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Deformations of Metrics and Biharmonic Maps

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Abstract. We construct biharmonic non-harmonic maps between Riemannian manifolds (M,g) and (N,h) by first making the ansatz that $\varphi\colon (M,g) \to (N,h)$ be a harmonic map and then deforming the metric on N by

$$\tilde{h}_\alpha = \alpha h + (1-\alpha)\mathrm{d}f \otimes \mathrm{d}f$$

to render φ biharmonic, where f is a smooth function with gradient of constant norm on (N,h) and $\alpha \in (0,1)$. We construct new examples of biharmonic non-harmonic maps, and we characterize the biharmonicity of some curves on Riemannian manifolds.

1 Introduction

Let (M,g) and (N,h) be two Riemannian manifolds. The energy functional of a map $\varphi \in C^\infty(M,N)$ is defined by

$$E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2 v^g\,, \tag{1}$$

where $|\mathrm{d}\varphi|$ is the Hilbert-Schmidt norm of the differential $\mathrm{d}\varphi$ and v^g is the volume element on (M,g). A map $\varphi \in C^\infty(M,N)$ is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (1)

$$\tau(\varphi) = \operatorname{trace}\nabla\mathrm{d}\varphi = \nabla^\varphi_{e_i}\mathrm{d}\varphi(e_i) - \mathrm{d}\varphi(\nabla^M_{e_i}e_i) = 0\,, \tag{2}$$

where $\{e_i\}_{i=1}^m$ is an orthonormal frame on (M,g), $m = \dim M$, ∇^M is the Levi--Civita connection of (M,g), and ∇^φ denote the pull-back connection on $\varphi^{-1}TN$.

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Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important role in many branches of mathematics and physics where they may serve as a model for liquid crystal (see [9]). One can refer to [6], [7], [8] for background on harmonic maps. A natural generalization of harmonic maps is given by integrating the square of the norm of the tension field. More precisely, the bi-energy functional of a map $\varphi \in C^\infty(M,N)$ is defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v^g\,. \tag{3}$$

A map $\varphi \in C^\infty(M,N)$ is called biharmonic if it is a critical point of the bi-energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (3)

$$\begin{aligned}\tau_2(\varphi) &= -\operatorname{trace} R^N(\tau(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi - \operatorname{trace}(\nabla^\varphi)^2\tau(\varphi)\\ &= -R^N(\tau(\varphi), \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) - \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi) + \nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi) = 0\,,\end{aligned} \tag{4}$$

where R^N is the curvature tensor of (N,h) (see [5], [12]). Clearly, harmonic maps are biharmonic. G.Y. Jiang [12] proved that if M is compact without boundary and the sectional curvature of (N,h) is negative, then any biharmonic map $\varphi \in C^\infty(M,N)$ is harmonic. So it is interesting to construct biharmonic non-harmonic maps. We refer the reader to [2], [5], [10], [11] for other examples and different approaches to their construction.

In this paper, we deform the codomain metric by $\tilde{h}_\alpha = \alpha h + (1-\alpha)\mathrm{d}f\otimes\mathrm{d}f$, where $\alpha \in (0,1)$ and $f \in C^\infty(N)$, in order to render a map biharmonic non-harmonic with respect to the new metric, we give a necessary and sufficient condition on f and α such that $\varphi\colon (M,g) \to (N,\tilde{h}_\alpha)$ is biharmonic non-harmonic. So by suitable choices of f, we are able to give new examples of biharmonic non-harmonic maps.

2 Deformations of Metrics

Let M be a Riemannian manifold equipped with Riemannian metric g, and f a smooth function on M. We define on M a Riemannian metric, denoted $\tilde{g}_\alpha$, by

$$\tilde{g}_\alpha = \alpha g + (1-\alpha)\mathrm{d}f \otimes \mathrm{d}f\,,$$

for some constant $\alpha \in (0,1)$. In the seminal work [4], we obtain the following results.

Theorem 1. *Let (M,g) be a Riemannian manifold and $\widetilde{\nabla}$ denote the Levi-Civita connection of $(M,\tilde{g}_\alpha)$. Then*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{(1-\alpha)\operatorname{Hess}_f(X,Y)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f\,,$$

where ∇ is the Levi-Civita connection of (M,g), Hess_f (resp. $\operatorname{grad} f$) is the Hessian (resp. the gradient vector) of f with respect to g, and

$$\|\operatorname{grad} f\|^2 = g(\operatorname{grad} f, \operatorname{grad} f)\,.$$

Proof. Let $X, Y, Z \in \Gamma(TM)$. From the Koszul formula (see [13]), we have

$$\begin{aligned} 2\tilde{g}_\alpha(\widetilde{\nabla}_X Y, Z) = 2\alpha g(\nabla_X Y, Z) + (1-\alpha)\Big\{ & X(Y(f)Z(f)) + Y(Z(f)X(f)) \\ & - Z(X(f)Y(f)) + Z(f)[X,Y](f) + Y(f)[Z,X](f) \\ & - X(f)[Y,Z](f)\Big\}. \end{aligned} \tag{5}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame on (M, g) at $x \in M$ (see [3]), where $m = \dim M$. By (5) we obtain

$$\begin{aligned} 2\tilde{g}_\alpha(\widetilde{\nabla}_X Y, e_i) = 2\alpha g(\nabla_X Y, e_i) + (1-\alpha)\Big\{ & X(Y(f)g(e_i, \operatorname{grad} f)) \\ & + Y(X(f)g(e_i, \operatorname{grad} f)) - e_i(g(X, \operatorname{grad} f)g(Y, \operatorname{grad} f)) \\ & + e_i(f)[X,Y](f) + Y(f)(\nabla_{e_i} X)(f) + X(f)(\nabla_{e_i} Y)(f)\Big\}, \end{aligned} \tag{6}$$

from equation (6), and the definition of Hessian (see [13]), we get

$$\begin{aligned} \tilde{g}_\alpha(\widetilde{\nabla}_X Y, e_i) = {} & \alpha g(\nabla_X Y, e_i) + (1-\alpha) g(\nabla_X Y, \operatorname{grad} f) g(e_i, \operatorname{grad} f) \\ & + (1-\alpha) \operatorname{Hess}_f(X, Y) g(e_i, \operatorname{grad} f), \end{aligned} \tag{7}$$

from equation (7), we obtain

$$\begin{aligned} \tilde{g}_\alpha(\widetilde{\nabla}_X Y, Z) = {} & \alpha g(\nabla_X Y, Z) + (1-\alpha) g(\nabla_X Y, \operatorname{grad} f) g(Z, \operatorname{grad} f) \\ & + (1-\alpha) \operatorname{Hess}_f(X, Y) g(Z, \operatorname{grad} f), \end{aligned} \tag{8}$$

by the definition of the Riemannian metric $\tilde{g}_\alpha$, and (8) we find that

$$\tilde{g}_\alpha(\widetilde{\nabla}_X Y, Z) = \tilde{g}_\alpha(\nabla_X Y, Z) + (1-\alpha) \operatorname{Hess}_f(X, Y) Z(f). \tag{9}$$

Hence Theorem 1 follows from (9), with the following

$$Z(f) = \frac{1}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \tilde{g}_\alpha(Z, \operatorname{grad} f). \qquad \square$$

Now consider the curvature tensor $\widetilde{R}$ of $(M, \tilde{g}_\alpha)$, writing R for the curvature tensor of (M, g). We have the following result:

Theorem 2. *For all* $X, Y, Z \in \Gamma(TM)$, *we have*

$$\begin{aligned} \widetilde{R}(X,Y)Z = R(X,Y)Z & + \frac{(1-\alpha) g(R(X,Y)\operatorname{grad} f, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \\ & - \frac{(1-\alpha)^2 \operatorname{Hess}_f(X, \operatorname{grad} f) \operatorname{Hess}_f(Y, Z)}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2} \operatorname{grad} f \\ & + \frac{(1-\alpha)^2 \operatorname{Hess}_f(Y, \operatorname{grad} f) \operatorname{Hess}_f(X, Z)}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2} \operatorname{grad} f \\ & + \frac{(1-\alpha) \operatorname{Hess}_f(Y, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \nabla_X \operatorname{grad} f \\ & - \frac{(1-\alpha) \operatorname{Hess}_f(X, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \nabla_Y \operatorname{grad} f. \end{aligned}$$

Proof. By the definition of the curvature tensor $\widetilde{R}$,

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z\,,$$

and Theorem 1 we obtain

$$\begin{aligned}
\widetilde{R}(X,Y)Z = {} & \widetilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& - \widetilde{\nabla}_Y \left(\nabla_X Z + \frac{(1-\alpha)\operatorname{Hess}_f(X,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& - \left(\nabla_{[X,Y]} Z + \frac{(1-\alpha)\operatorname{Hess}_f([X,Y],Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right),
\end{aligned} \tag{10}$$

the first term of (10) is given by

$$\begin{aligned}
& \widetilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& \quad = \nabla_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& \quad + \frac{(1-\alpha)\operatorname{Hess}_f\left(X, \nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f\,,
\end{aligned} \tag{11}$$

by equation (11), and the definition of Hessian, we obtain

$$\begin{aligned}
& \widetilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& = \nabla_X \nabla_Y Z + \frac{(1-\alpha) g(\nabla_X \nabla_Y \operatorname{grad} f, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \\
& \quad + \frac{(1-\alpha)\operatorname{Hess}_f(Y, \nabla_X Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \\
& \quad - \frac{(1-\alpha)^2 \operatorname{Hess}_f(X, \operatorname{grad} f)\operatorname{Hess}_f(Y,Z)}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2} \operatorname{grad} f \\
& \quad + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \nabla_X \operatorname{grad} f \\
& \quad + \frac{(1-\alpha)\operatorname{Hess}_f(X, \nabla_Y Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f\,.
\end{aligned} \tag{12}$$

Using the similar method, the second term of (10) is given by

$$\begin{aligned}
-\widetilde{\nabla}_Y\Big(\nabla_X Z + \frac{(1-\alpha)\operatorname{Hess}_f(X,Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f\Big)
&= -\nabla_Y\nabla_X Z - \frac{(1-\alpha)g(\nabla_Y\nabla_X \operatorname{grad} f, Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f\\
&\quad - \frac{(1-\alpha)\operatorname{Hess}_f(X,\nabla_Y Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f\\
&\quad + \frac{(1-\alpha)^2\operatorname{Hess}_f(Y,\operatorname{grad} f)\operatorname{Hess}_f(X,Z)}{(\alpha+(1-\alpha)\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f\\
&\quad - \frac{(1-\alpha)\operatorname{Hess}_f(X,Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\nabla_Y \operatorname{grad} f\\
&\quad - \frac{(1-\alpha)\operatorname{Hess}_f(Y,\nabla_X Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f. \qquad (13)
\end{aligned}$$

Theorem 2 follows from equations (10), (12) and (13). □

3 The biharmonicity of $\varphi\colon (M,g)\to(N,\tilde{h}_\alpha)$

We now consider the effects of a deformation of the codomain metric, as regards harmonic and biharmonic mappings.

Theorem 3. *Let $\varphi\colon (M,g)\to(N,h)$ be a harmonic map between two Riemannian manifolds and let the Riemannian metric $\tilde{h}_\alpha = \alpha h + (1-\alpha)\mathrm{d}f\otimes\mathrm{d}f$, where $\alpha\in(0,1)$ and $f\in C^\infty(N)$. We suppose that $\|\operatorname{grad} f\| = 1$. If the function $\Delta(f\circ\varphi)$ is a non--null constant on M, then the map $\varphi\colon (M,g)\to(N,\tilde{h}_\alpha)$ is proper biharmonic if and only if the gradient vector of f is Jacobi field along φ, i.e. $(\operatorname{grad} f)\circ\varphi\in\ker J_\varphi$ where J_φ is a Jacobi operator corresponding to φ.*

Proof. Let $\{e_i\}_{i=1}^m$ be a normal orthonormal frame on (M,g) at x, where $m=\dim M$. Then the map $\varphi\colon (M,g)\to(N,\tilde{h}_\alpha)$ is biharmonic if and only if

$$\tilde{\tau}_2(\varphi) = -\tilde{R}^N(\tilde{\tau}(\varphi),\mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) - \tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}\tilde{\tau}(\varphi) = 0\,, \qquad (14)$$

where $\tilde{R}^N$ is the Riemannian curvature with respect to $\tilde{h}_\alpha$, $\tilde{\tau}(\varphi)$ denotes the tension field of the map φ with respect to $\tilde{h}_\alpha$, and $\tilde{\nabla}^\varphi$ is the pull-back connection with respect to the metric $\tilde{h}_\alpha$. First, we compute the tension field $\tilde{\tau}(\varphi)$,

$$\begin{aligned}
\tilde{\tau}(\varphi) &= \tilde{\nabla}^\varphi_{e_i}\mathrm{d}\varphi(e_i) = \widetilde{\nabla}^N_{\mathrm{d}\varphi(e_i)}\mathrm{d}\varphi(e_i)\\
&= \tau(\varphi) + \frac{(1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i))}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2\circ\varphi}(\operatorname{grad} f)\circ\varphi\\
&= (1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i))(\operatorname{grad} f)\circ\varphi,
\end{aligned}$$

since $\Delta(f\circ\varphi) = \mathrm{d}f(\tau(\varphi)) + \operatorname{trace}\operatorname{Hess}_f(\mathrm{d}\varphi,\mathrm{d}\varphi)$ (see [3]), and $\tau(\varphi)=0$, we have $\tilde{\tau}(\varphi) = \lambda(\operatorname{grad} f)\circ\varphi$, with $\lambda = (1-\alpha)\Delta(f\circ\varphi)$ is a non-null constant. Now, we compute the first term of (14), from Theorem 2, we have

$$\begin{aligned}
&\widetilde{R}^N(\tilde{\tau}(\varphi), \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad = \lambda\Big\{R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad\quad + \frac{(1-\alpha)h(R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\operatorname{grad} f, \mathrm{d}\varphi(e_i))}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f \\
&\qquad\quad - \frac{(1-\alpha)^2\operatorname{Hess}_f(\operatorname{grad} f, \operatorname{grad} f)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \mathrm{d}\varphi(e_i))}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f \\
&\qquad\quad + \frac{(1-\alpha)^2\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \operatorname{grad} f)\operatorname{Hess}_f(\operatorname{grad} f, \mathrm{d}\varphi(e_i))}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f \\
&\qquad\quad + \frac{(1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \mathrm{d}\varphi(e_i))}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2}\nabla^N_{\operatorname{grad} f}\operatorname{grad} f \\
&\qquad\quad - \frac{(1-\alpha)\operatorname{Hess}_f(\operatorname{grad} f, \mathrm{d}\varphi(e_i))}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2}\nabla^N_{\mathrm{d}\varphi(e_i)}\operatorname{grad} f\Big\} \circ \varphi\,, \qquad (15)
\end{aligned}$$

since $\|\operatorname{grad} f\| = 1$, is constant on N, we obtain

$$\operatorname{Hess}_f(\operatorname{grad} f, X) = 0, \quad \nabla^N_{\operatorname{grad} f}\operatorname{grad} f = \frac{1}{2}\operatorname{grad}\|\operatorname{grad} f\|^2 = 0, \qquad (16)$$

for all $X \in \Gamma(TN)$, the equation (15) becomes

$$\begin{aligned}
&\widetilde{R}^N(\tilde{\tau}(\varphi), \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad = \lambda\Big\{R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad\quad + (1-\alpha)h(R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\operatorname{grad} f, \mathrm{d}\varphi(e_i))\operatorname{grad} f\Big\} \circ \varphi\,. \qquad (17)
\end{aligned}$$

The second term of (14) is given by

$$\begin{aligned}
\tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}\tilde{\tau}(\varphi) &= \lambda\tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}(\operatorname{grad} f) \circ \varphi \\
&= \lambda\tilde{\nabla}^\varphi_{e_i}(\tilde{\nabla}^N_{\mathrm{d}\varphi(e_i)}\operatorname{grad} f) \circ \varphi \\
&= \lambda\tilde{\nabla}^\varphi_{e_i}\Big\{(\nabla^N_{\mathrm{d}\varphi(e_i)}\operatorname{grad} f) \circ \varphi \\
&\quad + \frac{(1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), (\operatorname{grad} f) \circ \varphi)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2 \circ \varphi}(\operatorname{grad} f) \circ \varphi\Big\}, \qquad (18)
\end{aligned}$$

from equations (16) and (18), we find that

$$\begin{aligned}
\tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}\tilde{\tau}(\varphi) &= \lambda\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(\operatorname{grad} f) \circ \varphi \\
&\quad + (1-\alpha)\lambda\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \nabla^\varphi_{e_i}(\operatorname{grad} f) \circ \varphi)(\operatorname{grad} f) \circ \varphi\,, \qquad (19)
\end{aligned}$$

and note that

$$\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \nabla^\varphi_{e_i}(\operatorname{grad} f) \circ \varphi) = -h((\operatorname{grad} f) \circ \varphi, \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(\operatorname{grad} f) \circ \varphi)\,.$$

So, the map $\varphi\colon (M,g)\to(N,\tilde h_\alpha)$ is biharmonic if and only if

$$J_\varphi\big((\operatorname{grad} f)\circ\varphi\big)-(1-\alpha)h\big(J_\varphi\big((\operatorname{grad} f)\circ\varphi\big),(\operatorname{grad} f)\circ\varphi\big)(\operatorname{grad} f)\circ\varphi=0\,. \quad (20)$$

Note that, the equation (20) is equivalent to $J_\varphi\big((\operatorname{grad} f)\circ\varphi\big)=0$. □

Example 1. Let $M=\mathbb{R}^2$ and $N=\mathbb{H}^2=\{(y_1,y_2)\in\mathbb{R}^2|y_2>0\}$. We consider the harmonic map $\varphi\colon (M,\mathrm{d}x_1^2+\mathrm{d}x_2^2)\to(N,y_2^2(\mathrm{d}y_1^2+\mathrm{d}y_2^2))$, $(x_1,x_2)\mapsto(x_1,\sqrt{x_2^2+1})$, and let the function $f(y_1,y_2)=\frac{1}{2}y_2^2$. A straightforward calculation shows that $\|\operatorname{grad} f\|=1$, $\Delta(f\circ\varphi)=1$, $(\operatorname{grad} f)\circ\varphi=\Big(0,\frac{1}{\sqrt{x_2^2+1}}\Big)$ and $J_\varphi((\operatorname{grad} f)\circ\varphi)=0$. Thus, with respect to metric $\tilde h_\alpha=y_2^2(\alpha\mathrm{d}y_1^2+\mathrm{d}y_2^2)$, the map φ is biharmonic non-harmonic, with $\tilde\tau(\varphi)=\Big(0,\frac{1-\alpha}{\sqrt{x_2^2+1}}\Big)$.

Remark 1.

- Let $\varphi\colon (M,g)\to(N,h)$ be a harmonic map between two Riemannian manifolds and $\tilde h_\alpha=\alpha h+(1-\alpha)\mathrm{d}f\otimes\mathrm{d}f$, where $\alpha\in(0,1)$ and $f\in C^\infty(N)$ such that $\|\operatorname{grad} f\|=1$. Then the map $\varphi\colon (M,g)\to(N,\tilde h_\alpha)$ is harmonic if and only if $f\circ\varphi$ is harmonic on (M,g).
- Let (M,g) be a Riemannian manifold, and let f be a smooth function on M such that $\|\operatorname{grad} f\|=1$ and $\Delta f=k$, where $k\in\mathbb{R}$. Then, the identity map from (M,g) to $(M,\tilde g_\alpha)$ is biharmonic if and only if it is harmonic. Indeed; from Theorem 3 the identity map from (M,g) to $(M,\tilde g_\alpha)$ is a biharmonic map if and only if $\operatorname{Ricci}(\operatorname{grad} f)=0$, and by Bochner-Weitzenböck formula for smooth functions (see [14])

  $$\frac{1}{2}\Delta(\|\operatorname{grad} f\|^2)=\|\operatorname{Hess}_f\|^2+g(\operatorname{grad} f,\operatorname{grad}(\Delta f))+\operatorname{Ric}(\operatorname{grad} f,\operatorname{grad} f)\,,$$

  we obtain $\|\operatorname{Hess}_f\|=0$, so that $\Delta f=0$, that is the identity map from (M,g) to $(M,\tilde g_\alpha)$ is harmonic map.

4 The biharmonicity of the identity map $(M,\tilde g_\alpha)\to(M,\tilde g_\beta)$

Let (M,g) be a Riemannian manifold, $f\in C^\infty(M)$, $\alpha,\beta\in(0,1)$, and denote by

$$\widetilde{I}_{\alpha,\beta}\colon (M,\tilde g_\alpha)\to(M,\tilde g_\beta)\,,\qquad x\mapsto x$$

the identity map, where $\tilde g_\alpha=\alpha g+(1-\alpha)\mathrm{d}f\otimes\mathrm{d}f$ and $\tilde g_\beta=\beta g+(1-\beta)\mathrm{d}f\otimes\mathrm{d}f$.

Theorem 4. *If $\alpha\neq\beta$, and $\|\operatorname{grad} f\|=1$. Then the identity map $\widetilde{I}_{\alpha,\beta}$ is a proper biharmonic if and only if the function f is non-harmonic on M, and satisfying the following*

$$\begin{aligned}2\Delta f\operatorname{Ricci}(\operatorname{grad} f)=&-\frac{1}{\beta}\Delta^2 f\operatorname{grad} f-2\nabla_{\operatorname{grad}\Delta f}\operatorname{grad} f-\Delta f\operatorname{grad}\Delta f\\&+\frac{1-\alpha}{\beta}\Delta f g(\operatorname{grad} f,\operatorname{grad}\Delta f)\operatorname{grad} f\\&+\frac{1-\alpha}{\beta}\operatorname{Hess}_{\Delta f}(\operatorname{grad} f,\operatorname{grad} f)\operatorname{grad} f\,,\end{aligned}$$

where Δf is the Laplacian of f with respect to g, and $\Delta^2 f=\Delta(\Delta f)$.

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on M with respect to the metric g, such that $e_1 = \operatorname{grad} f$, it is easy to prove that $\left\{e_1, \frac{1}{\sqrt{\alpha}}e_i\right\}_{i=2}^m$ is a orthonormal frame on M with respect to the metric $\tilde{g}_\alpha$, where $m = \dim M$. Let $\widetilde{\nabla}^\alpha$ (resp. $\widetilde{\nabla}^\beta$) the Levi-Civita connection of $(M, \tilde{g}_\alpha)$ (resp. of $(M, \tilde{g}_\beta)$), then the tension field of $\widetilde{I}_{\alpha,\beta}$ is given by

$$\begin{aligned}\tau(\widetilde{I}_{\alpha,\beta}) &= \nabla^{\widetilde{I}_{\alpha,\beta}}_{e_1} \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_1) - \mathrm{d}\widetilde{I}_{\alpha,\beta}(\widetilde{\nabla}^\alpha_{e_1} e_1) + \frac{1}{\alpha}\sum_{i=2}^m \left\{\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i} \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i) - \mathrm{d}\widetilde{I}_{\alpha,\beta}(\widetilde{\nabla}^\alpha_{e_i} e_i)\right\} \\ &= \widetilde{\nabla}^\beta_{e_1} e_1 - \widetilde{\nabla}^\alpha_{e_1} e_1 + \frac{1}{\alpha}\sum_{i=2}^m \left\{\widetilde{\nabla}^\beta_{e_i} e_i - \widetilde{\nabla}^\alpha_{e_i} e_i\right\},\end{aligned}$$

using Theorem 1, with $\|\operatorname{grad} f\| = 1$, we have

$$\tau(\widetilde{I}_{\alpha,\beta}) = \frac{\alpha-\beta}{\alpha}\sum_{i=2}^m \operatorname{Hess}_f(e_i, e_i)\operatorname{grad} f\,, \tag{21}$$

since $\operatorname{Hess}_f(e_1, e_1) = 0$, the equation (21) becomes

$$\tau(\widetilde{I}_{\alpha,\beta}) = \frac{\alpha-\beta}{\alpha}\Delta f \operatorname{grad} f\,.$$

Note that $\widetilde{I}_{\alpha,\beta}$ is harmonic if and only if $\Delta f = 0$, i.e. the function f is harmonic on (M, g). We compute the bitension field of the identity $\widetilde{I}_{\alpha,\beta}$, for all $i = 1, \ldots, m$ we have

$$\widetilde{R}_\beta(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i) = \frac{\alpha-\beta}{\alpha}\Delta f \widetilde{R}_\beta(\operatorname{grad} f, e_i)e_i\,, \tag{22}$$

where $\widetilde{R}_\beta$ is the curvature tensor of $\widetilde{\nabla}^\beta$. From Theorem 2, and equation (22) with $\|\operatorname{grad} f\| = 1$, $\operatorname{Hess}_f(\operatorname{grad} f, X) = 0$, for all $X \in \Gamma(TM)$, and $\nabla_{\operatorname{grad} f}\operatorname{grad} f = 0$, we obtain the following

$$\begin{aligned}&\widetilde{R}_\beta(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i) \\ &\quad= \frac{\alpha-\beta}{\alpha}\Delta f\Big\{R(\operatorname{grad} f, e_i)e_i + (1-\beta)g(R(\operatorname{grad} f, e_i)\operatorname{grad} f, e_i)\operatorname{grad} f\Big\},\end{aligned} \tag{23}$$

from (23) and the definition of Ricci curvature, we get

$$\begin{aligned}\widetilde{R}(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_1))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_1) + \frac{1}{\alpha}\sum_{i=2}^m \widetilde{R}(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i)& \\ = \frac{\alpha-\beta}{\alpha^2}\Delta f\Big\{\operatorname{Ricci}(\operatorname{grad} f)& \\ - (1-\beta)\operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f\Big\}.&\end{aligned} \tag{24}$$

Let $i = 1, \ldots, m$, we compute

$$\begin{aligned}
\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\tau(\widetilde{I}_{\alpha,\beta})-\nabla^{\widetilde{I}_{\alpha,\beta}}_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\tau(\widetilde{I}_{\alpha,\beta})
&= \frac{\alpha-\beta}{\alpha}\Big\{\widetilde{\nabla}^{\beta}_{e_i}\widetilde{\nabla}^{\beta}_{e_i}\Delta f \operatorname{grad} f - \widetilde{\nabla}^{\beta}_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\Delta f \operatorname{grad} f\Big\} \\
&= \frac{\alpha-\beta}{\alpha}\Big\{\widetilde{\nabla}^{\beta}_{e_i}\nabla_{e_i}\Delta f \operatorname{grad} f - \nabla_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\Delta f \operatorname{grad} f\Big\} \\
&= \frac{\alpha-\beta}{\alpha}\Big\{\nabla_{e_i}\nabla_{e_i}\Delta f \operatorname{grad} f - \nabla_{\nabla_{e_i}e_i}\Delta f \operatorname{grad} f \\
&\quad + (1-\beta)\operatorname{Hess}_f(e_i,\nabla_{e_i}\Delta f \operatorname{grad} f)\operatorname{grad} f \\
&\quad - (1-\alpha)\operatorname{Hess}_f(e_i,e_i)\nabla_{\operatorname{grad} f}\Delta f \operatorname{grad} f\Big\}, \qquad (25)
\end{aligned}$$

a straightforward calculation shows that

$$\begin{aligned}
\nabla_{e_i}\nabla_{e_i}\Delta f \operatorname{grad} f - \nabla_{\nabla_{e_i}e_i}\Delta f \operatorname{grad} f
&= e_i(e_i(\Delta f))\operatorname{grad} f + 2e_i(\Delta f)\nabla_{e_i}\operatorname{grad} f \\
&\quad + \Delta f\nabla_{e_i}\nabla_{e_i}\operatorname{grad} f - (\nabla_{e_i}e_i)(\Delta f)\operatorname{grad} f \\
&\quad - \Delta f\nabla_{\nabla_{e_i}e_i}\operatorname{grad} f, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
(1-\beta)&\operatorname{Hess}_f(e_i,\nabla_{e_i}\Delta f \operatorname{grad} f)\operatorname{grad} f \\
&= -(1-\beta)\Delta f g(\operatorname{grad} f,\nabla_{e_i}\nabla_{e_i}\operatorname{grad} f)\operatorname{grad} f, \qquad (27)
\end{aligned}$$

and

$$\begin{aligned}
-(1-\alpha)&\operatorname{Hess}_f(e_i,e_i)\nabla_{\operatorname{grad} f}\Delta f \operatorname{grad} f \\
&= -(1-\alpha)\operatorname{Hess}_f(e_i,e_i)(\operatorname{grad} f)(\Delta f)\operatorname{grad} f, \qquad (28)
\end{aligned}$$

by equations (25)–(28), with $\|\operatorname{grad} f\| = 1$, we find that

$$\begin{aligned}
\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_1}\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_1}\tau(\widetilde{I}_{\alpha,\beta}) - \nabla^{\widetilde{I}_{\alpha,\beta}}_{\widetilde{\nabla}^{\alpha}_{e_1}e_1}\tau(\widetilde{I}_{\alpha,\beta}) + \frac{1}{\alpha}\sum_{i=2}^{m}\Big\{\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\tau(\widetilde{I}_{\alpha,\beta}) - \nabla^{\widetilde{I}_{\alpha,\beta}}_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\tau(\widetilde{I}_{\alpha,\beta})\Big\} \\
= \frac{\alpha-\beta}{\alpha^2}\Big\{(\alpha-1)\operatorname{Hess}_{\Delta f}(\operatorname{grad} f,\operatorname{grad} f)\operatorname{grad} f + \Delta^2 f \operatorname{grad} f \\
+ 2\nabla_{\operatorname{grad}\Delta f}\operatorname{grad} f + \Delta f \operatorname{trace}\nabla^2\operatorname{grad} f \\
- (1-\beta)\Delta f g(\operatorname{grad} f,\operatorname{trace}\nabla^2\operatorname{grad} f)\operatorname{grad} f \\
- (1-\alpha)\Delta f g(\operatorname{grad} f,\operatorname{grad}\Delta f)\operatorname{grad} f\Big\}, \qquad (29)
\end{aligned}$$

from equations (24), (29), and the following (see [1])

$$\operatorname{trace}\nabla^2\operatorname{grad} f = \operatorname{Ricci}(\operatorname{grad} f) + \operatorname{grad}(\Delta f),$$

the identity map $\widetilde{I}_{\alpha,\beta}$ is a proper biharmonic map if and only if

$$\begin{aligned}2\Delta f\,\mathrm{Ricci}(\operatorname{grad} f) - 2(1-\beta)\Delta f\,\mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f \\ + \Delta^2 f \operatorname{grad} f + 2\nabla_{\operatorname{grad}\Delta f}\operatorname{grad} f + \Delta f \operatorname{grad}\Delta f \\ - (2-\alpha-\beta)\Delta f g(\operatorname{grad} f, \operatorname{grad}\Delta f)\operatorname{grad} f \\ + (\alpha-1)\operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f = 0\,, \quad (30)\end{aligned}$$

with $\alpha \neq \beta$ and $\Delta f \neq 0$, taking its inner product with $\operatorname{grad} f$, we have

$$\begin{aligned}&-2(1-\beta)\Delta f\,\mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f) \\ &= \frac{1-\beta}{\beta}\Delta^2 f - \frac{(1-\beta)(1-\alpha)}{\beta}\operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f) \\ &\quad - \frac{(1-\beta)(1-\alpha-\beta)}{\beta}\Delta f g(\operatorname{grad} f, \operatorname{grad}\Delta f)\,. \quad (31)\end{aligned}$$

Theorem 4 follows from (30) and (31). □

Corollary 1. *If $\alpha \neq \beta$, $\|\operatorname{grad} f\| = 1$, $\Delta f = F(f)$, where F is a non-null function on $I \subset \mathbb{R}$, and $\mathrm{Ricci}(\operatorname{grad} f) = \lambda \operatorname{grad} f$, for some smooth function λ on M. Then the identity map $\widetilde{I}_{\alpha,\beta}$ is a proper biharmonic if and only if the function f satisfying the following*

$$2\beta\lambda F(f) + (\alpha+\beta)F(f)F'(f) + \alpha F''(f) = 0\,.$$

According to Corollary 1, we have the following example.

Example 2. Let $M = (0,\infty)\times\mathbb{R}^n$ equipped with the Riemannian metric

$$g = \mathrm{d}t^2 + \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2}{t}\,,$$

we set $f(t,x) = t$, for all $(t,x) \in M$. We have $\operatorname{grad} f = \partial_t$, $\|\operatorname{grad} f\| = 1$, $\Delta f = -\frac{n}{2t}$ and $\mathrm{Ricci}(\operatorname{grad} f) = -\frac{3n}{4t^2}\partial_t$, so that $F(s) = -\frac{n}{2s}$, for all $s \in I = (0,\infty)$ and $\lambda(t,x) = -\frac{3n}{4t^2}$ for all $(t,x) \in M$. Using the Corollary 1, Then the identity map $\widetilde{I}_{\alpha,\beta}$ is proper biharmonic if and only if $n \neq 4$ and $\alpha = \frac{2n\beta}{n+4}$.

5 Biharmonic curve in $(M, \tilde{g}_\alpha)$

Let $\gamma\colon I \subset \mathbb{R} \to (M,g)$, $t \mapsto \gamma(t)$ be a harmonic curve in a Riemannian manifold (M,g), such that $g(\dot\gamma,\dot\gamma) = 1$, and let f be a smooth function on M. In this section we suppose that the gradient vector of f at $\gamma(t)$ is parallel to the tangent vector $\dot\gamma(t)$. Thus, $(\operatorname{grad} f)_{\gamma(t)} = \rho(t)\dot\gamma(t)$, with $\rho(t) = (f\circ\gamma)'(t)$, for all $t \in I$. Since γ is harmonic we get the following formula

$$(\nabla_{\dot\gamma}\operatorname{grad} f)_t = \rho'(t)\dot\gamma(t)\,, \quad \forall t \in I. \quad (32)$$

We set $\tilde{g}_\alpha = \alpha g + (1-\alpha)\mathrm{d}f\otimes\mathrm{d}f$, where $\alpha \in (0,1)$. We have the following result:

Theorem 5. *The curve $\gamma\colon I \to (M, \tilde g_\alpha)$ is biharmonic if and only if the function f satisfying the following*

$$f(\gamma(t)) = \pm \int \sqrt{(at^2+bt+c)^2 - \frac{\alpha}{1-\alpha}}\,\mathrm{d}t\,,$$

where $a, b, c \in \mathbb{R}$, such that $(at^2+bt+c)^2 > \frac{\alpha}{1-\alpha}$, for all $t \in I$.

Proof. By Theorem 1, we have

$$\widetilde{\tau}(\gamma) = \tau(\gamma) + \frac{(1-\alpha)\operatorname{Hess}_f(\dot\gamma,\dot\gamma)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2\circ\gamma}(\operatorname{grad} f)\circ\gamma\,, \tag{33}$$

from the harmonicity condition of γ, and equations (32), (33), we obtain $\widetilde{\tau}(\gamma) = \lambda\dot\gamma$, where

$$\lambda = \frac{(1-\alpha)\rho\rho'}{\alpha+(1-\alpha)\rho^2}\,. \tag{34}$$

Now, the curve $\gamma\colon I \to (M, \tilde g_\alpha)$ is biharmonic if and only if

$$\widetilde{R}\Big(\widetilde{\tau}(\gamma), \mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)\Big)\mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big) + \widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\tau}(\gamma) = 0\,, \tag{35}$$

by the property of the curvature tensor, the first term on the left-hand side of (35) is

$$\widetilde{R}\Big(\widetilde{\tau}(\gamma), \mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)\Big)\mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \lambda\widetilde{R}(\dot\gamma,\dot\gamma)\dot\gamma = 0\,.$$

For the second term on the left-hand side of (35), we compute

$$\begin{aligned}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\tau}(\gamma) &= \widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\lambda\dot\gamma\\ &= \lambda'\dot\gamma + \lambda\widetilde{\nabla}_{\dot\gamma}\dot\gamma\\ &= (\lambda'+\lambda^2)\dot\gamma\,,\end{aligned} \tag{36}$$

with the same method of (36), we find that

$$\begin{aligned}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\tau}(\gamma) &= \widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}(\lambda'+\lambda^2)\dot\gamma\\ &= (\lambda''+2\lambda\lambda')\dot\gamma + (\lambda'+\lambda^2)\widetilde{\nabla}_{\dot\gamma}\dot\gamma\\ &= (\lambda''+3\lambda\lambda'+\lambda^3)\dot\gamma\,.\end{aligned}$$

So, the curve $\gamma\colon I \to (M, \tilde g_\alpha)$ is biharmonic if and only if $\lambda''+3\lambda\lambda'+\lambda^3 = 0$, that is the function λ is the form $(2at+b)/(at^2+bt+c)$, where $a, b, c \in \mathbb{R}$, such that $at^2+bt+c \neq 0$, for all $t \in I$. Thus, from (34) with $(at^2+bt+c)^2 > \frac{\alpha}{1-\alpha}$, for all $t \in I$, we obtain

$$\rho(t) = \pm\sqrt{(at^2+bt+c)^2 - \frac{\alpha}{1-\alpha}}\,, \quad \forall t \in I. \tag{37}$$

Theorem 5 follows from equation (37), with $\rho = (f\circ\gamma)'$. □

Remark 2. The curve $\gamma\colon I \to (M, \tilde{g}_\alpha)$ is proper biharmonic if and only if there exists $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 > 0$, and for all $i = 1, \dots, m$ ($m = \dim M$), and in any local coordinates (x_i) on M, such that

$$\sum_{j=1}^{m} g^{ij}(\gamma(t)) \frac{\partial f}{\partial x_j}\Big|_{\gamma(t)} = \pm\sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha}\,\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\Big|_t}, \quad \forall t \in I.$$

Using Theorem 5 and the previous Remark, we can construct many examples for proper biharmonic curves.

Example 3. Let $M = \mathbb{R}^n$ equipped with the Riemannian metric $g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$,

$$f(x) = \frac{2}{3}\sum_{i=1}^{n}(1 + x_i^2)^{\frac{3}{2}}, \quad \forall x = (x_1, \dots, x_n) \in M.$$

For $\alpha = \frac{n}{n+1}$, the curve

$$\gamma\colon I \to (M, \tilde{g}_\alpha), \quad t \mapsto \left(\frac{t}{\sqrt{n}}, \dots, \frac{t}{\sqrt{n}}\right),$$

is proper biharmonic.

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