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Lie commutators in a free diassociative algebra

A.S. Dzhumadil'daev, N.A. Ismailov, A.T. Orazgaliyev

Abstract. We give a criterion for Leibniz elements in a free diassociative algebra. In the diassociative case one can consider two versions of Lie commutators. We give criterions for elements of diassociative algebras to be Lie under these commutators. One of them corresponds to Leibniz elements. It generalizes the Dynkin-Specht-Wever criterion for Lie elements in a free associative algebra.

1 Introduction

Let A be an associative algebra and $A^{(-)}$ be its minus algebra under Lie bracket $[a,b] = ab - ba$ for any $a, b \in A$. Then $A^{(-)}$ is a Lie algebra. There are two well-known criterions, Dynkin-Specht-Wever and Friedrich criterions, which allow to determine Lie elements in a free associative algebra [4].

A diassociative algebra is a vector space with two bilinear associative operations $\dashv$ (left product) and $\vdash$ (right product) satisfying the following identities

$$a \dashv (b \dashv c) = a \dashv (b \vdash c) \tag{1}$$

$$(a \vdash b) \dashv c = a \vdash (b \dashv c) \tag{2}$$

$$(a \dashv b) \vdash c = (a \vdash b) \vdash c \tag{3}$$

Diassociative algebras were introduced by Loday in papers [6] and [8]. An algebra is called (right) *Leibniz* if it satisfies Leibniz identity

$$[[a,b],c] = [[a,c],b] + [a,[b,c]] \,.$$

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Affiliation:
A.S. Dzhumadil'daev – Al-Farabi Kazakh National University, Almaty, Kazakhstan and Saint Petersburg State University, Saint Petersburg, Russia
E-mail: dzhuma@hotmail.com
N.A. Ismailov – Astana IT University, Nur-Sultan, Kazakhstan and Saint Petersburg State University, Saint Petersburg, Russia
E-mail: nurlan.ismail@gmail.com
A.T.Orazgaliyev – Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
E-mail: aybol96@gmail.com

Leibniz algebras first appeared in [1] and independently later in [5]. There are two versions of Lie brackets for diassociative algebras.

$$[a,b] := a \dashv b - b \vdash a \quad (\textit{Leibniz bracket} \text{ or } \textit{dicommutator}),$$
$$a \blacklozenge b := a \dashv b - a \vdash b\,.$$

It is well-known that any diassociative algebra under Leibniz bracket becomes a Leibniz algebra. Leibniz algebras are noncommutative generalization of Lie algebras. In [7] Loday and Pirashvili gave generalization of the Poincaré-Birkhoff-Witt (PBW) theorem for Leibniz algebras. From their theorem follows that any identity satisfied by Leibniz brackets in every diassocaitve algebra is a consequence of Leibniz identity. Some analogues of important theorems in Lie algebras, as Engel's Theorem, Levi's Theorem, are valid for Leibniz algebras, see a survey [3]. In our paper we prove analogue of Dynkin-Specht-Wever criterion for Leibniz algebras.

We give a criterion that determines whether an element in $\mathrm{DiAs}(X)$ lies in the subalgebra generated by X and the bilinear operation $\blacklozenge$. In [2] Bremner and Dotsenko proved that any diassociative algebra under the product $a \blacklozenge b$ satisfies the following two identities

$$(a \blacklozenge (b \blacklozenge c)) \blacklozenge d = a \blacklozenge ((b \blacklozenge c) \blacklozenge d), \tag{4}$$
$$(a \blacklozenge b) \blacklozenge (c \blacklozenge d) = 0. \tag{5}$$

Furthermore, they showed that every identity satisfied by the product $a \blacklozenge b$ in diassociative algebra follows from (4) and (5). This result can be also obtained by our criterion.

All algebras are considered over a field of characteristic zero.

2 Statement of the result

Let X be a set and $\mathrm{DiAs}(X)$ be a free diassociative algebra on a set X. Let $a_1, \ldots, a_n \in X$ and $a = a_1 \ldots a_n$ be a monomial in $\mathrm{DiAs}(X)$ with some placement of parantheses and choice of operations. The *center* of a, denoted $c(x)$, is defined inductively as follows. If $a \in X$, then $c(a) = a$, otherwise $c(b_1 \dashv b_2) = c(b_1)$ and $c(b_1 \vdash b_2) = c(b_2)$.

Lemma 1. *([8]) If $a = a_1 \ldots a_n$ and $c(a) = a_i$ then*

$$a = (a_1 \vdash \cdots \vdash a_{i-1}) \vdash a_i \dashv (a_{i+1} \dashv \cdots \dashv a_n).$$

We write $a_1 \cdots a_{i-1}\widehat{a}_i a_{i+1} \cdots a_n$ as a normal form of a.

Lemma 2. *([8]) The set of monomials $a_1 \cdots a_{i-1}\widehat{a}_i a_{i+1} \cdots a_n$ with $1 \le i \le n$ for $a_1, \ldots, a_n \in X$ forms a basis of* $\mathrm{DiAs}(X)$.

In order to formulate the Leibniz criterion, we introduce so-called *Dynkin map D* on $\mathrm{DiAs}(X)$. The map $D\colon \mathrm{DiAs}(X) \to \mathrm{DiAs}(X)$ is the linear map, defined on the base elements as follows

$$\widehat{a}_1 a_2 \ldots a_n \mapsto [[\cdots[a_1, a_2]\cdots], a_n]$$

and

$$a_1a_2\cdots a_{i-1}\widehat{a}_ia_{i+1}\cdots a_n \mapsto -[[\cdots[[a_i,[[\cdots[a_1,a_2]\cdots],a_{i-1}]],a_{i+1}]\cdots],a_n]$$

if $i > 1$.

An element of $\mathrm{DiAs}(X)$ is called *Leibniz* element (or *Lie di-element*) of $\mathrm{DiAs}(X)$ if it can be expressed by elements of X in terms of Leibniz brackets. The main result of our paper is the following theorem.

Theorem 1. *Let f be an element of $\mathrm{DiAs}(X)$ of degree n. Then f is a Leibniz element of $\mathrm{DiAs}(X)$ if and only if $D(f) = nf$.*

For example, if

$$f = \widehat{a}_1a_2a_3 + a_1\widehat{a}_2a_3 - a_1a_3\widehat{a}_2 - a_2\widehat{a}_1a_3 - \widehat{a}_2a_3a_1 - a_3\widehat{a}_1a_2 + a_3a_2\widehat{a}_1 + a_3\widehat{a}_2a_1\,,$$

then we have

$$\begin{aligned} D(f) = {} & [[a_1,a_2],a_3] - [[a_2,a_1],a_3] + [a_2,[a_1,a_3]] + [[a_1,a_2],a_3] \\ & - [[a_2,a_3],a_1] + [[a_1,a_3],a_2] - [a_1,[a_3,a_2]] - [[a_2,a_3],a_1] \end{aligned}$$

(by Leibniz identity)

$$= 3([[a_1,a_2],a_3] - [[a_2,a_3],a_1])\,.$$

Then we check and obtain $f = D(f)/3$. Hence f is a Leibniz element. However, by Theorem 1, $g = \widehat{a}_1a_2a_3 - \widehat{a}_1a_3a_2 - a_2\widehat{a}_1a_3 + a_3a_2\widehat{a}_1$ is not a Leibniz element, because

$$D(g) = [[a_1,a_2],a_3] - [[a_1,a_3],a_2] + [[a_1,a_2],a_3] - [a_1,[a_3,a_2]]$$

(by Leibniz identity)

$$\begin{aligned} & = 3[[a_1,a_2],a_3] - 2[[a_1,a_3],a_2] \\ & = 3\widehat{a}_1a_2a_3 - 2\widehat{a}_1a_3a_2 - a_2\widehat{a}_1a_3 - 2a_2a_3\widehat{a}_1 - a_3\widehat{a}_1a_2 + 3a_3a_2\widehat{a}_1 \\ & \neq 3g\,. \end{aligned}$$

Remark 1. We note that if $ab = a \dashv b = a \vdash b$ and consider skew-symmetric Leibniz bracket, then we obtain well-known Dynkin-Specht-Wever criterion from Theorem 1.

Let $D^{(\blacklozenge)}(X)$ be a subalgebra of $\mathrm{DiAs}(X)$ under the product $\blacklozenge$ generated by set X. Below we give criterion for elements of $\mathrm{DiAs}(X)$ to be in subalgebra $D^{(\blacklozenge)}(X)$.

Theorem 2. *Let $f \in \mathrm{DiAs}(X)$ of the form $f = \sum_{i=1}^{n} \lambda_i a_1\cdots\widehat{a_i}\cdots a_n$. Then $f \in D^{(\blacklozenge)}(X)$ if and only if $\sum_{i=1}^{n}\lambda_i = 0$.*

Remark 2. Define sum of operations in $\mathrm{DiAs}(X)$ as follows

$$a \diamond b = a \dashv b + a \vdash b\,.$$

In [2] the identities satisfied by $a \diamond b$ in every diassociative algebra were studied. Additionally, it was proved that any element in $\mathrm{DiAs}(X)$ of degree more than three can be written by elements of X in terms of diamond product $a \diamond b$.

3 Proof of Theorem 1.

Lemma 3. *Let* $a \in \mathrm{DiAs}(X)$ *and* $b = b_1 \dots \widehat{b_j} \dots b_l$*. Then*

$$D(a \dashv b) = [[\cdots [D(a), b_1] \cdots], b_l] .$$

Proof. Since D is linear, it is sufficient to prove the statement for $a = a_1 \dots \widehat{a}_i \dots a_k$.

$$\begin{aligned} D(a \dashv b) &= D((a_1 \dots \widehat{a}_i \dots a_k) \dashv (b_1 \dots \widehat{b_j} \dots b_l)) \\ &= D(a_1 \dots \widehat{a}_i \dots a_k b_1 \dots b_l) \\ &= -[[\cdots [[[\cdots [[a_i, [[\cdots [a_1, a_2] \cdots], a_{i-1}]], a_{i+1}] \dots], a_k], b_1] \cdots], b_l] \\ &= [[\cdots [D(a), b_1] \cdots], b_l] . \end{aligned}$$

□

Lemma 4. *Let* $a = a_1 \dots \widehat{a}_i \dots a_k$ *and* b *be a Leibniz element of* $\mathrm{DiAs}(X)$*. Then*

$$D(a \vdash b) = -[b, [[\cdots [a_1, a_2] \dots], a_k]] .$$

Proof. Denote by $\theta(a)$ element $[[\cdots [a_1, a_2], \cdots], a_k]$. Assume that

$$b = [[\cdots [b_1, b_2] \cdots], b_l] .$$

We prove the statement by induction on l. If $b = b_1$, then

$$\begin{aligned} D(a \vdash b_1) &= D((a_1 \dots \widehat{a}_i \dots a_k) \vdash b_1) \\ &= D(a_1 \dots a_k \widehat{b_1}) = -[b_1, [[\cdots [a_1, a_2] \dots], a_k]] . \end{aligned}$$

Assume that our statement is true for brackets with less than $l >$ elements.

$$\begin{aligned} D(a \vdash [[\cdots [b_1, b_2] \cdots], b_l]) &= D(a \vdash ([[\cdots [b_1, b_2] \cdots], b_{l-1}] \dashv b_l)) \\ &\quad - D(a \vdash (b_l \vdash [[\cdots [b_1, b_2] \cdots], b_{l-1}])) \end{aligned}$$

(by identity (2))

$$\begin{aligned} &= D((a \vdash [[\cdots [b_1, b_2] \cdots], b_{l-1}]) \dashv b_l) \\ &\quad - D(a \vdash (b_l \vdash [[\cdots [b_1, b_2] \cdots], b_{l-1}])) \end{aligned}$$

(by Lemma 3)

$$\begin{aligned} &= [D(a \vdash [[\cdots [b_1, b_2] \cdots], b_{l-1}]), b_l] \\ &\quad - D(a \vdash (b_l \vdash [[\cdots [b_1, b_2] \cdots], b_{l-1}])) \end{aligned}$$

(by induction assumption)

$$\begin{aligned} &= -[[[[\cdots [b_1, b_2] \cdots], b_{l-1}], \theta(a)], b_l] \\ &\quad + [[[\cdots [b_1, b_2] \cdots], b_{l-1}], [\theta(a), b_l]] \end{aligned}$$

(by Leibniz identity)

$$\begin{aligned}&= -[[[[\cdots[b_1,b_2]\cdots],b_{l-1}],\theta(a)],b_l]\\&\quad+[[[[\cdots[b_1,b_2]\cdots],b_{l-1}],\theta(a)],b_l]\\&\quad-[[[\cdots[b_1,b_2]\cdots],b_l],\theta(a)]\\&= -[[[\cdots[b_1,b_2]\cdots],b_l],\theta(a)]\,.\end{aligned}$$

□

Proof of Theorem 1. Assume that $f = [[\cdots[a_1,a_2]\cdots],a_n]$. We prove the statement by induction on degree n. If $n=2$, then

$$D([a_1,a_2]) = D(a_1 \dashv a_2) - D(a_2 \vdash a_1) = 2[a_1,a_2]\,.$$

Assume that our statement is true for brackets with less than $n>2$ elements.

$$\begin{aligned}D([[\cdots[a_1,a_2],\cdots],a_n]) &= D([[\cdots[a_1,a_2],\cdots],a_{n-1}] \dashv a_n)\\&\quad - D(a_n \vdash [[\cdots[a_1,a_2],\cdots],a_{n-1}])\end{aligned}$$

(by Lemmas 3 and 4)

$$= [D([[\cdots[a_1,a_2],\cdots],a_{n-1}]),a_n] + [[\cdots[a_1,a_2],\cdots],a_n]$$

(by induction assumption)

$$\begin{aligned}&= (n-1)[[\cdots[a_1,a_2],\cdots],a_n] + [[\cdots[a_1,a_2],\cdots],a_n]\\&= n[[\cdots[a_1,a_2],\cdots],a_n]\,.\end{aligned}$$

□

4 Proof of Theorem 2

Lemma 5.

$$\begin{aligned}&a_1\cdots\widehat{a_i}\cdots a_n - a_1\cdots\widehat{a_{i+1}}\cdots a_n\\&\qquad= (-1)^i(\cdots((a_1 \blacklozenge (\cdots(a_{i-1}\blacklozenge(a_i\blacklozenge a_{i+1}))\cdots))\blacklozenge a_{i+2})\cdots)\blacklozenge a_n\,,\end{aligned}$$

where $1\le i\le n-1$.

Proof. It is easy to see that

$$\begin{aligned}a_1\cdots\widehat{a_i}a_{i+1} - a_1\cdots a_i\widehat{a_{i+1}} &= -\widehat{a_1}a_2\cdots a_{i+1} + a_1\cdots\widehat{a_i}a_{i+1}\\&\quad+\widehat{a_1}a_2\cdots a_{i+1} - a_1\cdots a_i\widehat{a_{i+1}}\\&= -a_1\blacklozenge a_2\cdots\widehat{a_i}a_{i+1} + a_1\blacklozenge a_2\cdots\widehat{a_{i+1}}\\&= -a_1\blacklozenge(a_2\cdots\widehat{a_i}a_{i+1} - a_2\cdots\widehat{a_{i+1}})\,,\end{aligned}$$

and

$$\begin{aligned}\widehat{a_i}a_{i+1}\cdots a_n - a_i\widehat{a_{i+1}}\cdots a_n &= \widehat{a_i}a_{i+1}\cdots a_n - a_i\cdots a_{n-1}\widehat{a_n} - a_i\widehat{a_{i+1}}\cdots a_n\\&\quad+a_i\cdots a_{n-1}\widehat{a_n}\\&= \widehat{a_i}a_{i+1}\cdots a_{n-1}\blacklozenge a_n - a_i\widehat{a_{i+1}}\cdots a_{n-1}\blacklozenge a_n\\&= (\widehat{a_i}a_{i+1}\cdots a_{n-1} - a_i\widehat{a_{i+1}}\cdots a_{n-1})\blacklozenge a_n\,.\end{aligned}$$

Induction on n based on these relations ends the proof. □

Proof of Theorem 2. Let $f \in \mathrm{DiAs}(X)$ and $f = \sum_{i=1}^{n} \lambda_i a_1 \cdots \widehat{a_i} \cdots a_n$. If $f \in D^{(\blacklozenge)}(X)$, then it is easy to verify that $\sum_{i=1}^{n} \lambda_i = 0$.

Suppose that $\sum_{i=1}^{n} \lambda_i = 0$. Then

$$\begin{aligned} f &= \sum_{i=1}^{n} \lambda_i a_1 \cdots \widehat{a_i} \cdots a_n \\ &= \lambda_1(\widehat{a_1} a_2 \cdots a_n - a_1 \widehat{a_2} a_3 \cdots a_n) \\ &\quad + (\lambda_1 + \lambda_2)(a_1 \widehat{a_2} a_3 \cdots a_n - a_1 a_2 \widehat{a_3} a_4 \cdots a_n) \\ &\quad + \ldots + (\lambda_1 + \ldots + \lambda_{n-1})(a_1 \cdots a_{n-2} \widehat{a_{n-1}} a_n - a_1 \cdots a_{n-1} \widehat{a_n}) \\ &\quad + (\lambda_1 + \ldots + \lambda_n) a_1 \cdots a_{n-1} \widehat{a_n} \\ &= \sum_{i=1}^{n-1} (\lambda_1 + \ldots + \lambda_i)(a_1 \cdots \widehat{a_i} \cdots a_n - a_1 \cdots \widehat{a_{i+1}} \cdots a_n) \,. \end{aligned}$$

By Lemma 5, $a_1 \cdots \widehat{a_i} \cdots a_n - a_1 \cdots \widehat{a_{i+1}} \cdots a_n \in D^{(\blacklozenge)}(X)$. Hence $f \in D^{(\blacklozenge)}(X)$. □

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References

[1] A. Bloch: A generalization of the concept of a Lie algebra. *Doklady Akademii Nauk – Russian Academy of Sciences* 165 (3) (1965) 471–473.

[2] M.R. Bremner, V. Dotsenko: Bilinear operations in the diassociative operad. preprint.

[3] I. Demir, K.C. Misra, E. Stitzinger: On some structures of Leibniz algebras. *Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics, Contemporary Mathematics* 623 (2014) 41–54.

[4] N. Jacobson: *Lie algebras.* Interscience Publishers, Wiley, New York (1962).

[5] J.-L. Loday: Une version non commutative des algébres de Lie: Les algébres de Leibniz. *L'Enseignement Mathématique* 39 (2) (1993) 269–293.

[6] J.-L. Loday: Algébres ayant deux opérations associatives: les digébres. *Comptes rendus de l'Académie des Sciences* 321 (1995) 141–146.

[7] J.-L. Loday, T. Pirashvili: Universal enveloping algebras of Leibniz algebras and (co)-homology. *Mathematische Annalen* 296 (1) (1993) 139–158.

[8] J.-L. Loday: Dialgebras. Chapter in: Dialgebras and related operads, Lecture Notes in Mathematics, Vol. 1763, J.-L. Loday, F. Chapoton, F. Goichot, and A. Frabetti. (2001).

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