Communications in Mathematics 28 (2020) 231–251
DOI: 10.2478/cm-2020-0018

Conservative algebras and superalgebras: a survey

Yury Popov

Abstract. We give a survey of results obtained on the class of conservative algebras and superalgebras, as well as on their important subvarieties, such as terminal algebras.

1 Introduction

In this section we define conservative algebras, which were introduced by Kantor as a generalization of Jordan algebras that can be studied with the help of the TKK construction. We consider the relation of the class of conservative algebras with the known varieties of nonassociative algebras, and recall the basic properties of this class. Throughout the paper, all spaces and algebras are assumed finite-dimensional over an algebraically closed field $\mathbb{F}$ of characteristic 0, if not said otherwise.

1.1 Origins: the TKK constructions for Jordan algebras

The Tits–Koecher–Kantor construction (TKK construction) is one of the main tools in the theory of Jordan algebras. The main idea of the original TKK construction is to associate (in an invertible way) to a unital Jordan algebra $(J, \circ)$ a $\mathbb{Z}$-graded Lie algebra $\mathrm{TKK}(J) = \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ preserving many important structural properties, such as simplicity and nilpotency, so that J can be studied with the aid of the Lie algebra theory.

Let us briefly recall the details of this construction. Let P be a bilinear operator on J be given by $P(x, y) = x \circ y$. Let L_a be the operator of the left multiplication by a in J. Let $\mathfrak{g}_{-1} = J$, let $\mathfrak{g}_0$ be the subspace of $\mathrm{End}(J)$ spanned by the operators

2020 MSC: 17A30, 17A70, 17A15
Key words: Conservative algebra, Jordan algebra, Tits-Koecher-Kantor construction, terminal algebra.
The work is supported by the Russian Science Foundation under grant 19-71-10016.
Affiliation:
Universidade Estadual de Campinas (UNICAMP), 651 Sergio Buarque de Holanda, 13083-859 Campinas, SP, Brazil
Saint Petersburg State University, 7-9 Universitetskaya emb., 199034 St. Petersburg, Russia
E-mail: yuri.ppv@gmail.com

L_a, $[L_a, L_b]$, $a, b \in J$, and let $\mathfrak{g}_1$ be a subspace of $\mathrm{Hom}(J \otimes J, J)$ spanned by the operators P, $[L_a, P]$, $a \in J$, where the product $[L_a, P]$ (and other nonzero products) are as follows:

$$[A, x] = A(x), \qquad [B, x](y) = B(x, y), \tag{1}$$

$$[A, B](x, y) = A(B(x, y)) - B(A(x), y) - B(x, A(y)). \tag{2}$$

for $x, y \in \mathfrak{g}_{-1}, A \in \mathfrak{g}_0, B \in \mathfrak{g}_1$.

One can verify that the elements $e = 1 \in \mathfrak{g}_{-1}$, $h = -L_1 \in \mathfrak{g}_0$, $f = -P \in \mathfrak{g}_1$ generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$, and the operator $\mathrm{ad}(h)$ only has eigenvalues $-2, 0, 2$ in $\mathfrak{g}$. Such subalgebra is called a *short subalgebra*.

Conversely, one can show that if $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-graded Lie algebra with a short subalgebra and $P \in \mathfrak{g}_1$, then the formula $a \circ b = [[a, P], b]$ defines a structure of a Jordan algebra on $\mathfrak{g}_{-1}$. One can prove that the TKK construction induces an equivalence between the category of Jordan unital algebras and the category of Lie algebras with a short subalgebra (both with surjective morphisms). In particular, this construction preserves simplicity in both ways. For more details, see, for example, [8].

The TKK construction was generalized and applied in various cases. For example, in [9] Kac, using the classification of simple Lie superalgebras and the TKK construction for Jordan superalgebras, classified simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero. Using the same technique, in [24] the authors found the irreducible finite-dimensional representations of simple Jordan superalgebras.

In 1972, wanting to study Lie algebras with more general $\mathbb{Z}$-gradings, Kantor introduced the notion of a conservative algebra as a generalization of Jordan algebras [11]. In the next subsection we consider this definition.

1.2 Main definitions

Unlike other classes of non-associative algebras, the class of conservative algebras is not defined by a set of identities.

For an algebra on a space V and with a multiplication $\cdot$ and an element $x \in V$ we denote by L_x the operator of left multiplication by x. Kantor defines conservative algebras as follows:

Definition 1. An algebra with an underlying vector space V and a multiplication $P(x, y) = xy$ is called a (left) conservative algebra if there exists a new multiplication $P^*(x, y) = x * y$ (called an *associated algebra structure*) on the underlying space of A such that

$$[L_b, [L_a, P]] = -[L_{a*b}, P], \text{ for all } a, b \in V, \tag{3}$$

where the commutation is defined by (2).

Let us explain informally this definition. The relation (2) may be considered as a transformation of a bilinear operator B under the action of an infinitesimal transformation $x \mapsto x + tA(x)$. Indeed, the right-hand side of (2) is the coefficient at the first degree of t in the series $e^{\varphi t}(B(e^{-At}(x), e^{-At}(y)))$. Thus, $\{[L_a, P] : a \in V\}$ is the set of all algebras which arise from the initial algebra P by the action of left shifts L_a, $a \in V$. Thus, the definition of a conservative algebra given above says that this set is transformed into itself under other actions of the left shifts L_a, $a \in V$.

The relation (3) can be written explicitly as an identity of degree 4 with respect to the multiplications $\cdot$ and $*$:

$$b\big(a(xy) - (ax)y - x(ay)\big) - a\big((bx)y\big) + \big(a(bx)\big)y + (bx)(ay) \\ - a\big(x(by)\big) + (ax)(by) + x\big(a(by)\big) = -(a*b)(xy) + \big((a*b)x\big)y + x\big((a*b)y\big).$$

Replacing the left multiplications with the right multiplications and modifying correspondingly the above relation, we can define right conservative algebras and obtain a similar theory.

Assuming that the underlying spaces are $\mathbb{Z}_2$-graded and introducing signs in appropriate places, we can get a notion of a conservative superalgebra [21]. One may also use the general approach to define conservative superalgebras. Namely, let $\Gamma := \Gamma_{\bar 0} \oplus \Gamma_{\bar 1}$ be the Grassmann superalgebra in generators

$$1, \qquad \xi_i, \ i \in \mathbb{N},$$
$$\Gamma_{\bar 0} = \langle 1, \xi_{i_1} \dots \xi_{i_{2k}} : k \in \mathbb{N}\} \rangle\,,$$
$$\Gamma_{\bar 1} = \big\langle \xi_{i_1} \dots \xi_{i_{2k-1}} : k \in \mathbb{N}\} \big\rangle\,.$$

Let $A := A_{\bar 0} \oplus A_{\bar 1}$ be a superalgebra and $\cdot$ and $*$ be two products on A. Consider its *Grassmann envelope* $\Gamma(A) := (A_{\bar 0} \otimes \Gamma_{\bar 0}) \oplus (A_{\bar 1} \otimes \Gamma_{\bar 1})$, and extend the products $\cdot$ and $*$ to $\Gamma(A)$ as follows:

$$(a \otimes f) \cdot (b \otimes g) = (-1)^{ab} ab \otimes fg,$$
$$(a \otimes f) * (b \otimes g) = (-1)^{ab} a * b \otimes fg$$

for all homogeneous $a, b \in A, f, g \in \Gamma$ $(p(a) = p(f), p(b) = p(g))$. Then $(A, \cdot)$ is a conservative superalgebra with an associated multiplication $*$ if and only if $(\Gamma(A), \cdot)$ is a conservative algebra with an associated multiplication $*$.

1.3 Examples and relations with varieties of nonassociative algebras

The class of conservative algebras is very vast. Let us consider some examples.

Lie algebras give obvious examples of conservative algebras. Indeed, let $\mathfrak{g}$ be a Lie algebra with a product P. Then the Jacobi identity and the anticommutativity imply that $[L_a, P] = 0$ for all $a \in \mathfrak{g}$. Thus, the left and right-hand sides of (3) are zero for arbitrary product P^* on $\mathfrak{g}$. Analogously one can show that any (left) Leibniz algebra is conservative.

As another example we have associative algebras. In this case

$$[L_a, P](x, y) = -xay; \quad [L_b, [L_a, P]](x, y) = xaby,$$

and (3) holds with $x * y := xy$. Analogously, Jordan algebras and quasiassociative algebras are quasiassociatve with the associatied multiplication equal to the original one.

The variety of Zinbiel algebras is defined by the relation $x(yz) = (xy + yx)z$. Every Zinbiel algebra is conservative with the associated multiplicaton $a * b = ab + ba$.

The relations of conservative algebras with other classes of nonassociative algebras were studied in [20]. Note that for all nonassociatve varieties mentioned above the associated multiplication can be expressed as $a * b = \alpha ab + \beta ba$ for $\alpha, \beta \in \mathbb{F}$.

As we have seen, associative and Jordan (super)algebras are conservative with the associated multiplication $P = P^*$. It is natural to ask what is the subclass of conservative (super)algebras with this additional restriction.

Definition 2. An algebra A is called a *noncommutative Jordan algebra* if A is flexible (that is, the identity $(xy)x = x(yx)$ holds in A) and its symmetrized algebra (the algebra on the space of A with the multiplication $x \circ y = \frac{1}{2}(xy+yx)$) is Jordan.

For alternative definitions and more information on noncommutative Jordan algebras see [26] and references therein.

Proposition 1 ([11]). *A flexible conservative algebra with the product P whose associated algebra has the same product $P^* = P$ is noncommutative Jordan.*

There is an example of a simple non-conservative noncommutative Jordan algebra. Namely, the simple non-Lie Malcev algebra of dimension 7 is not conservative [20].

The relation between conservative and noncommutative Jordan algebras is made clear in the following proposition:

Proposition 2 ([11]). *A conservative algebra with unity has associated product equal to the original one and is a noncommutative Jordan algebra.*

The following statement provides us with different examples of conservative superalgebras.

Proposition 3 ([21]). *Let Ω be a family of polynomial identities. Suppose that there exist $\alpha, \beta \in \mathbb{F}$ such that every Ω-algebra $(A, \cdot)$ is conservative with the associative multiplication given by $a * b = \alpha a \cdot b + \beta b \cdot a$. Then every Ω-superalgebra $(B, \bullet)$ is conservative with the associated multiplication given by*

$$a * b = \alpha a \bullet b + (-1)^{ab} \beta b \bullet a.$$

It follows that associative, quasi-associative, Jordan, Lie, Leibniz, and Zinbiel superalgebras are conservative.

1.4 Operator relations

Let V be a vector space. By $\mathcal{U}(V)$ we denote the space of all bilinear operators on V. Let $P \in \mathcal{U}(V)$ be a conservative algebra on V (during the text we occasionally identify a bilinear operator P with the algebra structure that it defines on V, and similarly for superalgebras). Considering both parts of (3) as operators acting on $y \in V$, we obtain the following operator relation:

$$[L_b,[L_a,L_x]] - [L_b,L_{ax}] - [L_a,L_{bx}] + L_{a(bx)} + [L_{a*b},L_x] - L_{(a*b)x} = 0. \qquad (4)$$

Therefore, the space $\mathcal{U}_0(V) := \langle L_a, [L_a,L_b] : a,b \in V\rangle$ is a subalgebra of $\mathfrak{gl}(V)$. Moreover, since (2) gives an action of the Lie algebra $\mathfrak{gl}(V)$ on $\mathcal{U}(V)$, we immediately get

$$[[L_b,L_a],P] = [L_{b*a-a*b},P], \qquad (5)$$

which implies that $\mathcal{U}_1(V) := \langle P, [L_a,P] : a \in V\rangle$ is a $\mathcal{U}_0(V)$-submodule of $\mathcal{U}(V)$. This also implies that the operators $[L_b,L_a] - L_{b*a-a*b},\ a,b \in V$, are derivations of P, called *inner derivations*.

Definition 3. Let P be a (super)algebra such that $\mathcal{U}_1(P)$ is a $\mathcal{U}_0(P)$-submodule of $\mathcal{U}_1(P)$. Then P is called *rigid* or *quasi-conservative* [14].

In other words, an algebra (V,P) is rigid if there exist a multiplication P^* and a bilinear form φ on V such that

$$[L_a,[L_b,P]] = -[L_{b*a},P] + \varphi(a,b)P$$

for all $a,b \in V$. Analogously to the relation (1.2), this relation can be expanded to an identity of degree 4 involving the initial multiplication, the associated multiplication and the form φ. In [4] the 2-dimensional complex rigid algebras were classified.

1.5 Jacobi elements and quasiunities

Definition 4. An element a in an algebra M is called a *Jacobi element* provided that

$$a(xy) = (ax)y + x(ay)$$

holds for all $x,y \in M$.

In other words, a is a Jacobi element if L_a is a derivation of M. The relation above can be rewritten in the following forms:

$$[L_a,L_x] = L_{ax} \text{ for every } x \in M, \qquad (6)$$

$$[L_a,M] = 0. \qquad (7)$$

Denote by J the space of all Jacobi elements of an algebra M. Let

$$N := \{a \in M : L_a = 0\}$$

be the *left annihilator* of M. Obviously, $N \subseteq J$. An ideal I of M is called a *Jacobi ideal* provided that $I \subseteq J$.

The following statement is immediate from the definitions and (6).

Lemma 1. *Let M be an algebra, and let J and N be as above. Then J is a subalgebra of M; N is an ideal of J, and the quotient algebra J/N is isomorphic to a subalgebra of the Lie algebra of derivations of M. If M possesses a unity then $J = 0$; and if M is a Lie algebra then $J = M$.*

Definition 5. An element $e \in M$ is said to be a *left quasiunity* if the equality

$$e(xy) = (ex)y + x(ey) - xy$$

holds for all $x, y \in M$.

This condition is equivalent to the relations

$$\begin{aligned} [L_e, L_x] &= L_{ex-x} \text{ for every } x \in M, \\ [L_e, M] &= -M. \end{aligned} \tag{8}$$

Note that a left quasiunity is uniquely determined modulo a Jacobi element. Obviously, a left unity is a left quasiunity. But, in general, the converse is not true (see examples in the next section). The importance of the Jacobi subspace is showed by the following result:

Theorem 1. *Let P be a conservative algebra. The associated algebra P^* is defined up to an arbitrary algebra with values in J. Moreover, the following relations hold:*

$$\begin{aligned} P^*(a,b) &\equiv 0 \pmod{J}, \quad a \in J, \\ P^*(a,b) &\equiv -ba \pmod{J}, \quad b \in J. \end{aligned}$$

If P has a left quasiunity e, then

$$P^*(e,a) \equiv a, \quad P^*(a,e) \equiv 2a - ea \pmod{J}.$$

2 TKK construction for conservative algebras

Since the introduction of the original TKK construction, there have been many attempts to generalize it for various algebraic systems of Jordan type, such as structurable algebras [1], Jordan triple systems [25], Jordan pairs [23] and other classes of systems. In [11] Kantor discovered a version of the TKK construction that can be applied to any algebra. In this section we consider this construction and show how it can be applied to the classification of simple conservative algebras and other algebraic systems.

2.1 A universal graded Lie algebra

The general TKK construction is best formulated in terms of the other construction also introduced by Kantor, that of a *universal graded Lie algebra*. Let us recall the details of this construction, which is of independent interest.

Let V be a vector space of dimension n. In the paper [10] Kantor defined the universal graded Lie algebra $\widetilde{U} = \widetilde{U}^{(n)} = \sum_{i=-\infty}^{\infty} \widetilde{U}_i$. Let us recall the definition of $\widetilde{U}$. The graded component $\widetilde{U}_{-1}$ is identified with V, and the subalgebra $\widetilde{U}^- = \sum_{i=-\infty}^{-1} \widetilde{U}_i$ is the free Lie algebra generated by $\widetilde{U}_{-1}$ with the grading by degree. The space $\widetilde{U}_{k-1}, k > 0$ is the space of all k-linear operators on the space $\widetilde{U}_{-1}$. The commutation between the spaced $\widetilde{U}_{k-1}$ and $\widetilde{U}_{l-1}$, where $k, l > 0$, is defined as follows. Let

$$A = A(x_1, \dots, x_k) \in \widetilde{U}_{k-1}, \ B = B(x_1, \dots, x_l) \in \widetilde{U}_{l-1}.$$

Define

$$[A,B] = A\,\square\,B - B\,\square\,A, \tag{9}$$

where

$$A\,\square\,B(x_1,\dots,x_{k+l-1}) = A(B(x_1,\dots,x_l),x_{l+1},\dots,x_{l+k-1}) +$$
$$\sum_{s=1}^{k-1}\sum_{i_1<\dots<i_s}^{l+s-1} A(x_{i_1},\dots,x_{i_s},B(x_1,\dots,\widehat{x_{i_1}},\dots,\widehat{x_{i_s}},\dots,x_{l+s}),x_{l+s+1},\dots,x_{l+k-1}),$$

where $\sum_{i_1<\dots<i_s}^{l+s-1}$ means the summation by all ordered sets of s indices, each of them not greater than $l+s-1$. Note that this notion generalizes the formulas (1), (2).

To define the commutator of the spaces $\widetilde{U}_{l-1}$ and $\widetilde{U}_{-k}$ recall that every element $a_1 * \dots * a_k$ of the free Lie algebra can be uniquely represented as a sum of the monomials $a_1 \dots a_{i_k}$ in the free associative algebra generated by the space $\widetilde{U}_{-1}$ (for example, $a_1 * a_2 = a_1a_2 - a_2a_1$). Define the commutator of an operator $A = A(x_1,\dots,x_l) \in \widetilde{U}_{l-1}$ and an associative monomial $\beta = b_1 \dots b_k$ by

$$[A,\beta] = A(b_1,\dots,b_k,x_1,\dots,x_{l-k}),\ l \geq k,$$
$$[A,\beta] = A(b_1,\dots,b_l) * b_{l+1} * \dots * b_k,\ l < k.$$

For example, for a linear operator $A \in \widetilde{U}_0$ and $a_1, a_2 \in \widetilde{U}_{-1}$ we define

$$[A, a_1 * a_2] = A(a_1) * a_2 - a_1 * A(a_2).$$

One can show that $\widetilde{U}$ is indeed a graded Lie algebra. Now let us formulate the universal property of $\widetilde{U}$.

Definition 6. Let $U = \sum_{i=-\infty}^{\infty} U_i$ be a graded Lie algebra. We say that U is an *algebra of type A*, if the following conditions hold:

1. The subalgebra $U_- = \sum_{i=-\infty}^{-1} U_i$ is generated by U_{-1};
2. There are no ideals of U contained in the subalgebra $U_+ = \sum_{i=0}^{\infty} U_i$.

If U satisfies as well the following conditions:

1. The subalgebra U_+ is generated by U_1;
2. There are no ideals of U contained in the subspace $\sum_{i=-\infty}^{-2} U_i$,

then we say that U is an *algebra of type α*.

Let $U = \sum_{i=-\infty}^{\infty} U_i$ be a graded Lie algebra such that $U_{-1} = V$. Consider a graded mapping $F\colon U_+ \to \widetilde{U}_+$ defined as follows: to a vector $a \in U_k, k \geq 0$, corresponds a $(k+1)$-linear operator $F(a)$ on V given by

$$F(a)(x_1,\dots,x_{k+1}) = [\dots[a,x_1],\dots],x_{k+1}].$$

The main result of the paper [10] is the following theorem:

Theorem 2 ([10]). *Let V be a vector space of dimension n, and denote $\widetilde{U} = \widetilde{U}^{(n)}$. The following assertions hold*:
1) The algebra $\widetilde{U}$ is a Lie algebra of type A;
2) Let $U = \sum_{i=-\infty}^{\infty} U_i$ be a graded Lie algebra of type A such that $U_{-1} = V$. Then the mapping $F\colon U_+ \to \widetilde{U}_+$ is an injective homomorphism with the image U_+^*, the subspace $U_+^* + \widetilde{U}_-$ is a subalgebra of $\widetilde{U}$, and $U \cong (U_+^* + \widetilde{U}_-)/D$, for some ideal D of $U_+^* + \widetilde{U}_-$ contained in the subspace $\sum_{i=-\infty}^{-2} \widetilde{U}_i$.

Moreover, every graded Lie algebra of type A can be obtained in this way. That is, let $U_+ = \sum_{i=0}^{\infty} U_i$ be a graded subalgebra of $\widetilde{U}_+$ and be $D \subseteq \sum_{i=-\infty}^{-2} \widetilde{U}_i$ be a graded subspace such that

$$[U_+, D] \subseteq D,\ [U_k, \widetilde{U}_{-1}] \subseteq U_{k-1} \text{ for all } k > 0.$$

Then $U = \widetilde{U}_- + U$ is a subalgebra of $\widetilde{U}$, D is an ideal of U, and U/D is a graded Lie algebra of type A.

Later on, this construction was generalized for the super case in [12].

2.2 The general TKK construction

Now we are ready to define the general TKK construction.

Definition 7. Let A be an algebra (of finite dimension n) with a product P. The Lie algebra TKK(A) is defined as follows: $\mathrm{TKK}(A) = \mathcal{L}_0(A)/D$, where $\mathcal{L}_0(A)$ is the subalgebra of the universal Lie algebra $\widetilde{U}^{(n)}$ generated by the subspace $U_{-1} = A$ and the element $P \in U_1$, and D is the maximal ideal of $\mathcal{L}_0(A)$ that is contained in $\widetilde{U} = \sum_{i=-\infty}^{-2} \widetilde{U}_i$.

By above, $U = \mathrm{TKK}(A)$ is a graded Lie algebra of type α with $U_{-1} = A$. By construction, the space U_0 is generated by operators $[a, P] = L_a,\ a \in A$, that is, it is spanned by the operators of the form

$$L_a,\ [L_a, L_b],\ [[L_a, L_b], L_c], \dots,$$

and the space U_1 is spanned by the bilinear operators

$$P,\ [L_a, P],\ [L_a, [L_b, P]],\ [L_a, [L_b, [L_c, P]]] \dots$$

Conservative algebras behave well in relation to this construction. In particular, the relations (4), (5) show that the spaces U_0, U_1 are the same as in the case of Jordan algebras:

Lemma 2 ([11]). *Let A be a conservative algebra. Then in* TKK(A) *we have*

$$U_0 = \langle L_a, [L_a, L_b], a, b \in A\rangle,\ U_1 = \langle A, [L_a, A], a \in A\rangle.$$

The general TKK construction is functorial. Let $\omega\colon A \twoheadrightarrow B$ be a surjective homomorphsim of finite-dimensional algebras with kernel D. Consider the subspace $D^* = D_-^* + D_+^*$ of $\mathcal{L}_0(A)$ given as follows. The subspace D_-^* is an ideal of a free Lie algebra $\widetilde{U}_-$ generated by the subspace $D \subseteq \widetilde{U}_{-1}$ (hence, it is clear that $\widetilde{U}_-/D_-^*$ is a free Lie algebra generated by $\widetilde{U}_{-1}/D = B$). The subspace D_+^* is the subspace of all operators $A_k(x_1, \dots, x_k), k > 0$ such that $A_k(x_1, \dots, x_k) \in D$ for all $x_1, \dots, x_k \in A$.

Proposition 4. *The space D is an ideal of $\mathcal{L}_0(A)$, the quotient algebra of $\mathcal{L}_0(A)$ by D is isomorphic to $\mathcal{L}_0(B)$, and quotienting by maximal ideals having only graded components up to -2 we obtain the following commuting diagram*:

$$\begin{array}{ccc} A & \xrightarrow{\quad\omega\quad} & B \\ {\scriptstyle \mathrm{TKK}}\downarrow & & \downarrow{\scriptstyle \mathrm{TKK}} \\ \mathrm{TKK}(A) & \xrightarrow{\omega^*} & \mathrm{TKK}(B) \end{array}$$

In general, this construction produces infinite-dimensional Lie algebras (but with finite-dimensional graded pieces). We single out an important subclass of algebras whose associated Lie algebras are finite-dimensional.

Definition 8. An algebra A is called *an algebra of order* k, if $\mathrm{TKK}(A) = U = \bigoplus_{i=-k}^{k} U_i$. If A is of order k for some natural k, then we say that A is an *algebra of a finite order.*

The finiteness conditions can often be expressed in purely algebraic terms. For example, if $\mathrm{TKK}(A) = \sum_{i=-1}^{\infty} U_i$, then by definition the space $\sum_{i=-\infty}^{-2} \widetilde{U}_i$ must be an ideal of $\mathcal{L}_0(A)$. Therefore, for all $a, b \in A$ the commutator

$$[A, a * b] = A(a, b) - A(b, a)$$

must be zero, that is, A must be commutative. On the other hand, for a commutative algebra A all algebra structures $B = [L_{a_1}, [L_{a_2}, \ldots, [L_{a_k}, A]]]$ which span the space $U_1 \subseteq \mathcal{L}_0(A)$ are commutative, hence, they satisfy $[B, a * b] = 0$ for all $a, b \in A$. Therefore, the space $\sum_{i=-\infty}^{-2} \widetilde{U}_i$ is an ideal of $\mathcal{L}_0(A)$. In other words, the followings holds:

Lemma 3. *An algebra A is commutative if and only if* $\mathrm{TKK}(A) = \sum_{i=-1}^{\infty} U_i$.

It is interesting to note that one can define radicals for algebras of finite order:

Theorem 3 ([11]). *A solvable algebra of a finite order is nilpotent. Any algebra of finite order contains a maximal nilpotent ideal such that the quotient by it is a sum of simple algebras.*

Jordan algebras also have a natural characterization in terms of the TKK construction:

Proposition 5 ([11]). *An algebra A is Jordan if and only if* $\mathrm{TKK}(A) = \sum_{i=-1}^{1} U_i$. *Moreover, in this case the Kantor's construction coincides with the original TKK construction.*

In other words, Kantor's construction is a natural generalization on the original TKK construction and Jordan algebras are exactly algebras of order 1.

As we have seen in Proposition 4, if $\mathrm{TKK}(A)$ is a simple algebra, then A must be simple as well. In certain cases, the converse also holds:

Proposition 6 ([11]). *Let A be a simple conservative algebra with a left unit. Then* $\mathrm{TKK}(A)$ *is a simple Lie algebra.*

The condition of having a left unit is a rather strong one. However, it cannot be relaxed without imposing other restrictions. Take, for example, a simple Lie algebra $\mathfrak{g}$. Then the space $U_0 \subseteq \mathrm{TKK}(\mathfrak{g})$ consists of the operators of left multiplication L_a, $a \in \mathfrak{g}$, and U_1 is spanned by $\mathfrak{g}$. In this case $\mathrm{TKK}(\mathfrak{g})$ has an ideal $\sum_{i=-\infty}^{0} U_i$.

However, for algebras of finite order the simplicity is preserved by this construction:

Theorem 4 ([11]). *Let A be a simple algebra such that* $\mathrm{TKK}(A)$ *is finite-dimensional. Then* $\mathrm{TKK}(A)$ *is a simple Lie algebra.*

Therefore, the classification of simple conservative algebras of finite order can be done if one describes $\mathbb{Z}$-gradings on simple finite-dimensional Lie algebras. This is done in the following proposition:

Proposition 7. *Let $U = \sum_{i=-k}^{k} U_i$ be a finite-dimensional simple Lie algebra of type α over an algebraically closed field of characteristic 0. Then the algebra $U_+ = \sum_{i=0}^{k} U_i$ is parabolic with respect to a subset Σ of the set of simple roots, and the grading is defined as follows: U_k is the sum of root spaces U_α, where in the decomposition of α in the sum of simple roots the sum of the coefficients with indices in Σ is equal to k.*

Therefore, to classify simple conservative algebras of finite order it suffices to consider all possible simple root subsets in all irreducible root systems, construct the corresponding graded Lie algebras and consider the conservative algebras that may appear. Using the statements above, Kantor described simple finite-dimensional conservative algebras of order 2 with a left unit over an algebraically closed field of characteristic 0 (note that this generalizes the classification on simple finite-dimensional Jordan algebras, which are of order 1 in this terminology).

Example 1. Consider the algebra $A_{lk} - A_{kl}, l \geq k$. The underlying vector space of $A_{lk} - A_{kl}$ is the space of tetrads of the form (φ, ψ, a, b), where φ and ψ are $l \times l$-matrices, and a and b are $(l-k) \times l$-matrices. Defining

$$\begin{aligned} \varphi_1 &= (\varphi, 0, 0, 0), \\ \varphi_2 &= (0, \varphi, 0, 0), \\ a^1 &= (0, 0, a, 0), \\ a^2 &= (0, 0, 0, a), \end{aligned}$$

we can write the multiplication in this algebra as follows:

$$\begin{gathered} \varphi_1\psi_1 = (\varphi\psi + \psi\varphi)_1, \qquad \varphi_2\psi_2 = (\varphi\psi + \psi\varphi)_2, \\ \varphi_1\psi_2 = -(\varphi^T\psi)_2, \qquad \varphi_2\psi_1 = -(\varphi^T\psi)_1, \\ \varphi_i a^j = a^j\varphi_i = \delta_{ij}(a\varphi)^j, \qquad a^1b^2 = -(a^Tb)_2, \qquad a^2b^1 = -(a^Tb)_1, \\ a^1b^1 = 0, \qquad a^2b^2 = 0. \end{gathered}$$

One can show that the algebra $A_{lk} - A_{kl}$ is a simple conservative algebra of order 2 with left unit and its TKK algebra is a simple Lie algebra of type A. Moreover, any simple conservative algebra of order 2 with left unit such that its TKK algebra is a classical simple Lie algebra is a subalgebra of $A_{lk} - A_{kl}$ for suitable l, k.

This construction can be easily generalized to the case of an arbitrary algebraic system:

Definition 9 ([14]). Let ω be an algebraic system, that is, a set $\{\omega_\alpha\}_{\alpha\in A}$ of multilinear operations ω_α on on a vector space V. The algebra TKK(ω) is the algebra U^*/D, where U^* is the subalgebra of the universal Lie algebra $\widetilde{U}$ associated to the space V generated by $U_{-1} = V$ and all elements ω_α (considered as multilinear operations of the corresponding arity), and D is the maximal ideal of U^* contained in the space $\sum_{i=-\infty}^{-2} U_i^*$.

This construction is also functorial (in the sense above) and was later on used in the classification of various simple algebraic systems. For example, Kac and Cantarini classified simple linearly compact n-Lie superalgebras [6] and $N = 6$ 3-algebras [7] constructing a TKK functor from these varieties to graded Lie algebras and proving that it preserves simplicity [5].

2.3 Connection with structurable algebras

A unital algebra A with an involution τ is called *structurable* if it satisfies the following identity:

$$[V_{x,1}, V_{z,w}] = V_{V_{x,1}z,w} - V_{z,V_{x,1}w}, \text{ where } V_{x,y}(z) = (x\tau(y))z + (z\tau(y))x - (z\tau(x))y.$$

Note that these operators are natural generalizations of the quadratic multiplication operators $U_{x,y}$ on Jordan algebras [8]. Structurable algebras were defined by Allison in [1] as a natural generalization of unital Jordan algebras. Apart from Jordan algebras (with trivial involution) and unital associative algebras with involution, this class includes as well the tensor product of composition algebras and the algebras of hermitian form (a generalization of the algebra of symmetric bilinear form, where instead of the field $\mathbb{F}$ as "scalars" we take an associative algebra with involution, and instead of a vector space V we take an A-module).

There exists a version of the TKK construction for structurable algebras, that given a structurable algebra A returns a $\mathbb{Z}$-graded algebra $U = \sum_{i=-2}^{2} U_i$. This construction preserves simplicity in both ways, and every simple finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 can be realized as TKK(A) for A structurable and simple.

There is a connection between the classes of structurable and conservative algebras of order 2:

Proposition 8 ([2]). *Let A be a structurable algebra with an involution τ. Then the algebra $(A, *)$, where $x * y = xy + y(x - \tau(x))$, is a conservative algebra of order 2 with left unit 1. Moreover, for any conservative algebra B of order 2 with left unit there exists a unique structurable algebra A such that B can be obtained from A in the way above.*

This correspondence preserves simplicity in both ways. Moreover, this correspondence preserves the TKK construction, that is, for a structurable algebra A we have $\mathrm{TKK}(A) = \mathrm{TKK}(A, *)$. The classification of simple finite-dimensional structurable algebras over algebraically closed fields of characteristic $\neq 2, 3, 5$ (and the correspondence of resulting simple algebras with simple finite-dimensional conservative algebras of order 2) was obtained by Smirnov in [28].

3 Trace function and series of conservative algebras

A useful method of studying finite-dimensional nonassociative algebras is by means of introducing trace functions and bilinear forms that express some properties of the algebras (such as the Killing form for Lie algebras). In this section we consider a trace function and a related bilinear form defined for finite-dimensional conservative algebras by Kantor in [14]. Moreover, we consider a way to deform a given finite-dimensional conservative algebra with a left unit using a trace function.

3.1 Trace function and bilinear form

Definition 10. The *trace function* on a finite-dimensional conservative algebra A is the function Tr given by $\mathrm{Tr}(x) = \frac{1}{\dim A} \operatorname{tr} L_x$.

The normalization is chosen so that for a left unit e the equality $\mathrm{Tr}(e) = 1$ holds.

The following result is an immediate consequence of (4):

Theorem 5 ([14]). *Let A be a conservative algebra. Then for all $a, b, c \in A$ we have* $\mathrm{Tr}(a(bc) - (a * b)c) = 0$.

The expression $a(bc) - (a * b)c$ is sometimes called the *generalized associator* of a, b, and c.

Consider the bilinear form $(x, y) = \mathrm{Tr}(xy)$. This form is not always symmetric and nondegenerate. For example, for any element a in the Jacobi subspace of A by (6) we have

$$(a, x) = \frac{1}{\dim A} \operatorname{tr} L_{ax} = \frac{1}{\dim A} \operatorname{tr}[L_a, L_x] = 0,$$

so the Jacobi subspace J always lies in the radical of the form $(\cdot, \cdot)$. However, for the class of algebras of finite length we have the following result:

Theorem 6. *Let A be a simple conservative algebra of a finite order with a left unit. Then the bilinear form $(x, y) = \mathrm{Tr}(xy)$ on A is symmetric and nondegenerate.*

In particular, a simple conservative algebra of a finite order with a left unit has zero Jacobi subspace.

3.2 The algebras A_λ

Let A be a conservative algebra with multiplication $A(x, y) = xy$ and left unit e. Consider the family of algebras $A_\lambda(x, y) = x'y$, where $x' = x + \lambda \mathrm{Tr}(x)e$.

Note that for $\lambda \neq -1$ the algebra A_λ has a let unit $e_\lambda = (1 + \lambda)^{-1}e$. The following result can be obtained by a direct calculation:

Theorem 7 ([14]). *Let $A(x,y) = xy$ be a conservative algebra with left unit e. Then the algebras $A_\lambda, \lambda \neq -1$, are conservative with the associated multiplication given by*

$$A^*_\lambda(a,b) = A^*(a,b) + \lambda \operatorname{Tr}(a)b + \lambda \operatorname{Tr}(b)a - \frac{\lambda^2}{1+\lambda} \operatorname{Tr}(a) \operatorname{Tr}(b)e - \frac{\lambda}{1+\lambda} \operatorname{Tr}(ab)e.$$

Moreover, we have $A^*_\lambda(a,b)' = A^*(a',b')$, so all associated algebras A^*_λ are isomorphic.

The class of algebras A_λ cannot be extended by the same procedure. Particularly, $(A_\lambda)_\mu = A_{\lambda+\mu+\lambda\mu}$ for $\lambda, \mu \in \mathbb{F}$. In particular, $A = (A_\lambda)_{-\frac{\lambda}{1+\lambda}}$ for $\lambda \neq -1$.

In the case $\lambda = -1$ the algebra A_{-1} has no left unit and is not conservative. It turns out that the algebra A_{-1} is rigid:

Theorem 8 ([14]). *Let $A(x,y) = xy$ be a conservative algebra with left unit e. Then the algebra A_{-1} with the multiplication $A_{-1}(x,y) = xy - \operatorname{Tr}(x)y$ is rigid (quasi-conservative), and the multiplication in the associated algebra and the bilinear form φ are as follows*:

$$A^*_{-1}(a,b) = A^*(a,b) - \operatorname{Tr}(a)b - \operatorname{Tr}(A^*(a,b))e + 2\operatorname{Tr}(a)\operatorname{Tr}(b)e,$$
$$\varphi(a,b) = \operatorname{Tr}(A(a,b)) - \operatorname{Tr}(a)\operatorname{Tr}(b).$$

The algebra A_{-1} is not conservative, hence, is not isomorphic to other A_λ. The non-isomorphism of all other algebras A_λ can be in many cases proved with the help of the following result:

Theorem 9 ([14]). *If A has a unique left unit and the operator $R\colon x \mapsto xe$ has more than one nonzero eigenvalue (counted with multiplicity), then the algebras A_λ are pairwise nonisomorphic.*

Since in Jordan (more generally, noncommutative Jordan) algebras left units are necessarily right units as well, we get the following corollary:

Corollary 1. *If A is a Jordan (noncommutative Jordan) algebra, then all algebras A_λ are pairwise nonisomorphic.*

Corollary 2. *If A is a simple conservative algebra of second order with a left unit, then all algebras A_λ are pairwise nonisomorphic.*

Let us compare the algebras $\operatorname{TKK}(A_\lambda) = \sum_{i=-\infty}^{\infty} U_i^\lambda$ with the algebra $\operatorname{TKK}(A)$. All algebras A_λ are defined on the same space as $A = A_0$, so we may identify the spaces $U_{-1} \equiv U_{-1}^\lambda$. Moreover, the components U_0^λ also coincide for all $\lambda \neq -1$. Indeed, the operator L_a^λ of the left multiplication by a in A_λ is

$$L_a^\lambda = L_a + \lambda \operatorname{Tr}(a) \operatorname{id}.$$

As $L_e = \operatorname{id}$, the spaces U_0^λ coincide for all $\lambda \neq -1$. For $\lambda = -1$, the space U_0^{-1} is of dimension $\dim U_0 - 1$ and consists of linear operators of trace zero.

We can also identify the spaces U_1^λ with the help of the following lemma:

Lemma 4 ([14]). *The mapping* $F\colon a \mapsto a - \frac{\lambda}{\lambda+1} e$ *identifies the Jacobi spaces* J *and* J_λ *of the algebras* A *and* A_λ.

As the space U_1^λ is spanned by the operators $[L_a^\lambda, A_\lambda]$, by (7), we can identify U_1 with A/N_λ (we denote this mapping by $a \mapsto \tilde{a}$). Therefore, by the lemma above, the mapping F identifies the spaces U_1 and U_1^λ.

Moreover, by a direct computation we can get

$$L_{b'} F^{-1}(\tilde{a}) = F^{-1} L_{b'}(\tilde{a}),$$

where $F^{-1}\colon a \mapsto a' = a + \lambda \operatorname{Tr}(a)e$. As the algebra U_0 is generated by operators $L_a,\ a \in A$, this means that the representations of U_0 on the spaces U_1 and U_1^λ are isomorphic.

However, the dimensions of the spaces $U_k^\lambda,\ k \geq 2$, can be different from that of U_k. Consider a unital Jordan algebra A of dimension n. Then

$$[L_a^\lambda, A_\lambda] = [L_a, A](x,y) - \lambda \operatorname{Tr}(ax)y$$

and, since $U_2^\lambda = [U_1^\lambda, U_1^\lambda]$, the formula (9) implies that U_2^λ is spanned by trilinear operators of the form

$$[L_a, A](x,y) - \lambda \operatorname{Tr}(ax)y - [L_a, A](y,x) + \lambda \operatorname{Tr}(ay)x = \lambda(\operatorname{Tr}(ay)x - \operatorname{Tr}(ax)y).$$

If $\lambda \neq 0$, then this operator is zero if and only if x and y are in the radical of the bilinear form $(a,b) = \operatorname{Tr}(ab)$. Therefore, $\dim U_2 = \binom{n}{2} - \binom{k}{2}$, where k is the dimension of the radical of the form $(\cdot,\cdot)$. Generally, for $\lambda \neq 0$ the algebras $\operatorname{TKK}(A_\lambda)$ are infinite-dimensional.

4 A universal conservative algebra

In the theory of conservative algebras of great importance is the conservative algebra $\mathcal{U}(n)$ of bilinear mappings on an n-dimensional space. This algebra plays a role analogous to the one the algebra $\mathfrak{gl}_n$ plays in finite-dimensional Lie algebra theory, that is, for any finite-dimensional conservative algebra A there exists a homomorphism (with a known kernel) $A \to \mathcal{U}(n)$ for certain $n \leq \dim A$. In this section we consider the construction of the algebra $\mathcal{U}(n)$ introduced in [16], recall basic properties of this algebra and state the universality theorem for $\mathcal{U}(n)$.

4.1 The algebra $\mathcal{U}(V)$

Let V be a vector space over $\mathbb{F}$. The space of the algebra $\mathcal{U}(V)$ is the space of all bilinear operators $V \times V \to V$ on V. To define the multiplication $\triangle$ in $\mathcal{U}(V)$ we fix a nonzero vector $u \in V$. Then for $A, B \in \mathcal{U}(V)$ we set

$$(A \triangle_u B)(x,y) = [L_u^A, B](x,y) = A(u, B(x,y)) - B(A(u,x),y) - B(x, A(u,y)),$$

where $L_u^A\colon x \mapsto A(u,x)$ is the left multiplication with respect to A. Consider the natural action of the group $\operatorname{GL}(V)$ on $\mathcal{U}(V)$:

$$\varphi(A)(x,y) = \varphi(A(\varphi^{-1}(x), \varphi^{-1}(y))).$$

A direct computation shows that the mapping $A \mapsto \varphi(A)$ is an isomorphism between $(\mathcal{U}(V), \triangle_a)$ and $(\mathcal{U}(V), \triangle_{\varphi(a)})$. Therefore, different nonzero vectors a give rise to isomorphic algebras, which we denote by $\mathcal{U}(V)$. In particular (see [22]), we have an injective homomorphism of a $\mathrm{GL}(V, a) = \{\varphi \in \mathrm{GL}(V) : \varphi(a) = a\}$ to $\mathrm{Aut}(\mathcal{U}(V), \triangle_a)$. If $V = V_n$ is a finite-dimensional space of dimension n, then we denote $\mathcal{U}(V)$ by $\mathcal{U}(n)$.

The square of the multiplication A in the algebra $\mathcal{U}(V)$ is called the *Kantor square* of A. The Kantor square gives us a map K from a variety V of algebras to some class $K(V)$. The Kantor squares of multiplications satisfying certain conditions (such as associativity, Leibniz identity and others) were studied in [17].

Let $b \in V$, and A, B, C be bilinear operators on V. The following relation in $(\mathcal{U}(V), \triangle_a)$ can be obtained by a direct computation:

$$[L_A, \triangle_b](B, C) = B \triangle_{A(a,b)} C. \tag{10}$$

For $n > 1$ the algebra $\mathcal{U}(n)$ does not belong to a well-known class of algebras (such as associative, Lie, Jordan, Leibniz algebras). However, with the help of the relation (10) one easily proves the following result:

Theorem 10. *Let V be a space, and let $a \in V$. The algebra $(\mathcal{U}(V), \triangle_a)$ is conservative, and the associated multiplication can be given by*

$$A \bigtriangledown_a^1 B(x, y) = -B(a, A(x, y)), \tag{11}$$

or

$$A \bigtriangledown_a^2 B(x, y) = \frac{1}{3}(A^* \triangle_a B + \widetilde{B} \triangle_a A), \tag{12}$$

where $A^*(x, y) = A(x, y) + A(y, x)$ and $\widetilde{B}(x, y) = 2B(y, x) - B(x, y)$.

By (7) and (10), the Jacobi subspace J of $(\mathcal{U}(V), \triangle_a)$ consists precisely of those $A(x, y) \in \mathcal{U}(V)$ for which $A(a, a) = 0$, so we may identify the spaces $\mathcal{U}(V)/J$ and V by the mapping $A \mapsto A(a, a)$. In particular, for the algebra $\mathcal{U}(n)$ we have $\mathrm{codim}(J) = n$.

If the mapping $\mathcal{U}(V) \to \mathfrak{gl}(V)$ given by $A \mapsto L_a^A$, is surjective (which is always the case if V is of countable dimension) then any operator A such that $L_a^A = -\mathrm{id}$ is a left unity of $(\mathcal{U}(V), \triangle_a)$.

Properties of the algebra $\mathcal{U}(2)$ were studied in various articles. For example, in the paper [20] the authors described the derivations and subalgebras of codimension one of $\mathcal{U}(2)$ and its simple terminal subalgebras W_2, S_2 (see in the next section), and in the article [22] the one-sided ideals, automorphisms and idempotents of $\mathcal{U}(2)$ were described. Note that by definition the classification of idempotents of $(\mathcal{U}(2), \triangle_u)$ corresponds to the classification of 2-dimensional algebras with u as a left quasiunit.

4.2 Universality of the algebra $\mathcal{U}(n)$

Let M be a conservative algebra on a space V with the Jacobi subspace J. Consider the space W, which we define as $W = V/J$ if M has a left quasiunity, and $W = V/J \oplus E$ in the opposite case, where E is a one-dimensional space spanned by a vector ϵ.

Assume that M possesses a quasiunity. Define the *adjoint mapping* $\mathrm{ad}\colon M \to \mathcal{U}(W)$ as follows:

$$\mathrm{ad}(a)(\alpha,\beta) = (\beta * a) * \alpha + \beta * (\alpha a) - (\beta * \alpha) * a.$$

If M does not have a quasiunity, we define the adjoint mapping $\mathrm{ad}\colon M \to \mathcal{U}(W)$ by the formula above and the following equations:

$$\mathrm{ad}(a)(\alpha,\epsilon) = a * \alpha + \alpha a - \alpha * a,$$
$$\mathrm{ad}(a)(\epsilon,\beta) = \beta * a,\ \ \mathrm{ad}(a)(\epsilon,\epsilon) = a.$$

Using (8), one can check that if M has a quasiunity e, then the uniquely defined element $\epsilon = e \mod J$ satisfies the relations above. One can verify that the adjoint mapping is well-defined and does not depend on the choice of an associated multiplication.

Theorem 11 ([16]). *Let M be a conservative algebra on a vector space V with the Jacobi subspace J. Let either $W = V/J$ or $W = V/J \oplus \langle\epsilon\rangle$ as above. The adjoint mapping* $\mathrm{ad}\colon M \to (\mathcal{U}(W), \triangle_{-\epsilon})$ *is a homomorphism whose kernel is the maximal Jacobi ideal. In particular, if V is finite-dimensional and J is of codimension n, then we have a homomorphism* $\mathrm{ad}\colon M \to \mathcal{U}(k)$*, where $k = n$ if M has a quasiunity and $k = n+1$ otherwise.*

In particular, every unital Jordan algebra of dimension n is a subalgebra of $\mathcal{U}(n)$. The analog of this theorem for superalgebras was proved in [21].

The above theorem can be rewritten using the language of the category theory. Let us call the subspace of the elements that satisfy the condition

$$a(xy) = (ax)y + x(ay) - \mu(a)xy,$$

where μ is a scalar-valued function, the *extended Jacobi subspace* and denote it by $\widehat{J}$. It is obvious that $\widehat{J} = J \oplus \mathbb{F}e$ if M is an algebra with a left quasiunity e, and $\widehat{J} = J$ otherwise.

Consider the category S_n of the conservative algebras that do not contain ideals in the Jacobi subspace J and satisfy the condition $\operatorname{codim} \widehat{J} = n-1$. The morphisms in this category are embeddings.

Theorem 12 ([16]). *The algebra $\mathcal{U}(n)$ is the final object in the category S_n.*

Example 2. We check that the adjoint homomorphism applied to $\mathcal{U}(V)$ itself is the identity mapping. We have already seen that $\mathcal{U}(V)$ has a left unity. It is also easy to check that its maximal Jacobi ideal is zero (see, for example, [21]). Therefore, in our case the space W is $\mathcal{U}(V)/J$ that we identify with V by the mapping $A \mapsto A(a,a)$. Now, let $A, B, C \in \mathcal{U}(V)$, and let $B(a,a) = u$, $C(a,a) = v$. We want to show that $\mathrm{ad}(A)(B,C)(a,a) = A(u,v)$, which would establish the required isomorphism. Recall that the adjoint mapping does not depend on the choice of the associated multiplication. Therefore, we choose as the associated multiplication the product $\bigtriangledown_1$ given by (11). Now, a direct computation shows the desired equality:

$$\begin{aligned}\mathrm{ad}(A)(B,C)(a,a) &= \big((C \bigtriangledown_a A) \bigtriangledown_a B + C \bigtriangledown_a (B \bigtriangleup_a A) - (C \bigtriangledown_a B) \bigtriangledown_a A\big)(a,a)\\ &= \Big(B(a, A(a, C(a,a))) - \big(B(a, A(a, C(a,a))) - A(B(a,a), C(a,a))\\ &\quad - A(a, B(a, C(a,a)))\big) - A\big(a, B(a, C(a,a))\big)\Big)\\ &= A(B(a,a), C(a,a)) = A(u,v).\end{aligned}$$

5 Terminal algebras

Since the class of conservative algebras is very large, it is hard to study from the point of view of structure theory and for now even the basic general questions about it remain unanswered (see next section), so it is a good idea to study its subclasses which are sufficiently wide but have nice properties that make them easier to study (for example, being an algebraic variety). One such class is conservative algebras of finite order, considered in Section 2.2. In this section we consider another class, that of terminal algebras [13], which is a subvariety of conservative algebras.

Let B and C be bilinear operations on a vector space V. Define their commutator $[B, C]$ as a trilinear operation given by

$$\begin{aligned}[B,C](x,y,z) &= B(C(x,y),z) + B(x, C(y,z)) + B(y, C(x,z))\\ &\quad - C(B(x,y),z) - C(x, B(y,z)) - C(y, B(x,z)) \qquad (13)\end{aligned}$$

(note that this is a particular case of the formulas (9)).

Definition 11. An algebra (V, P), where V is a vector space and P is a multiplication, is called a *terminal algebra* if for all $a \in V$ we have

$$[[[P,a],P],P] = 0.$$

Using the formulas (1), (2), (13), we can expand this relation, obtaining an identity of degree 4. Therefore, the class of terminal algebras is a variety.

The following characterization of terminal algebras provides a description of this class as a subclass of the class of conservative algebras.

Theorem 13 ([13]). *Let A be an algebra with a multiplication $P(x,y) = xy$. The following statements are equivalent*:
1) A is terminal;
2) A is conservative and the multiplication in the associated superalgebra P^* can be defined by

$$P^*(x,y) = \frac{2}{3}P(x,y) + \frac{1}{3}P(y,x);$$

3) $\mathrm{TKK}(A) = \sum_{i=-\infty}^{1} U_i$.

The last point of the theorem suggests a way of constructing terminal algebras. Take a $\mathbb{Z}$-graded Lie algebra $U = \sum_{i=-\infty}^{1} U_i$ and let $a \in U_1$. Then the space U_{-1}

with the product $A(x,y) = [[x,a],y]$ is a terminal algebra, and in fact any terminal algebra can be obtained in this way (particularly, $U = \mathrm{TKK}(A)$).

Proposition 3 and Proposition 5 imply that the class of commutative terminal algebras coincides with Jordan algebras. Aside from Jordan algebras, the class of terminal algebras includes all Lie algebras, all (left) Leibniz algebras and some other types of algebras. We give examples of terminal non-Jordan algebras, which are subalgebras of the universal conservative algebra $\mathcal{U}(n)$:

- The algebra W_n is the subalgebra of $\mathcal{U}(n)$ whose space consists of all commutative multiplications on V_n. The algebra W_n is terminal: for commutative A, B the associated multiplication (12) reduces to

$$A \bigtriangledown_2 B = \frac{1}{3}\left((A + A^T) \bigtriangleup B + (2B - B^T) \bigtriangleup A\right) = \frac{1}{3}(2A \bigtriangleup B + B \bigtriangleup A).$$

  $(W_n, \bigtriangleup_a)$ has left units: these are the elements $A \in W_n$ such that $A(a,x) = -x$ for all $x \in V_n$. The Jacobi space and the left units for W_n are the same as for $\mathcal{U}(n)$ itself.

- The algebra S_n is the subalgebra of $\mathcal{U}(n)$ whose space consists of commutative multiplications A on V_n such that $\mathrm{tr}(L_x^A) = 0$ for all $x \in V_n$.

- The algebra H_n is the subalgebra of $\mathcal{U}(2n)$ whose space consists of commutative multiplications A on V_{2n} "preserving" a nondegenerate bilinear skew-symmetric form $(\cdot,\cdot)$, that is,

$$(A(x,y), z) = (x, A(y,z))$$

  for all $x, y, z \in V_{2n}$.

All assertions and calculations made for W_n hold also for S_n and H_n, except that the latter two algebras have no left units (but have left quasiunits).

Kantor described simple finite-dimensional terminal algebras with left quasiunits:

Theorem 14 ([13]). *Let A be a simple finite-dimensional terminal algebra with a left quasiunit over an algebraically closed field of characteristic 0. Then A is either Jordan or isomorphic to one of the algebras W_n, S_n, H_n. In the last case, the Lie algebras corresponding to these algebras are simple infinite-dimensional Lie algebras with the same notation.*

The algebraic and geometric classification of nilpotent terminal algebras in the dimension up to 4 was obtained in the paper [18], and 5-dimensional one generated nilpotent terminal algebras were classified in [19].

In the paper [15] Kantor considered the generalization of terminal algebras for an arbitrary arity.

Definition 12. An l-linear operation A is said to be *terminal* if

$$\mathrm{TKK}(A) = \sum_{i=-\infty}^{l-1} U_i,$$

where $\mathrm{TKK}(A)$ is defined as in Definition 9.

In particular, he considered terminal trilinear operations. He found an identity equivalent to the definition of a terminal trilinear operation and described the simple finite-dimensional terminal triple systems.

6 Open problems

In this section we formulate questions related to conservative algebras that are still open.

As we have seen, conservative algebras are usually studied with the help of the TKK construction and there is not many results regarding the structure theory of conservative algebras. Many problems from the list below are consequences of the fact that the class of conservative algebras does not form a variety.

- It is not clear if subalgebras and homomorphic images of conservative algebras are conservative. As direct products of conservative algebras are conservative, the affirmative answer to this question would imply (by Birkhoff's HSP theorem [3]) that the class of conservative algebras forms a variety, which is not true (see below). However, there is no explicit counterexample by now. By definition of the general TKK construction and Proposition 4, the class of conservative algebras of finite order is closed with respect to finite direct products and homomorphic images, but it is not clear if it is closed with respect to taking subalgebras.

- A related open problem is to study identities for certain subclasses of conservative algebras. In particular, it is interesting if one can express the finiteness conditions $\mathrm{TKK}(A) = \sum_{i=-k}^{k} U_i$ for all k as identities with respect to the original multiplication (this would imply that a subalgebra of a conservative algebra of finite order also has finite order). Also, one can study the identities of the algebras $\mathcal{U}(n)$, particularly, find the minimal degree of a nonassociative identity for $\mathcal{U}(n)$ and study if the variety generated by all $\mathcal{U}(n), n \geq 1$, coincides with the class of all nonassociative algebras.

- At the present moment, no attempts have been made to study the structure theory of conservative algebras in general, for example, to define a notion of a radical (in the Kurosh-Amitsur sense) for the class of conservative algebras and study the semisimple algebras with respect to this radical. The radical was defined for algebras of finite order by Kantor in [11], see Section 2.2. It seems an interesting task to construct a radical for terminal algebras and other subclasses of conservative algebras.

- By now, there is no defined notion of a conservative representation. Considering a conservative algebra A as an algebraic system with two bilinear multiplications P and P^*, we can define a conservative representation by extending the operators of left and right multiplications with respect to P and P^* to linear operators on a vector space V such that the split null extension $A \oplus V$ is a conservative algebra (for this general approach see [8]). It would be interesting to obtain the description of irreducible conservative representations of certain classes of simple finite-dimensional conservative algebras and see if classical results such as complete reducibility of finite-dimensional representations hold for these algebras. On the other hand, Theorem 11 suggests defining a representation of a conservative algebra A as a homomorphism $A \to \mathcal{U}(V)$ for some vector space V. One can obtain upper bounds on n such that $J \hookrightarrow \mathcal{U}(n)$ for a simple finite-dimensional Jordan algebra J.

- It would also be interesting to study the algebraic properties of universal conservative algebras $\mathcal{U}(n)$ for $n > 2$, continuing the work of the papers [20], [22] (that is, to find one-sided ideals, automorphisms, maximal subalgebras, etc. of these algebras)

- We can also study conservative algebras from a geometric point of view, that is, to study their degenerations and deformations. Theorem 8 implies that the set Cons_n of all conservative algebras, considered as a subset of an algebraic variety $V_n^* \otimes V_n^* \otimes V_n$ (see, for example, [18] for details) is not closed (therefore, conservative algebras do not form a variety), so it is interesting to describe its closure. Moreover, it is not clear if a degeneration of a conservative (or, more generally, rigid) is always conservative (rigid).

- One can also study finite-dimensional conservative superalgebras. For example, it would be interesting to obtain a classification of simple finite-dimensional conservative superalgebras of order two and construct the super version of the correspondence between structurable algebras and conservative algebras of order 2 with a left unit given in Proposition 8. Note that the classification of simple finite-dimensional structurable algebras over an algebraically closed field of characteristic 0 was obtained in [27].

References

[1] B. Allison: A class of nonassociative algebras with involution containing the class of Jordan algebras. *Mathematische Annalen* 237 (2) (1978) 133–156.

[2] B. Allison, W. Hein: Isotopes of some nonassociative algebras with involution. *Journal of Algebra* 69 (1981) 120–142.

[3] G. Birkhoff: On the structure of abstract algebras. *Mathematical Proceedings of the Cambridge Philosophical Society* 31 (4) (1935) 433–454.

[4] A. Calderón Martín, A. Fernández Ouaridi, I. Kaygorodov: The classification of 2-dimensional rigid algebras. *Linear and Multilinear Algebra* 68 (4) (2020) 828–844.

[5] N. Cantarini, V. Kac: Classification of linearly compact simple rigid superalgebras. *International Mathematics Research Notices* 17 (2010) 3341–3393.

[6] N. Cantarini, V. Kac: Classification of simple linearly compact n-Lie superalgebras. *Communications in Mathematical Physics* 298 (3) (2010) 833–853.

[7] N. Cantarini, V. Kac: Classification of simple linearly compact $N = 6$ 3-algebras. *Transformation Groups* 16 (3) (2011) 649–671.

[8] N. Jacobson: *Structure and Representations of Jordan Algebras.* American Mathematical Society, Providence, R.I (1969).

[9] V. Kac: Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras. *Communications in Algebra* 13 (5) (1977) 1375–1400.

[10] I. Kantor: Graded Lie algebras (Russian). *Trudy Seminara po Vektornomu i Tenzornomu Analizu s ikh Prilozheniyami k Geometrii, Mekhanike i Fizike* 15 (1970) 227–266.

[11] I. Kantor: Certain generalizations of Jordan algebras (Russian). *Trudy Seminara po Vektornomu i Tenzornomu Analizu s ikh Prilozheniyami k Geometrii, Mekhanike i Fizike* 16 (1972) 407–499.

[12] I. Kantor: A universal graded Lie superalgebra (Russian). *Trudy Seminara po Vektornomu i Tenzornomu Analizu s ikh Prilozheniyami k Geometrii, Mekhanike i Fizike* 20 (1981) 162–175.

[13] I. Kantor: On an extension of a class of Jordan algebras. *Algebra and Logic* 28 (2) (1989) 117–121.

[14] I. Kantor: Some problems in $\mathfrak{L}$-functor theory (Russian). *Algebra and Logic* 16 (1989) 54–75.

[15] I. Kantor: Terminal trilinear operations. *Algebra and Logic* 28 (1) (1989) 25–40.

[16] I. Kantor: An universal conservative algebra. *Siberian Mathematical Journal* 31 (3) (1990) 388–395.

[17] I. Kaygorodov: On the Kantor product. *Journal of Algebra and Its Applications* 16 (5) (2017).

[18] I. Kaygorodov, M. Khrypchenko, Yu. Popov: The algebraic and geometric classification of nilpotent terminal algebras. *arXiv:1909.00358* (2019).

[19] I. Kaygorodov, A. Khudoyberdiyev, A. Sattarov: One generated nilpotent terminal algebras. *Communications in Algebra* 48 (10) (2020) 4355–4390.

[20] I. Kaygorodov, A. Lopatin, Yu. Popov: Conservative algebras of 2-dimensional algebras. *Linear Algebra and its Applications* 486 (2015) 255–274.

[21] I. Kaygorodov, Yu. Popov, A. Pozhidaev: The universal conservative superalgebra. *Communications in Algebra* 47 (10) (2019) 4064–4074.

[22] I. Kaygorodov, Yu. Volkov: Conservative algebras of 2-dimensional algebras, II. *Communications in Algebra* 45 (8) (2017) 3413–3421.

[23] O. Loos: *Jordan Pairs, Lecture Notes in Mathematics*. Springer Verlag, Berlin (1975).

[24] C. Martínez, E. Zelmanov: Representation theory for Jordan superalgebras I. *Transactions of the American Mathematical Society* 362 (2) (2010) 815–846.

[25] K. Meyberg: *Lectures on Algebras and Triple Systems*. Lecture notes, University of Virginia, Charlottesville (1972).

[26] Yu. Popov: Representations of noncommutative Jordan superalgebras I. *Journal of Algebra* 544 (2020) 329–390.

[27] A. Pozhidaev, I. Shestakov: Structurable superalgebras of Cartan type. *Journal of Algebra* 323 (2010) 3230–3251.

[28] O. Smirnov: Simple and semisimple structurable algebras. *Algebra and Logic* 29 (5) (1990) 377–379.

Received: 19 September, 2019
Accepted for publication: 14 October, 2019
Communicated by: Ivan Kaygorodov