyclically X,Y,Z in condition (ii) and combine the resulting equations, using (i), to get that $g(\nabla_X Y,Z)$ is uniquely determined. Now use that g is nondegenerate.]

3 Lie groups and Lie algebras

A *Lie group* is a manifold G which is also an abstract group such that the multiplication map

$$\begin{aligned} G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \,, \end{aligned}$$

and the inversion map

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth maps.

Exercise 8. Identify the unit circle $\mathbb{S}^1$ with the set of norm one complex numbers. This set is a group under complex multiplication. Check that in this way $\mathbb{S}^1$ becomes a Lie group.

Remark 1. Given an element g in a Lie group G, the left multiplication by g gives a map $L_g \colon G \to G$ which is a diffeomorphism (with inverse $L_{g^{-1}}$). The same applies to the right multiplication R_g.

The *Lie algebra* of a Lie group G is the real vector subspace of "left-invariant vector fields":

$$\mathcal{L}(G) := \{X \in \mathfrak{X}(G) : (L_g)_*(X) = X, \forall g \in G\}.$$

By Exercise 3, $\mathfrak{g} = \mathcal{L}(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$. Moreover, the natural map

$$\begin{aligned} \mathfrak{g} &\to T_e G \\ X &\mapsto X_e \,, \end{aligned}$$

where e denotes the neutral element of the group structure, turns out to be a linear isomorphism (check this!), as any left-invariant vector field is determined uniquely by its value at any given point. Hence the dimension of $\mathfrak{g}$ as a vector space coincides with the dimension of G as a manifold.

Example 4. The general linear group $\mathrm{GL}_n(\mathbb{R})$ consists of the $n \times n$ regular matrices over $\mathbb{R}$. This is the inverse image of the open set $\mathbb{R} \setminus \{0\}$ under the smooth map (actually it is a polynomial map) given by the determinant $\det \colon \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$. Thus $\mathrm{GL}_n(\mathbb{R})$ is an open set in the euclidean space $\mathrm{Mat}_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ and hence it is a manifold too. The multiplication is given by a polynomial map and the inversion by a rational map, and hence they are both smooth.

Since $\mathrm{GL}_n(\mathbb{R})$ is open in $\mathbb{R}^{n^2}$ there is the global chart with standard coordinates $\{x_{ij} : 1 \le i, j \le n\}$, where $x_{ij}(A)$ denotes the (i,j)-entry of the matrix A. Then the tangent space at any point $A \in G = \mathrm{GL}_n(\mathbb{R})$ can be identified with the vector space of $n \times n$-matrices $\mathrm{Mat}_n(\mathbb{R})$ by means of the map:

$$\begin{aligned} T_A G &\to \mathrm{Mat}_n(\mathbb{R}) \\ v &\mapsto \big(v(x_{ij})\big)_{i,j=1}^{n} \,. \end{aligned} \tag{5}$$

(Note that $v = \sum_{i,j} v(x_{ij}) \frac{\partial}{\partial x_{ij}}\big|_A$.)

Let $\mathfrak{g}$ be the Lie algebra of $G = \mathrm{GL}_n(\mathbb{R})$, and let $X \in \mathfrak{g}$. For any $A = \big(a_{ij}\big) \in G$, the left-invariance of X gives:

$$\begin{aligned} X_A(x_{ij}) &= (L_A)_*(X_{I_n})(x_{ij}) = X_{I_n}(x_{ij} \circ L_A) \\ &= X_{I_n}\left(\sum_{k=1}^n a_{ik}x_{kj}\right) \quad \text{(the } (i,j)\text{-entry of } AB \text{ is } \sum_{k=1}^n a_{ik}b_{kj}\text{)} \\ &= \sum_{k=1}^n a_{ik} X_{I_n}(x_{kj}) = \sum_{k=1}^n X_{I_n}(x_{kj}) x_{ik}(A). \end{aligned}$$

Therefore, we obtain

$$Xx_{ij} = \sum_{k=1}^n X_{I_n}\big(x_{kj}\big) x_{ik}. \tag{6}$$

(Here I_n denotes the identity matrix, which is the neutral element of G.)

Consider now two elements $X, Y \in \mathfrak{g}$. X and Y are determined by X_{I_n} and Y_{I_n} respectively, and these can be identified as in (5) with the matrices $A := \big(X_{I_n}(x_{ij})\big)$ and $B := \big(Y_{I_n}(x_{ij})\big)$. Then we get

$$\begin{aligned} [X,Y]_{I_n}(x_{ij}) &= X_{I_n}(Yx_{ij}) - Y_{I_n}(Xx_{ij}) \\ &= X_{I_n}\left(\sum_{k=1}^n b_{kj}x_{ik}\right) - Y_{I_n}\left(\sum_{k=1}^n a_{kj}x_{ik}\right) \qquad \text{(because of (6))} \\ &= \sum_{k=1}^n b_{kj}a_{ik} - \sum_{k=1}^n a_{kj}b_{ik} = x_{ij}\big([A,B]\big)\,. \end{aligned}$$

Denote by $\mathfrak{gl}_n(\mathbb{R})$ the Lie algebra defined on the vector space $\mathrm{Mat}_n(\mathbb{R})$ with the usual Lie bracket $[A,B] = AB - BA$. Then the above computation shows that the linear map

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{gl}_n(\mathbb{R}) \\ X &\mapsto \big(X_{I_n}(x_{ij})\big)_{i,j=1}^n\,, \end{aligned}$$

is a Lie algebra isomorphism.

Moreover, for any $A \in \mathfrak{gl}_n(\mathbb{R})$ the map

$$\begin{aligned} \gamma_A \colon \mathbb{R} &\to G \\ t &\mapsto \exp(tA) := \sum_{n=0}^\infty \frac{t^n A^n}{n!}\,, \end{aligned}$$

is a smooth group homomorphism. Besides, $\dot\gamma(0) = A$ (with the identification above $T_{I_n}G \simeq \mathfrak{gl}_n(\mathbb{R})$). We conclude that the smooth map

$$\begin{aligned} \Phi_A \colon \mathbb{R} \times G &\to G \\ (t,B) &\mapsto B\exp(tA) \end{aligned}$$

is the global flow of the left-invariant vector field X with $X_{I_n} = A$.

We may substitute $\mathrm{GL}_n(\mathbb{R})$ by $\mathrm{GL}(V)$ (the group of linear automorphisms of V) for a finite-dimensional real vector space V, and $\mathfrak{gl}_n(\mathbb{R})$ by $\mathfrak{gl}(V)$ (the Lie algebra of linear endomorphisms of V with the natural Lie bracket) in the Example above. Once we fix a basis of V we get isomorphisms $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{R})$ (of Lie groups) and $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(\mathbb{R})$ (of Lie algebras).

In general, left-invariant vector fields on a Lie group G are always complete and there is a smooth map from the Lie algebra $\mathfrak{g}$ to G, called the *exponential*:

$$\exp\colon \mathfrak{g} \to G, \tag{7}$$

which restricts to a diffeomorphism of a neighbourhood of $0 \in \mathfrak{g}$ ($\mathfrak{g} \simeq \mathbb{R}^n$) onto an open neighbourhood of $e \in G$ such that for any $X \in \mathfrak{g}$, its global flow is given by the map

$$\begin{aligned} \Phi\colon \mathbb{R} \times G &\to G \\ (t, g) &\mapsto g \exp(tX)\,. \end{aligned}$$

Any open neighbourhood of $e \in G$ generates, as an abstract group, the connected component of e, which is a normal subgroup. In particular, the subgroup generated by $\exp(\mathfrak{g})$ is the connected component G_0 of e.

Remark 2. In this situation, for any $t \in \mathbb{R}$, Φ_t is the diffeomorphism $R_{\exp(tX)}$ (right multiplication by $\exp(tX)$). Note that for any $h \in G$, the left invariance of any $X \in \mathfrak{g}$ gives $(L_h)_*(X) = X$, so for any $g, h \in G$,

$$\Big((L_h)_*(X)\Big)_{hg} f = X_g(f \circ L_h) = \frac{d}{dt}\bigg|_{t=0} (f \circ L_h)(g \exp(tX))\,,$$

and this certainly agrees with

$$X_{hg} f = \frac{d}{dt}\bigg|_{t=0} f(hg \exp(tX))\,.$$

Also, condition (2) implies that the smooth map $\mathbb{R} \to G$, $t \mapsto \exp(tX)$ is a group homomorphism (called a *one-parameter subgroup*).

We may consider too the real subspace of "right-invariant" vector fields:

$$\mathfrak{g}_{\mathrm{right}} := \{X \in \mathfrak{X}(G) : \big(R_g\big)_*(X) = X \ \forall g \in G\}\,.$$

Again, the natural map $\mathfrak{g}_{\mathrm{right}} \to T_eG$, $X \mapsto X_e$, is a linear isomorphism.

Proposition 1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider the linear isomorphisms*

$$\begin{aligned} \varphi_{\mathrm{left}}\colon \mathfrak{g} &\to T_eG \\ X &\mapsto X_e \end{aligned} \qquad\qquad \begin{aligned} \varphi_{\mathrm{right}}\colon \mathfrak{g}_{\mathrm{right}} &\to T_eG \\ \hat{X} &\mapsto \hat{X}_e. \end{aligned}$$

Then the map

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{g}_{\text{right}} \\ X &\mapsto -\varphi_{\text{right}}^{-1} \circ \varphi_{\text{left}}(X) \end{aligned}$$

is an isomorphism of Lie algebras.

In other words, given $v \in T_eG$, let $X \in \mathfrak{g}$ (respectively $\hat{X} \in \mathfrak{g}_{\text{right}}$) be the left (respectively right) invariant vector field with $X_e = v$ (respectively $\hat{X}_e = v$). Then the map $X \mapsto -\hat{X}$ is a Lie algebra isomorphism.

Proof. The smooth map

$$\begin{aligned} \Phi^{\text{right}} \colon \mathbb{R} \times G &\to G \\ (t, g) &\mapsto \exp(tX)g\,. \end{aligned}$$

is the global flow of $\hat{X}$, because for any $f \in \mathcal{C}^\infty(G)$:

$$\begin{aligned} \hat{X}_g(f) &= \Big((R_g)_*(\hat{X}_e)\Big)(f) = \hat{X}_e(f \circ R_g) \\ &= v(f \circ R_g) = X_e(f \circ R_g) \\ &= \frac{d}{dt}\Big|_{t=0} (f \circ R_g)(\exp(tX)) = \frac{d}{dt}\Big|_{t=0} f(\exp(tX)g)\,. \end{aligned}$$

Now consider the inversion map $\iota \colon G \to G$, $g \mapsto g^{-1}$. It is a diffeomorphism, so $\iota_* \colon \mathfrak{X}(G) \to \mathfrak{X}(G)$ is an isomorphism of Lie algebras.

For $g \in G$, X and $\hat{X}$ as in the Proposition, and $f \in \mathcal{C}^\infty(G)$ we get:

$$\begin{aligned} \big(\iota_* X\big)_{g^{-1}} f = X_g(f \circ \iota) &= \frac{d}{dt}\Big|_{t=0} (f \circ \iota)(g \exp(tX)) \\ &= \frac{d}{dt}\Big|_{t=0} f\big((g \exp(tX))^{-1}\big) \\ &= \frac{d}{dt}\Big|_{t=0} f(\exp(-tX)g^{-1}) = -\hat{X}_{g^{-1}}(f)\,. \end{aligned}$$

We conclude that $\iota_*(X) = -\hat{X}$, so X and $-\hat{X}$ are ι-related. In particular, this shows that ι_* takes left invariant vector fields to right invariant vector fields, and hence it gives an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_{\text{right}}$. □

For any element g in a Lie group G, conjugation by g gives a diffeomorphism:

$$\begin{aligned} \iota_g \colon G &\to G \\ x &\mapsto gxg^{-1}\,, \end{aligned}$$

and this induces a Lie algebra isomorphism $\mathrm{Ad}_g := (\iota_g)_* \colon \mathfrak{X}(G) \to \mathfrak{X}(G)$ that

preserves $\mathfrak{g}$. Indeed, for $X \in \mathfrak{g}$ and $h \in G$

$$\begin{aligned}(L_h)_*(\mathrm{Ad}_g X) &= (L_h)_* \circ (\iota_g)_*(X) = (L_h)_* \circ (L_g)_* \circ (R_{g^{-1}})_*(X) \\ &= (R_{g^{-1}})_* \circ (L_h)_* \circ (L_g)_*(X) \\ &\qquad \text{(because } L_x R_y = R_y L_x \text{ for any } x, y \in G\text{)} \\ &= (R_{g^{-1}})_* X \quad \text{(as } X \text{ is left invariant)} \\ &= (R_{g^{-1}})_* \circ (L_g)_*(X) = \mathrm{Ad}_g X.\end{aligned}$$

We obtain in this way the so called *adjoint representation*:

$$\begin{aligned}\mathrm{Ad}\colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g\,,\end{aligned}$$

which is a homomorphism of Lie groups (i.e., a smooth group homomorphism).

Given a homomorphism of Lie groups $\varphi\colon G \to H$ and a left invariant vector field $X \in \mathfrak{g}$ (the Lie algebra of G), the map $t \mapsto \varphi(\exp(tX))$ gives a Lie group homomorphism $\mathbb{R} \to H$ and hence the map

$$\begin{aligned}\Phi\colon \mathbb{R} \times H &\to H \\ (t, h) &\mapsto h\varphi(\exp(tX))\end{aligned}$$

is the global flow of a left invariant vector field $Y \in \mathfrak{h}$ (the Lie algebra of H), which is φ-related to X. This induces a Lie algebra homomorphism $\varphi_*\colon \mathfrak{g} \to \mathfrak{h}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi_*} & \mathfrak{h} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\varphi} & H \end{array} \tag{8}$$

is commutative.

In particular, we obtain a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}_*} & \mathfrak{gl}(\mathfrak{g}) \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \end{array} \tag{9}$$

where, on the right exp denotes the standard exponential of linear endomorphisms.

For $X, Y \in \mathfrak{g}$ we get:

$$\begin{aligned}\mathrm{Ad}_{\exp(tX)} Y &= \big(\iota_{\exp(tX)}\big)_* Y \\ &= \big(R_{\exp(-tX)}\big)_* \circ \big(L_{\exp(tX)}\big)_*(Y) \\ &= \big(R_{\exp(-tX)}\big)_*(Y) \quad \text{(as } Y \text{ is left invariant).}\end{aligned}$$

Since $\Phi_t = R_{\exp(tX)}$ is the global flow of X (Remark 2), we obtain from (3) the equation

$$\left.\frac{d}{dt}\right|_{t=0} \left(R_{\exp(-tX)}\right)_*(Y) = [X,Y]\,. \tag{10}$$

Therefore the map $t \mapsto \mathrm{Ad}_{\exp(tX)}Y$ is the one-parameter group in $\mathfrak{g} \simeq T_eG$ whose derivative at 0 is $[X,Y]$. Alternatively, $t \mapsto \mathrm{Ad}_{\exp(tX)}$ is the one-parameter group in the Lie group $\mathrm{GL}(\mathfrak{g})$ whose derivative at 0 is the linear endomorphism $\mathrm{ad}_X\colon Y \mapsto [X,Y]$ ($\mathrm{ad}_X \in \mathfrak{gl}(\mathfrak{g})$, the Lie algebra of $\mathrm{GL}(\mathfrak{g})$).

We conclude that

$$\mathrm{Ad}_{\exp(tX)} = \exp(t\mathrm{ad}_X) \tag{11}$$

for any $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Hence we obtain $(\mathrm{Ad})_* = \mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, and the diagram in (9) becomes the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \mathrm{ad}\ } & \mathfrak{gl}(\mathfrak{g}) \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\ \mathrm{Ad}\ } & \mathrm{GL}(\mathfrak{g}) \end{array} \tag{12}$$

Remark 3. On the left hand side of (12) we have an arbitrary Lie group and its Lie algebra, while on the right hand side we have a very concrete Lie group: the general linear Lie group on a vector space, and its Lie algebra.

4 Invariant affine connections on homogeneous spaces

4.1 Homogeneous spaces

Let G be a Lie group with Lie algebra $\mathfrak{g}$, M a manifold and let

$$\begin{aligned} \tau\colon G \times M &\to M \\ (g,p) &\mapsto \tau(g,p) = g \cdot p \end{aligned}$$

be a smooth action. That is, τ is both a smooth map and a group action.

If the group action τ is transitive (i.e., for any $p, q \in M$ there is an element $g \in G$ such that $g \cdot p = q$), then M is said to be a *homogeneous space*. (Formally, we should consider homogeneous spaces as triples (M, G, τ).)

Example 5. The n-dimensional sphere

$$\mathbb{S}^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

is a homogeneous space relative to the natural action of the special orthogonal group: $SO(n+1) \times \mathbb{S}^n \to \mathbb{S}^n$.

On the other hand, $\mathbb{S}^{2n-1}$ can be identified with the set

$$\left\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 = 1\right\},$$

and hence it is a homogeneous space relative to the natural action of the special unitary group: $SU(n) \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$.

Given a homogeneous space M, fix an element $p \in M$ and consider the corresponding *isotropy subgroup*:

$$H := \{g \in G : g \cdot p = p\}\,.$$

This is a closed subgroup of G and hence (see e.g. [7, Theorem 5.3.2]) it is a Lie group too, with Lie algebra

$$\mathfrak{h} := \{X \in \mathfrak{g} : \exp(tX) \in H \ \forall t \in \mathbb{R}\}\,.$$

(Recall the map $\exp\colon \mathfrak{g} \to G$ in equation (7).) The Lie algebra $\mathfrak{h}$ is then a subalgebra of $\mathfrak{g}$.

Moreover, the set of left cosets G/H is a manifold with a suitable atlas such that the bijection

$$\begin{aligned} G/H &\to M \\ gH &\mapsto g \cdot p \end{aligned}$$

is a diffeomorphism (see e.g. [7, Proposition 5.4.12]).

For any $X \in \mathfrak{g}$ we can define a global flow:

$$\begin{aligned} \Phi\colon \mathbb{R} \times M &\to M \\ (t, m) &\mapsto \exp(tX) \cdot m\,. \end{aligned}$$

The associated vector field in $\mathfrak{X}(M)$ will be denoted by X^+. Hence we have

$$X^+_m(f) = \left.\frac{d}{dt}\right|_{t=0} f\big(\exp(tX) \cdot m\big)$$

for any $m \in M$ and $f \in \mathcal{C}^\infty(M)$. The next result summarizes some of the main properties of these vector fields.

Proposition 2. *Let M be a homogenous space of the Lie group G as above. Then the following properties hold:*

(i) *For any $q \in M$, let $\pi_q\colon G \to M$ be the smooth map $g \mapsto g \cdot q$. Then for any $X \in \mathfrak{g}$ and $f \in \mathcal{C}^\infty(M)$*

$$X^+_q(f) = X_e(f \circ \pi_q).$$

In other words, $X^+_q = (\pi_q)_*(X_e)$.

(ii) *For any $X \in \mathfrak{g}$, let $\hat{X}$ be the right invariant vector field on G with $X_e = \hat{X}_e$ (see Proposition 1). Then $\hat{X}$ and X^+ are π_q-related for any $q \in M$. Moreover, X^+ is determined by this property.*

(iii) *The map*

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{X}(M) \\ X &\mapsto -X^+ \end{aligned}$$

is a Lie algebra homomorphism.

(iv) *For any $g \in G$, denote by $\tau(g)$ the diffeomorphism $M \to M$, $q \mapsto g \cdot q$. Then for any $g \in G$ and $X \in \mathfrak{g}$ we have*

$$\tau(g)_*(X^+) = \big(\mathrm{Ad}_g X\big)^+ .$$

(v) *Fix, as before, a point $p \in M$, then $\pi := \pi_p \colon G \to M$ induces a linear surjection*

$$\begin{aligned} \pi_* \colon \mathfrak{g}(\simeq T_e(G)) &\to T_pM \\ X &\mapsto \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tX)) = X_p^+ , \end{aligned}$$

with $\ker \pi_* = \mathfrak{h}$ (the Lie algebra of the isotropy subgroup H at p).

Proof. For (i), we proceed as follows:

$$X_q^+(f) = \left.\frac{d}{dt}\right|_{t=0} f\big(\exp(tX)\cdot q\big) = \left.\frac{d}{dt}\right|_{t=0} f\big(\pi_q(\exp(tX))\big) = X_e(f\circ\pi_q).$$

For (ii), take any $f \in \mathcal{C}^\infty(M)$ and $g \in G$ to obtain

$$\begin{aligned} \big(X^+ f\big)\circ\pi_q(g) &= \big(X^+ f\big)(g\cdot q) = X^+_{g\cdot q}(f) \\ &= X_e(f\circ\pi_{g\cdot q}) = X_e(f\circ\pi_q\circ R_g) \\ &= \hat{X}_g(f\circ\pi_q) = \big(\hat{X}(f\circ\pi_q)\big)(g) . \end{aligned}$$

Hence $(X^+ f)\circ\pi_q = \hat{X}(f\circ\pi_q)$, as required.

By Exercise 3, for any $X, Y \in \mathfrak{g}$, the vector fields $[\hat{X}, \hat{Y}]$ in G and $[X^+, Y^+]$ in M are π_q-related for any $q \in M$. But $[\hat{X}, \hat{Y}] = -\widehat{[X,Y]}$ by Proposition 1, so $-\widehat{[X,Y]}$ and $[X^+, Y^+]$ are π_q-related for any $q \in M$, and this shows that $[X^+, Y^+] = -[X,Y]^+$, thus proving (iii).

Now, for any $q \in M$ and $f \in \mathcal{C}^\infty(M)$ we obtain:

$$\begin{aligned} \big(\tau(g)_*(X^+)\big)_{g\cdot q}(f) &= X_q^+(f\circ\tau(g)) = X_e(f\circ\tau(g)\circ\pi_q) \\ &= X_e(f\circ\pi_{g\cdot q}\circ\iota_g) \\ &\qquad (\text{as } \pi_{g\cdot q}\circ\iota_g(h) = \pi_{g\cdot q}(ghg^{-1}) = (gh)\cdot q = \tau(g)\circ\pi_q(h)) \\ &= \big(\mathrm{Ad}_g X\big)_e(f\circ\pi_{g\cdot q}) = \big(\mathrm{Ad}_g X\big)^+_{g\cdot q}(f), \end{aligned}$$

and this proves (iv).

Finally,

$$\begin{aligned} \ker\pi_* &= \left\{ X \in \mathfrak{g} : \left.\frac{d}{dt}\right|_{t=0} \pi\big(\exp(tX)\big) = 0 \right\} \\ &= \left\{ X \in \mathfrak{g} : \left.\frac{d}{dt}\right|_{t=0} \exp(tX)\cdot p = 0 \right\} \end{aligned}$$

contains $\mathfrak{h}$. But π_* is surjective, and hence $\ker\pi_* = \mathfrak{h}$ by dimension count. □

Therefore, under the conditions of the Proposition above, the map

$$\begin{aligned} \phi\colon \mathfrak{g}/\mathfrak{h} &\to T_pM \\ X+\mathfrak{h} &\mapsto X_p^+ \end{aligned} \tag{13}$$

is a linear isomorphism that allows us to identify T_pM with $\mathfrak{g}/\mathfrak{h}$. Under this identification, for any $h \in H$, the map $\tau(h)_*\colon T_pM \to T_pM$ corresponds, because of Proposition 2(iv), to the map

$$\begin{aligned} j(h)\colon \mathfrak{g}/\mathfrak{h} &\to \mathfrak{g}/\mathfrak{h} \\ X+\mathfrak{h} &\mapsto (\mathrm{Ad}_h X)+\mathfrak{h}. \end{aligned} \tag{14}$$

That is, we get a commutative diagram

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h} & \xrightarrow{j(h)} & \mathfrak{g}/\mathfrak{h} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ T_pM & \xrightarrow[\tau(h)_*]{} & T_pM \end{array} \tag{15}$$

The map $j\colon H \to GL(\mathfrak{g}/\mathfrak{h})$, $h \mapsto j(h)$, is the natural representation of H on $\mathfrak{g}/\mathfrak{h}$ (induced by the adjoint representation).

In what follows, when we refer to a homogeneous space $M \simeq G/H$, we mean that G is a Lie group acting smoothly and transitively on M, and that a point $p \in M$ has been fixed with H as isotropy subgroup.

Remark 4. Proposition 2(v) shows that for any $q \in M$, the subspace $\{X_q^+ : X \in \mathfrak{g}\}$ fills the whole T_qM. In other words, the space $\{X^+ : X \in \mathfrak{g}\}$ spans $\mathfrak{X}(M)$ as a $\mathcal{C}^\infty(M)$-module.

4.2 Invariant affine connections on homogeneous spaces

Definition 7. Let $M \simeq G/H$ be a homogeneous space. An affine connection ∇ on M is said to be *invariant* if for any $X, Y \in \mathfrak{X}(M)$ and $g \in G$,

$$\tau(g)_*(\nabla_X Y) = \nabla_{\tau(g)_*(X)}\tau(g)_*(Y)\,.$$

Proposition 3. *Let ∇ be an invariant affine connection on a homogeneous space $M \simeq G/H$. Then for any $X \in \mathfrak{g}$ (the Lie algebra of G) and any $g \in G$ we have:*

$$\boldsymbol{L}_{(\mathrm{Ad}_g X)^+} = \tau(g)_* \circ \boldsymbol{L}_{X^+} \circ \tau(g^{-1})_*\,.$$

Proof. For any $X \in \mathfrak{g}$, recall from Proposition 2(iv) that $(\mathrm{Ad}_g X)^+ = \tau(g)_*(X^+)$, so it is enough to prove that $\boldsymbol{L}_{\tau(g)_*U} \circ \tau(g)_* = \tau(g)_* \circ \boldsymbol{L}_U$ for any $U \in \mathfrak{X}(M)$, and this is a straightforward consequence of the invariance of ∇. □

Therefore, the invariant affine connection ∇, which is determined by the Nomizu operators $\boldsymbol{L}_{X^+}$ for $X \in \mathfrak{g}$ because of Remark 4, is actually determined by the

endomorphisms $\boldsymbol{L}_{X^+}\big|_p$, because for any $q \in M$, there is an element $g \in G$ such that $q = g \cdot p$, and for $Y \in \mathfrak{g}$ we have

$$\begin{aligned}\boldsymbol{L}_{X^+}(Y^+_{g\cdot p}) &= \boldsymbol{L}_{X^+}\big(\tau(g)_*(\mathrm{Ad}_{g^{-1}}Y)^+_p\big) \quad \text{(Proposition 2(iv))}\\ &= \tau(g)_*\boldsymbol{L}_{(\mathrm{Ad}_{g^{-1}}X)^+}\big((\mathrm{Ad}_{g^{-1}}Y)^+_p\big).\end{aligned}$$

If we identify T_pM with $\mathfrak{g}/\mathfrak{h}$ via the map ϕ in (13) we may think of $\boldsymbol{L}_{X^+}\big|_p$ as a linear endomorphism of $\mathfrak{g}/\mathfrak{h}$.

Hence ∇ is determined by the linear map, also denoted by $\boldsymbol{L}$:

$$\begin{aligned}\boldsymbol{L}\colon \mathfrak{g} &\to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})\\ X &\mapsto \boldsymbol{L}_X\colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}\\ &\qquad Y+\mathfrak{h} \mapsto \phi^{-1}\big(\boldsymbol{L}_{X^+}(Y^+_p)\big)\,.\end{aligned} \tag{16}$$

Proposition 4. *Let ∇ be an invariant affine connection on the homogeneous space $M \simeq G/H$. Then the linear map $\boldsymbol{L}$ just defined satisfies the following properties:*

(a) $\boldsymbol{L}_{\mathrm{Ad}_h X} = j(h) \circ \boldsymbol{L}_X \circ j(h^{-1})$ *for any $X \in \mathfrak{g}$ and $h \in H$.*

(b) $\boldsymbol{L}_X(Y+\mathfrak{h}) = [X,Y]+\mathfrak{h}$ *for any $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.*

In other words, $\boldsymbol{L}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ is a homomorphism of H-modules that extends the natural (adjoint) representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$.

Actually, (a) is a direct consequence of Proposition 3 and the commutativity of (15), and (b) follows because for $X \in \mathfrak{h}$, $X^+_p = 0$, so $\big(\nabla_{X^+}Y^+\big)_p = 0$, and hence

$$\big(\boldsymbol{L}_{X^+}(Y^+)\big)_p = \big(\nabla_{X^+}Y^+\big)_p - [X^+,Y^+]_p = [X,Y]^+_p\,.$$

Conversely, given an $\mathbb{R}$-linear map $\boldsymbol{L}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ satisfying the properties (a) and (b) above, we may define, for $X, Y \in \mathfrak{g}$ and $g \in G$:

- $\nabla_{X^+_p}Y^+ := \phi\big(\boldsymbol{L}_X(Y+\mathfrak{h})\big) - [X,Y]^+_p \in T_pM$ (well defined by property (b)),
- $\nabla_{X^+_{g\cdot p}}Y^+ := \tau(g)_*\big(\nabla_{(\mathrm{Ad}_{g^{-1}}X)^+_p}(\mathrm{Ad}_{g^{-1}}Y)^+\big)$ (well defined by property (a)),

and this defines an invariant affine connection on M.

We summarize the above arguments in the next result, which goes back to Vinberg [23, Theorem 2].

Theorem 1. *The invariant affine connections on a homogeneous space $M \simeq G/H$ are in bijection with the $\mathbb{R}$-linear maps $\boldsymbol{L}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ satisfying the properties (a) and (b) in Proposition 4.*

Remark 5. Condition (a) above means that the map $\boldsymbol{L}$ in the Theorem is a homomorphisms of H-modules, and this implies that it is a homomorphism of $\mathfrak{h}$-modules:

$$\boldsymbol{L}_{[U,X]} = [\tilde{j}(U), \boldsymbol{L}_X]$$

for any $U \in \mathfrak{h}$ and $X \in \mathfrak{g}$, where $\tilde{j}(U)$ is the endomorphism of $\mathfrak{g}/\mathfrak{h}$ such that $\tilde{j}(U)(Y+\mathfrak{h}) = [U,Y]+\mathfrak{h}$ for any $Y \in \mathfrak{g}$. Indeed, property (a) gives

$$\boldsymbol{L}_{\mathrm{Ad}_{\exp(tU)}X} \circ j(\exp(tU)) = j(\exp(tU)) \circ \boldsymbol{L}_X$$

in $\mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ for any $t \in \mathbb{R}$. Taking the derivative at $t=0$ we get

$$\boldsymbol{L}_{[U,X]} + \boldsymbol{L}_X \circ \tilde{j}(U) = \tilde{j}(U) \circ \boldsymbol{L}_X\,,$$

as required.

The torsion and curvature tensors of an invariant affine connection on a homogeneous space $M \simeq G/H$ can now be expressed in terms of the linear map $\boldsymbol{L}$ in (16), because, by invariance, it is enough to compute them at the point p.

Torsion For $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned} T(X^+,Y^+) &:= \nabla_{X^+}Y^+ - \nabla_{Y^+}X^+ - [X^+,Y^+] \\ &= \boldsymbol{L}_{X^+}(Y^+) - \boldsymbol{L}_{Y^+}(X^+) + [X^+,Y^+] \\ &= \boldsymbol{L}_{X^+}(Y^+) - \boldsymbol{L}_{Y^+}(X^+) - [X,Y]^+ \quad \text{(by Proposition 2(iii))}, \end{aligned}$$

so at the point p we get

$$T(X^+,Y^+)_p = \phi\big(\boldsymbol{L}_X(Y+\mathfrak{h}) - \boldsymbol{L}_Y(X+\mathfrak{h}) - ([X,Y]+\mathfrak{h})\big),$$

which corresponds, through the map ϕ in (13) to

$$T(X+\mathfrak{h},Y+\mathfrak{h}) := \boldsymbol{L}_X(Y+\mathfrak{h}) - \boldsymbol{L}_Y(X+\mathfrak{h}) - \big([X,Y]+\mathfrak{h}\big). \tag{17}$$

Curvature Again, for $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned} R(X^+,Y^+) &:= [\nabla_{X^+},\nabla_{Y^+}] - \nabla_{[X^+,Y^+]} \\ &= [\boldsymbol{L}_{X^+} + \mathrm{ad}_{X^+}, \boldsymbol{L}_{Y^+} + \mathrm{ad}_{Y^+}] - \nabla_{[X^+,Y^+]} \\ &= [\boldsymbol{L}_{X^+},\boldsymbol{L}_{Y^+}] + [\mathrm{ad}_{X^+},\boldsymbol{L}_{Y^+}] + [\boldsymbol{L}_{X^+},\mathrm{ad}_{Y^+}] \\ &\quad + \mathrm{ad}_{[X^+,Y^+]} - \nabla_{[X^+,Y^+]} \\ &= [\boldsymbol{L}_{X^+},\boldsymbol{L}_{Y^+}] + [\mathrm{ad}_{X^+},\boldsymbol{L}_{Y^+}] + [\boldsymbol{L}_{X^+},\mathrm{ad}_{Y^+}] - \boldsymbol{L}_{[X^+,Y^+]}\,. \end{aligned}$$

Now, given a manifold M with an affine connection, an *affine transformation* is a diffeomorphism φ such that $\varphi_*(\nabla_X Y) = \nabla_{\varphi_*(X)}\varphi_*(Y)$ for any vector fiels $X, Y \in \mathfrak{X}(M)$, and a vector field $X \in \mathfrak{X}(M)$ is called an *infinitesimal affine transformation* if its local flow Φ_t is an affine transformation for $-\epsilon < t < \epsilon$ for some $\epsilon > 0$.

If $X \in \mathfrak{X}(M)$ is an infinitesimal affine transformation, for any $Y, Z \in \mathfrak{X}(M)$ we get:

$$\begin{aligned} [X,\nabla_Y Z] &= \left.\frac{d}{dt}\right|_{t=0} (\Phi_{-t})_*(\nabla_Y Z) \quad \text{(because of (3))} \\ &= \left.\frac{d}{dt}\right|_{t=0} \big(\nabla_{(\Phi_{-t})_*(Y)}(\Phi_{-t})_*(Z)\big) \\ &= \nabla_{[X,Y]}Z + \nabla_Y[X,Z]\,. \end{aligned} \tag{18}$$

The last equality above is proved as follows. Locally (i.e., in local coordinates)

$$Y(t,q) := (\Phi_{-t})_* Y = \sum_i f_i(t,q) \frac{\partial}{\partial x_i}$$

and

$$Z(t,q) := (\Phi_{-t})_* Z = \sum_i g_i(t,q) \frac{\partial}{\partial x_i}\,.$$

Hence

$$\nabla_{Y(t,q)} Z(t,q) = \sum_{i,j} f_i(t,q) \frac{\partial g_j(t,q)}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j} f_i(t,q) g_j(t,q) \Gamma_{i,j}(q) \frac{\partial}{\partial x_k}\,.$$

(Note that the Christoffel symbols do not depend on t.) Now take the derivative with respect to t at $t = 0$ using Leibniz's rule to get the required equality.

Returning to our homogeneous space $M \simeq G/H$ with an invariant affine connection ∇, for any $X \in \mathfrak{g}$, X^+ is an infinitesimal affine transformation, because its global flow is $\tau(\exp(tX))$ and ∇ is invariant. Hence, by (18), we get for any $X, Y \in \mathfrak{g}$:

$$\begin{aligned}[\mathrm{ad}_{X^+}, \boldsymbol{L}_{Y^+}] &= [\mathrm{ad}_{X^+}, \nabla_{Y^+} + \mathrm{ad}_{Y^+}] = \nabla_{[X^+,Y^+]} + \mathrm{ad}_{[X^+,Y^+]} \\ &= \boldsymbol{L}_{[X^+,Y^+]} = -\boldsymbol{L}_{[X,Y]^+},\end{aligned}$$

so the formula above for $R(X^+, Y^+)$ becomes:

$$\begin{aligned}R(X^+, Y^+) &= [\boldsymbol{L}_{X^+}, \boldsymbol{L}_{Y^+}] + \boldsymbol{L}_{[X^+,Y^+]} - \boldsymbol{L}_{[Y^+,X^+]} - \boldsymbol{L}_{[X^+,Y^+]} \\ &= [\boldsymbol{L}_{X^+}, \boldsymbol{L}_{Y^+}] + \boldsymbol{L}_{[X^+,Y^+]} = [\boldsymbol{L}_{X^+}, \boldsymbol{L}_{Y^+}] - \boldsymbol{L}_{[X,Y]^+},\end{aligned}$$

which at the point $p \in M$ corresponds via ϕ in (13) to

$$R(X + \mathfrak{h}, Y + \mathfrak{h}) = [\boldsymbol{L}_X, \boldsymbol{L}_Y] - \boldsymbol{L}_{[X,Y]} \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}). \tag{19}$$

Remark 6. Equation (19) shows that the connection ∇ is flat (i.e., its curvature is trivial) if and only if the map $\boldsymbol{L} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ is a homomorphism of Lie algebras, that is, $\boldsymbol{L}$ gives a representation of $\mathfrak{g}$ in the quotient space $\mathfrak{g}/\mathfrak{h}$.

5 Nomizu's Theorem

5.1 Reductive homogeneous spaces

Definition 8. The homogeneous space $M \simeq G/H$ is said to be *reductive* if there is a decomposition as a direct sum of vector spaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$ for any $h \in H$. (Such decomposition is called a *reductive decomposition*.)

Remark 7. Because of (11), the condition $\mathrm{Ad}_H(\mathfrak{m}) \subseteq \mathfrak{m}$ implies $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, and the converse holds if H is connected.

A big deal of information on the reductive homogneous space $M \simeq G/H$ is located in the following two products defined on $\mathfrak{m}$ ($\simeq \mathfrak{g}/\mathfrak{h} \simeq T_pM$):

Binary product:

$$X \cdot Y := [X, Y]_{\mathfrak{m}}\,, \tag{20}$$

the projection of $[X, Y]$ on $\mathfrak{m}$

Ternary product:

$$[X, Y, Z] := [[X, Y]_{\mathfrak{h}}, Z]\,, \tag{21}$$

where $[X, Y]_{\mathfrak{h}}$ denotes the projection of $[X, Y]$ on $\mathfrak{h}$.

Exercise 9. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra and $\mathfrak{m}$ a subspace such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

1. Prove the following properties:

   (LY1) $X \cdot X = 0$,

   (LY2) $[X, X, Y] = 0$,

   (LY3) $\sum_{(X,Y,Z)} \big([X, Y, Z] + (X \cdot Y) \cdot Z\big) = 0$,

   (LY4) $\sum_{(X,Y,Z)} [X \cdot Y, Z, T] = 0$,

   (LY5) $[X, Y, U \cdot V] = [X, Y, U] \cdot V + U \cdot [X, Y, V]$,

   (LY6) $[X, Y, [U, V, W]] = [[X, Y, U], V, W] + [U, [X, Y, V], W] + [U, V, [X, Y, W]]$,

   for any $X, Y, Z, T, U, V, W \in \mathfrak{m}$, where $\sum_{(X,Y,Z)}$ stands for the cyclic sum on X, Y, Z.

   A vector space endowed with a binary and a ternary multilinear products satisfying these properties is called a *Lie-Yamaguti algebra*. These algebras were named 'general Lie triple systems' by Yamaguti [26]. The name 'Lie--Yamaguti algebras' was given by Kinyon and Weinstein [13]. Irreducible Lie-Yamaguti algebras have been studied in [4], [5].

2. Prove that if $(\mathfrak{m}, \cdot, [\dots])$ is a Lie-Yamaguti algebra, then there is a Lie algebra $\mathfrak{g}$ containing $\mathfrak{m}$ and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ complementing $\mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ such that $X \cdot Y = [X, Y]_{\mathfrak{m}}$ and $[X, Y, Z] = [[X, Y]_{\mathfrak{h}}, Z]$ for any $X, Y, Z \in \mathfrak{m}$.

   [Hint: Let $\mathfrak{h}$ be the subspace of $\mathrm{End}(\mathfrak{m})$ spanned by the endomorphismsm $[X, Y, \cdot]$ for $X, Y \in \mathfrak{m}$, and define $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with a suitable bracket.]

5.2 Invariant affine connections on reductive homogeneous spaces

Let $M \simeq G/H$ be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In this case we will identify the quotient space $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{m}$. Hence any linear map $\boldsymbol{L} \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ satisfying properties (a) and (b) in Proposition 4 is determined by the linear map

$$\begin{aligned} \tilde{\boldsymbol{L}} \colon \mathfrak{m} &\to \mathrm{End}_{\mathbb{R}}(\mathfrak{m}) \\ X &\mapsto \tilde{\boldsymbol{L}}_X \colon \mathfrak{m} \to \mathfrak{m} \\ &\qquad Y \mapsto \tilde{\phi}^{-1}\big(\boldsymbol{L}_{X^+}(Y_p^+)\big), \end{aligned}$$

where $\tilde{\phi}\colon \mathfrak{m} \to T_pM$ is the composition of the identification $\mathfrak{m} \simeq \mathfrak{g}/\mathfrak{h}$ and the map $\phi\colon \mathfrak{g}/\mathfrak{h} \to T_pM$ in (13).

Note that the restriction of the map $\boldsymbol{L}$ in (16) to $\mathfrak{h}$ is determined by property (b) in Proposition 4, while property (a) translates, thanks to (14), into the condition

$$\tilde{\boldsymbol{L}}_{\mathrm{Ad}_h X} = \mathrm{Ad}_h\big|_{\mathfrak{m}} \circ \tilde{\boldsymbol{L}}_X \circ \mathrm{Ad}_{h^{-1}}\big|_{\mathfrak{m}} \tag{22}$$

for any $h \in H$ and $X \in \mathfrak{m}$.

But any such map $\tilde{\boldsymbol{L}}$ determines (and is determined by) the bilinear map

$$\begin{aligned} \alpha\colon \mathfrak{m} \times \mathfrak{m} &\to \mathfrak{m} \\ (X, Y) &\mapsto \alpha(X, Y) := \tilde{\boldsymbol{L}}_X(Y)\,, \end{aligned}$$

and Equation (22) is equivalent to the condition

$$\alpha\big(\mathrm{Ad}_h X, \mathrm{Ad}_h Y\big) = \mathrm{Ad}_h\big(\alpha(X, Y)\big)$$

for any $h \in H$ and $X, Y \in \mathfrak{m}$.

This proves the following theorem of Nomizu, which is the main result of this course:

Theorem 2 (Nomizu, 1954). *The invariant affine connections on a reductive homogeneous space $M \simeq G/H$ with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ are in bijection with the vector space of nonassociative multiplications $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that H acts by automorphisms, i.e., $\mathrm{Ad}_H\big|_{\mathfrak{m}} \subseteq \mathrm{Aut}(\mathfrak{m}, \alpha)$.*

Remark 8. The vector space $\mathfrak{m}$ is a module for the group H, and the vector space of nonassociative multiplications as in the Theorem above is naturally isomorphic to $\mathrm{Hom}_H(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$.

Moreover, the condition $\mathrm{Ad}_H\big|_{\mathfrak{m}} \subseteq \mathrm{Aut}(\mathfrak{m}, \alpha)$ implies that $\mathrm{ad}_{\mathfrak{h}}\big|_{\mathfrak{m}}$ is contained in $\mathrm{Der}(\mathfrak{m}, \alpha)$. The converse is valid if H is connected.

Remark 9. Nomizu [21] proved this result in a very different way. He defined the product $\alpha(X, Y)$ by extending locally X and Y to some invariant vector fields $\tilde{X}$ and $\tilde{Y}$ defined on a neighbourhood of p and imposing $\alpha(X, Y) := \nabla_{\tilde{X}_p} \tilde{Y} \in T_pM \simeq \mathfrak{m}$.

The torsion and curvature tensors are determined in equations (17) and (19), which now become simpler:

$$\begin{aligned} T(X, Y) &= \alpha(X, Y) - \alpha(Y, X) - X \cdot Y\,, \\ R(X, Y)Z &= \alpha(X, \alpha(Y, Z)) - \alpha(Y, \alpha(X, Z)) - \alpha(X \cdot Y, Z) - [X, Y, Z]\,, \end{aligned}$$

for any $X, Y, Z \in \mathfrak{m}$, where $X \cdot Y = [X, Y]_{\mathfrak{m}}$ and $[X, Y, Z] = [[X, Y]_{\mathfrak{h}}, Z]$ as in (20) and (21).

There are always two distinguished invariant affine connections in this case:

- **Natural connection** (or canonical connection of the first kind), given by

$$\alpha(X,Y) = \frac{1}{2} X \cdot Y \,,$$

for $X, Y \in \mathfrak{m}$, which is symmetric ($T(X,Y) = 0$ for any $X, Y \in \mathfrak{m}$).

- **Canonical connection** (of the second kind), given by $\alpha(X,Y) = 0$ for any $X, Y \in \mathfrak{m}$.

Remark 10. These two connections coincide if and only if $X \cdot Y = 0$ for any $X, Y \in \mathfrak{m}$, and this is equivalent to the fact that the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a $\mathbb{Z}/2\mathbb{Z}$-grading: $\mathfrak{g}_{\bar{0}} = \mathfrak{h}$ is a subalgebra, $\mathfrak{g}_{\bar{1}} = \mathfrak{m}$ is a $\mathfrak{g}_{\bar{0}}$-module, and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{0}}$.

This is the case for the *symmetric spaces*, which are the coset spaces G/H for a Lie group G endowed with an order 2 automorphism σ such that

$$G_0^\sigma \subseteq H \subseteq G^\sigma \,,$$

where $G^\sigma = \{g \in G : \sigma(g) = g\}$ is the subgroup of fixed elements by σ and G_0^σ is the connected component of G^σ containing the neutral element. In this case $\mathfrak{h} = \{X \in \mathfrak{g} : \sigma_*(X) = X\}$ is the Lie algebra of H, and $\mathfrak{m} := \{X \in \mathfrak{g} : \sigma_*(X) = -X\}$ is a complementary subspace. Since $(\sigma_*)^2 = (\sigma^2)_*$ is the identity, and $\sigma_*\big|_{\mathfrak{g}}$ is an automorphism, this gives a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, which is clearly a $\mathbb{Z}/2\mathbb{Z}$-grading. (Gradings on Lie algebras are the subject of [9].)

The invariant affine connections on symmetric spaces, and the associated algebras, have been studied in [2], [17].

Nomizu's Theorem allows us to transfer geometric conditions to algebra, or algebraic conditions to geometry.

For example, given a homogeneous space $M \simeq G/H$, and a vector field $X \in \mathfrak{g}$, the curve $\gamma\colon t \mapsto \exp(tX) \cdot p$ is a geodesic with respect to an invariant affine connection ∇ if and only if

$$\nabla_{X^+_{\exp(tX)\cdot p}} X^+ = 0$$

for any $t \in \mathbb{R}$. (Recall that $\dot\gamma(t) = X^+_{\exp(tX)\cdot p}$ by the definition of X^+.) But

$$\nabla_{X^+_{g\cdot p}} Y^+ = \tau(g)_*\big(\nabla_{(\mathrm{Ad}_{g^{-1}} X)^+_p} (\mathrm{Ad}_{g^{-1}} Y)^+\big)$$

by Proposition 2, since ∇ is invariant. From $\mathrm{Ad}_{\exp(-tX)} X = X$ (see (11)), we check that $\nabla_{X^+_{\exp(tX)\cdot p}} X^+ = 0$ for any $t \in \mathbb{R}$ if and only if $\nabla_{X^+_p} X^+ = 0$, or $\boldsymbol{L}_{X^+}(X^+_p) = 0$.

The last condition is equivalent to $\boldsymbol{L}_X(X + \mathfrak{h}) = 0$, where $\boldsymbol{L}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ is the linear map in (16). Hence, if $M \simeq G/H$ is reductive and α is the multiplication on $\mathfrak{m}$ determined by ∇ we obtain the following result:

Proposition 5. *Under the hypotheses above, the multiplication α attached to ∇ is anticommutative (i.e., $\alpha(X,X) = 0$ for any $X \in \mathfrak{m}$) if and only if the geodesics through p are exactly the curves $t \mapsto \exp(tX) \cdot p$ for $X \in \mathfrak{m}$.*

Note that given a tangent vector $v \in T_pM$ there is a unique geodesic $\mu(t)$ through p: $\mu(0) = p$, such that $\dot{\mu}(0) = v$ (in general, μ is defined only locally). The geodesics through other points are given by 'translation'. Actually, if $q = g \cdot p$, then

$$\tau(g^{-1})_* \nabla_{X_q^+} X^+ = \nabla_{\tau(g^{-1})_*(X_q^+)} \tau(g^{-1})_*(X^+) = \nabla_{(\mathrm{Ad}_{g^{-1}} X)_p^+} (\mathrm{Ad}_{g^{-1}} X)_p^+ ,$$

so if $\nabla_{X_p^+} X^+ = 0$ for any $X \in \mathfrak{g}$, also $\nabla_{X_q^+} X^+ = 0$ for any $X \in \mathfrak{g}$ and $q \in M$.

Corollary 1. *The natural connection is the only symmetric invariant affine connection such that the curves $t \mapsto \exp(tX) \cdot q$ are geodesic for any $X \in \mathfrak{m}$ and $q \in M$.*

Proof. If α is anticommutative, then we have $T(X,Y) = 2\alpha(X,Y) - X \cdot Y$. So the connection is symmetric (zero torsion) if and only if $\alpha(X,Y) = \frac{1}{2} X \cdot Y$. □

Exercise 10. Let $M \simeq G/H$ be a homogeneous space endowed with an invariant pseudo-metric g. This means that M is pseudo-Riemannian with the pseudo-metric g satisfying

$$g(\tau(x)_*(u), \tau(x)_*(v))_{x \cdot q} = g(u,v)_q$$

for any $x \in G$ and $u, v \in T_qM$. Hence, for any $q \in M$, $X \in \mathfrak{g}$, and $U, V \in \mathfrak{X}(M)$ we have:

$$\begin{aligned} X_q^+(g(U,V)) &= \left.\frac{d}{dt}\right|_{t=0} g(U,V)_{\exp(tX)\cdot q} \\ &= \left.\frac{d}{dt}\right|_{t=0} g\big(\tau(\exp(-tX))_*(U), \tau(\exp(-tX))_*(V)\big)_q \\ &= g\big([X^+,U],V\big)_q + g\big(U,[X^+,V]\big)_q \quad \text{(see (3))}. \end{aligned}$$

In particular, for $X \in \mathfrak{h}$, $X_p^+ = 0$, so $g\big([X^+,U],V\big)_p + g\big(U,[X^+,V]\big)_p = 0$.

1. Prove that the Levi-Civita connection of (M,g) is invariant.
2. Check that the Nomizu operator $\boldsymbol{L}_{X^+}$ for $X \in \mathfrak{g}$ satisfies
$$g\big(\boldsymbol{L}_{X^+}(U),V)\big) + g\big(U, \boldsymbol{L}_{X^+}(V)\big) = 0$$
for any $U, V \in \mathfrak{X}(M)$.
3. Conclude that for $X, Y, Z \in \mathfrak{g}$ we have
$$2g\big(\boldsymbol{L}_{X^+}(Y^+), Z^+\big) = g\big([X,Y]^+, Z^+\big) - g\big([X,Z]^+, Y^+\big) - g\big(X^+, [Y,Z]^+\big).$$
4. Prove that if $\boldsymbol{L}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ is the associated linear map in Theorem 1 and we identify $g_p : T_pM \times T_pM \to \mathbb{R}$ with a bilinear map
$$\bar{g}\colon \mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathbb{R}$$
by means of (13), then for any $X, Y, Z \in \mathfrak{g}$ we get
$$\begin{aligned} &2\bar{g}\big(\boldsymbol{L}_X(Y+\mathfrak{h}), Z+\mathfrak{h}\big) \\ &\quad = \bar{g}\big([X,Y]+\mathfrak{h}, Z+\mathfrak{h}\big) - \bar{g}\big(Y+\mathfrak{h}, [X,Z]+\mathfrak{h}\big) - \bar{g}\big(X+\mathfrak{h}, [Y,Z]+\mathfrak{h}\big). \end{aligned}$$

5. Deduce that if M is reductive with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and we identify $\bar{g}$ with a bilinear map (also denoted by $\bar{g}$) $\mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$, the multiplication α in $\mathfrak{m}$ attached by Nomizu's Theorem 2 with the Levi-Civita connection on M is determined by the equation:

$$2\bar{g}\big(\alpha(X,Y),Z\big) = \bar{g}(X\cdot Y,Z) - \bar{g}(X\cdot Z,Y) - \bar{g}(X,Y\cdot Z)$$

for any $X, Y, Z \in \mathfrak{m}$. Conclude from (2) that

$$\bar{g}(\alpha(X,Y),Z) + \bar{g}(Y,\alpha(X,Z)) = 0$$

for any $X, Y, Z \in \mathfrak{m}$.

6. Since the Levi-Civita connection is symmetric, $\alpha(X,Y) = \frac{1}{2}X\cdot Y + \mu(X,Y)$, where $\mu(X,Y)$ is a commutative multiplication in $\mathfrak{m}$. Prove that this commutative multiplication is determined by the equation:

$$2\bar{g}\big(\mu(X,Y),Z\big) = \bar{g}(Z\cdot X,Y) + \bar{g}(X,Z\cdot Y)$$

for any $X, Y, Z \in \mathfrak{m}$.

7. Prove that the Levi-Civita connection coincides with the natural connection if and only if

$$\bar{g}(Y\cdot X,Z) = \bar{g}(Y,X\cdot Z)$$

for any $X, Y, Z \in \mathfrak{m}$ (i.e., $\bar{g}$ is associative relative to the product $X\cdot Y$). In this case M is said to be a *naturally reductive homogeneous manifold* (see [20]).

6 Examples

6.1 Left invariant affine connections on Lie groups

Any Lie group G is a reductive homogeneous space under the action of G into itself by left multiplication. The isotropy subgroup at any point is trivial. Hence G is a reductive homogeneous space with $\mathfrak{h} = 0$ and $\mathfrak{m} = \mathfrak{g}$. Also, for any $X \in \mathfrak{g}$, the vector field X^+ is nothing else but the right invariant vector field $\hat{X} \in \mathfrak{g}_{\text{right}}$ with $\hat{X}_e = X_e$ (see Proposition 1).

According to Nomizu's Theorem (Theorem 2) we have a bijection

$$\left\{\begin{matrix}\text{Left invariant affine}\\ \text{connections on } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{bilinear maps}\\ \alpha\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\end{matrix}\right\}$$
$$\nabla \mapsto \alpha(X,Y) := \phi^{-1}\big((\boldsymbol{L}_{X^+}(Y^+)_e\big)$$

where $\phi\colon \mathfrak{g} \to T_eM$, $X \mapsto X_e$, as in (13).

For $X, Y \in \mathfrak{g}$ we get

$$\big(\boldsymbol{L}_{X^+}(Y^+)\big)_e = \boldsymbol{L}_{\hat{X}}(\hat{Y}_e) = \boldsymbol{L}_{\hat{X}}(Y_e) = \big(\boldsymbol{L}_{\hat{X}}(Y)\big)_e = \big(\nabla_{\hat{X}}Y - [\hat{X},Y]\big)_e\,,$$

and

$$\big(\nabla_{\hat{X}}Y\big)_e = \nabla_{\hat{X}_e}Y = \nabla_{X_e}Y = \big(\nabla_X Y\big)_e\,.$$

Besides, $\nabla_X Y \in \mathfrak{g}$ by left invariance, and $[\hat{X},Y] = 0$ since the corresponding flows commute: the flow of $\hat{X}$ is $L_{\exp(tX)}$ and the flow of Y is $R_{\exp(tY)}$ and left and right multiplications commute by associativity of the multiplication in the group G.

Here we use the following result:

Lemma 1. *Let M be a manifold, $X, Y \in \mathfrak{X}(M)$ with flows Φ_t and Ψ_t respectively. If $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for some $\epsilon > 0$ and $-\epsilon < s, t < \epsilon$, then $[X, Y] = 0$.*

Proof. Just notice that for any $f \in \mathcal{C}^\infty(M)$ and any point $q \in M$

$$\begin{aligned}
[X,Y]_q(f) &= \left.\frac{d}{dt}\right|_{t=0} \big((\Phi_{-t})_*(Y)\big)_q(f) = \left.\frac{d}{dt}\right|_{t=0} Y_{\Phi_t(q)}(f \circ \Phi_{-t}) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\left.\frac{d}{ds}\right|_{s=0} \big(f \circ \Phi_{-t} \circ \Psi_s(\Phi_t(q))\big) \right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\left.\frac{d}{dt}\right|_{t=0} \big(f \circ \Phi_s(q)\big) \right) \quad \text{(because } \Phi_{-t} \text{ and } \Psi_s \text{ commute)} \\
&= \left.\frac{d}{dt}\right|_{t=0} \big(Y_q(f)\big) = 0 \quad \text{(as } Y_q(f) \text{ does not depend on } t\text{).}
\end{aligned}$$

□

Therefore $\big(\boldsymbol{L}_{X^+}(Y^+)\big)_e = \big(\nabla_X Y\big)_e$ and $\phi^{-1}\big((\boldsymbol{L}_{X^+}(Y^+)_e\big) = \nabla_X Y$. Hence the bijection in Nomizu's Theorem becomes a very natural one:

$$\begin{Bmatrix} \text{Left invariant affine} \\ \text{connections on } G \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{bilinear maps} \\ \alpha\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \end{Bmatrix} \qquad (23)$$
$$\nabla \mapsto \alpha(X,Y) = \nabla_X Y$$

Remark 11.

- The canonical connection is then the connection that satisfies $\nabla_X Y = 0$ for any $X, Y \in \mathfrak{g}$. That is, it is the affine connection in which the left invariant vector fields are parallel.
- A left invariant affine connection ∇ on a Lie group G is symmetric if and only if $\nabla_X Y - \nabla_Y X = [X, Y]$ for any $X, Y \in \mathfrak{g}$. This means that the algebra $(\mathfrak{g}, \alpha)$ (with multiplication $\alpha(X, Y) = \nabla_X Y$) is *Lie-admissible* with associated Lie algebra $\mathfrak{g}$:

  A nonassociative algebra (A, α) is said to be *Lie-admissible* if the new algebra defined on A but with multiplication $\alpha^-(x, y) := \alpha(x, y) - \alpha(y, x)$ is a Lie algebra (the associated Lie algebra). For instance, associative algebras and Lie algebras are Lie-admissible.
- A left invariant affine connection ∇ on a Lie group G is flat if and only if $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$ for any $X, Y \in \mathfrak{g}$.

Exercise 11. A nonassociative algebra (A, α) is *left-symmetric* if the associator, defined as the trilinear map $(x, y, z) := \alpha(\alpha(x, y), z) - \alpha(x, \alpha(y, z))$ (the associator measures the lack of associativity of α!), is symmetric in the first two components.

1. Prove that any left-symmetric algebra is Lie-admissible.
2. Deduce that a left invariant affine connection on a Lie group G is symmetric and flat if and only if the corresponding nonassociative algebra $(\mathfrak{g}, \alpha)$ (where $\alpha(X, Y) = \nabla_X Y$) is left-symmetric and the algebra $(\mathfrak{g}, \alpha^-)$ is just the Lie algebra $\mathfrak{g}$. (Therefore, the left invariant symmetric (i.e., torsion-free) and flat affine connections on a Lie group are in a natural bijection with the set of left-symmetric algebra structures on the corresponding Lie algebra.)

The reader may consult the survey paper [6] for more results on (and applications of) left-symmetric algebras.

6.2 Bi-invariant affine connections on Lie groups

Let ∇ be a left invariant affine connection on a Lie group G. Then ∇ is also right invariant if for any $g \in G$ and $X, Y \in \mathfrak{g}$,

$$(R_{g^{-1}})_*(\nabla_X Y) = \nabla_{(R_{g^{-1}})_*(X)}(R_{g^{-1}})_*(Y).$$

But by left invariance $(R_{g^{-1}})_*(X) = (R_{g^{-1}})_* \circ (L_g)_*(X) = \mathrm{Ad}_g X$, and similarly for Y and $\nabla_X Y$. Hence ∇ is bi-invariant if and only if

$$\mathrm{Ad}_g(\nabla_X Y) = \nabla_{\mathrm{Ad}_g X}(\mathrm{Ad}_g Y)$$

for any $g \in G$ and $X, Y \in \mathfrak{g}$. Therefore the bijection in (23) restricts to a bijection

$$\left\{\begin{matrix}\text{Bi-invariant affine}\\ \text{connections on } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{bilinear maps}\\ \alpha\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\\ \text{with } \mathrm{Ad}_G \subseteq \mathrm{Aut}(\mathfrak{g},\alpha)\end{matrix}\right\}$$
$$\nabla \mapsto \alpha(X,Y) = \nabla_X Y\,.$$

Note that the space on the right can be canonically identified with $\mathrm{Hom}_G(\mathfrak{g}\otimes_{\mathbb{R}}\mathfrak{g},\mathfrak{g})$, which is contained in $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes_{\mathbb{R}}\mathfrak{g},\mathfrak{g})$ (and they coincide if G is connected).

Remark 12.

- If ∇ is a symmetric bi-invariant affine connection on a Lie group G, and $\alpha\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is the associated multiplication ($\alpha(X,Y) = \nabla_X Y$), then we know from Remark 11 that $(\mathfrak{g},\alpha)$ is Lie-admissible with associated Lie algebra $\mathfrak{g}$. But here α lies in $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes_{\mathbb{R}}\mathfrak{g},\mathfrak{g})$. This means that

  $$[X,\alpha(Y,Z)] = \alpha([X,Y],Z) + \alpha(Y,[X,Z])$$

  for any $X, Y, Z \in \mathfrak{g}$. That is, ad_X is a derivation of $(\mathfrak{g},\alpha)$. Then the algebra $(\mathfrak{g},\alpha)$ is flexible and Lie-admissible. (See Exercise 12). Flexible Lie-admissible algebras are studied in [19].

- If ∇ is a symmetric and flat bi-invariant affine connection on a Lie group G, and $\alpha : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is the associated multiplication: $\alpha(X,Y) = \nabla_X Y$, then we know from Remark 11 that $(\mathfrak{g},\alpha)$ is left-symmetric and flexible, but then the algebra $(\mathfrak{g},\alpha)$ is associative [18]. (See Exercise 12.)

Exercise 12. Let (A,α) be a nonassociative algebra. By simplicity, write $\alpha(x,y) = xy$ for any $x, y \in A$. Recall from Exercise 11 that the associator of the elements $x, y, z \in A$ is the element $(x,y,z) = (xy)z - x(yz)$. The algebra (A,α) is said to be *flexible* if $(x,y,x) = 0$ for any $x, y \in A$. That is, the associator is alternating in the first and third arguments.

1. Assume that this algebra is Lie-admissible (i.e.; the bracket $[x,y] := xy - yx$ satisfies the Jacobi identity). Prove that it is flexible if and only if the map $\mathrm{ad}_x\colon y \mapsto [x,y]$ is a derivation of (A,α) for any $x \in A$: $[x,yz] = [x,y]z + y[x,z]$ for any $x, y, z \in A$.

2. Check that the Lie algebra $\mathfrak{su}(n)$ of the special unitary group $SU(n)$, endowed with the multiplication given by:

$$\alpha(X,Y) = \mu XY - \bar{\mu} YX - \frac{\mu - \bar{\mu}}{n}\mathrm{trace}(XY),$$

where $\mu \in \mathbb{C}$ satisfies $\mu + \bar{\mu} = 1$, is a flexible and Lie-admissible algebra.

This example plays a key role in the classification of the invariant connections on compact simple Lie groups by Laquer [16].

3. Prove that an algebra (A, α) is flexible and left-symmetric if and only if it is associative.

Simple non-Lie Malcev algebras [11], and other interesting nonassociative algebras [3], [8], [10], [12], [22], appear too related to some specific reductive homogeneous spaces.

Nomizu's Theorem has been extended in order to study invariant affine connections on principal fibre bundles which admits a fibre-transitive Lie group of automorphisms [24], but this is out of the scope of this introductory survey.

References

[1] D.V. Alekseevskiĭ, R.V. Gamkrelidze, V.V. Lychagin, A.M. Vinogradov: *Geometry I: Basic ideas and concepts of differential geometry*. Springer, Berlin (1991). Encyclopaedia Math. Sci. 28.

[2] P. Benito, C. Draper, A. Elduque: On some algebras related to simple Lie triple systems. *Journal of Algebra* 219 (1) (1999) 234–254.

[3] P. Benito, C. Draper, A. Elduque: Lie-Yamaguti algebras related to $\mathfrak{g}_2$. *Journal of Pure and Applied Algebra* 202 (1–3) (2005) 22–54.

[4] P. Benito, A. Elduque, F. Martín-Herce: Irreducible Lie-Yamaguti algebras. *Journal of Pure and Applied Algebra* 213 (5) (2009) 795–808.

[5] P. Benito, A. Elduque, F. Martín-Herce: Irreducible Lie-Yamaguti algebras of generic type. *Journal of Pure and Applied Algebra* 215 (2) (2011) 108–130.

[6] D. Burde: Left-symmetric algebras, or pre-Lie algebras in geometry and physics. *Central European Journal of Mathematics* 4 (3) (2006) 323–357.

[7] L. Conlon: *Differentiable manifolds, Second edition*. Birkhäuser, Boston, MA (2001).

[8] C. Draper, A. Garvín, F.J. Palomo: Invariant affine connections on odd-dimensional spheres. *Annals of Global Analysis and Geometry* 49 (3) (2016) 213–251.

[9] A. Elduque, M. Kochetov: *Gradings on simple Lie algebras*. American Mathematical Soc. (2013).

[10] A. Elduque, H.C. Myung: Color algebras and affine connections on S^6. *Journal of Algebra* 149 (1) (1992) 234–261.

[11] A. Elduque, H.C. Myung: The reductive pair (B_3, G_2) and affine connections on S^7. *Journal of Pure and Applied Algebra* 86 (2) (1993) 155–171.

[12] A. Elduque, H.C. Myung: The reductive pair (B_4, B_3) and affine connections on S^{15}. *Journal of Algebra* 227 (2) (2000) 504–531.

[13] M.K. Kinyon, A. Weinstein: Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces. *American Journal of Mathematics* 123 (3) (2001) 525–550.

[14] S. Kobayashi, K. Nomizu: *Foundations of differential geometry I*. Interscience Publishers, New York-London (1963).

[15] S. Kobayashi, K. Nomizu: *Foundations of differential geometry II*. Interscience Publishers, New York-London-Sidney (1969).

[16] H.T. Laquer: Invariant affine connections on Lie groups. *Transactions of the American Mathematical Society* 331 (2) (1992) 541–551.

[17] H.T. Laquer: Invariant affine connections on symmetric spaces. *Proceedings of the American Mathematical Society* (1992) 447–454.

[18] P.A. Medina: Flat left-invariant connections adapted to the automorphism structure of a Lie group. *Journal of Differential Geometry* 16 (3) (1981) 445–474.

[19] H.C. Myung: *Malcev-admissible algebras*. Birkhäuser Boston, Inc., Boston, MA (1986). Progress in Mathematics, 64.

[20] S. Nagai: The classification of naturally reductive homogeneous real hypersurfaces in complex projective space. *Archiv der Mathematik* 69 (6) (1997) 523–528.

[21] K. Nomizu: Invariant affine connections on homogeneous spaces. *American Journal of Mathematics* 76 (1) (1954) 33–65.

[22] A.A. Sagle: Jordan algebras and connections on homogeneous spaces. *Transactions of the American Mathematical Society* 187 (1974) 405–427.

[23] E.B. Vinberg: Invariant linear connections in a homogeneous space. *Trudy Moskovskogo Matematicheskogo Obshchestva* 9 (1960) 191–210.

[24] H.C. Wang: On invariant connections over a principal fibre bundle. *Nagoya Mathematical Journal* 13 (1958) 1–19.

[25] F.W. Warner: *Foundations of differentiable manifolds and Lie groups*. Springer-Verlag, New York-Berlin (1983). Corrected reprint of the 1971 edition. Graduate Texts in Mathematics 94.

[26] K. Yamaguti: On the Lie triple system and its generalization. *Journal of Science of the Hiroshima University, Series A (Mathematics, Physics, Chemistry)* 21 (3) (1958) 155–160.

Received: 19 July 2019
Accepted for publication: 5 September 2019
Communicated by: Pasha Zusmanovich