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Reconciliation of discrete and continuous versions of some dynamic inequalities synthesized on time scale calculus

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Abstract. The aim of this paper is to synthesize discrete and continuous versions of some dynamic inequalities such as Radon's Inequality, Bergström's Inequality, Schlömilch's Inequality and Rogers-Hölder's Inequality on time scales in comprehensive form.

1 Introduction

We present discrete versions of some classical inequalities. The inequality from (1) is called Bergström's Inequality, Titu Andreescu's Inequality or Engel's Inequality in literature as given in [4], [5], [6], [15].

Theorem 1. *If $n \in \mathbb{N}$, $x_k \in \mathbb{R}$ and $y_k > 0$, $k \in \{1, 2, \dots, n\}$, then*

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} y_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{y_k}, \tag{1}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n}$.

The upcoming result is called Radon's Inequality as given in [16].

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Theorem 2. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$ and $\beta \geq 0$, then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^{\beta+1}}{\left(\sum\limits_{k=1}^{n} y_k\right)^{\beta}} \leq \sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^{\beta}}. \tag{2}$$

Inequality (2) is widely studied by many authors because it is used in practical applications.

The following inequality is generalized Radon's Inequality as given in [9].

Theorem 3. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$, $\beta \geq 0$ and $\gamma \geq 1$, then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{n} y_k^{\gamma}\right)^{\beta+\gamma-1}} \leq \sum_{k=1}^{n} \frac{x_k^{\beta+\gamma}}{y_k^{\beta}}, \tag{3}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n}$.

The following inequality is a refinement of Radon's Inequality as given in [11].

Theorem 4. *If $m, n \in \mathbb{N}$, $n > m$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$, $\beta \geq 0$ and $\gamma \geq 1$, then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{n} y_k^{\gamma}\right)^{\beta+\gamma-1}} \leq \frac{\left(\sum\limits_{k=1}^{m} x_k y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{m} y_k^{\gamma}\right)^{\beta+\gamma-1}} + \sum_{k=m+1}^{n} \frac{x_k^{\beta+\gamma}}{y_k^{\beta}}. \tag{4}$$

We shall prove these results on time scales. The calculus of time scales was initiated by Stefan Hilger as given in [13]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced in order to unify continuous and discrete analysis and to combine them in one comprehensive form. In the calculus of time scales, results are extended. This is studied as delta calculus, nabla calculus and diamond-α calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. This hybrid theory is also widely applied on dynamic inequalities (see [1], [2], [7], [17], [18]). The basic work on dynamic inequalities is done by Agarwal, Anastassiou, Bohner, Peterson, O'Regan, Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2 Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [7], [8].

For $t \in \mathbb{T}$, the *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}\,.$$

The mapping $\mu\colon \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}\,.$$

The mapping $\nu\colon \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f\colon \mathbb{T} \to \mathbb{R}$, the delta derivative f^Δ is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|\,,$$

for all $s \in U$, then f is said to be *delta differentiable* at t, and $f^\Delta(t)$ is called the *delta derivative* of f at t.

A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [7], [8].

Definition 1. A function $F\colon \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f\colon \mathbb{T} \to \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a)\,.$$

The following results of nabla calculus are taken from [3], [7], [8].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f\colon \mathbb{T}_k \to \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, with nabla derivative $f^\nabla(t)$, if there exists $f^\nabla(t) \in \mathbb{R}$ such that given any $\epsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \le \epsilon|\rho(t) - s|\,,$$

for all $s \in V$.

A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* (*ld-continuous*), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [3], [7], [8].

Definition 2. A function $G\colon \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g\colon \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a)\,.$$

Now we present short introduction of diamond-α derivative as given in [1], [19].

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, the diamond-$\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t)\,, \qquad 0 \le \alpha \le 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0,1)$.

Theorem 5 ([19]). *Let $f, g\colon \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$ and we write $f^\sigma(t) = f(\sigma(t))$, $g^\sigma(t) = g(\sigma(t))$, $f^\rho(t) = f(\rho(t))$ and $g^\rho(t) = g(\rho(t))$. Then*

(*i*) *$f \pm g\colon \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t)\,.$$

(*ii*) *$fg\colon \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^{\Delta}(t) + (1-\alpha)f^\rho(t)g^{\nabla}(t)\,.$$

(*iii*) *For $g(t)g^\sigma(t)g^\rho(t) \neq 0$, $\frac{f}{g}\colon \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^\sigma(t)g^\rho(t) - \alpha f^\sigma(t)g^\rho(t)g^{\Delta}(t) - (1-\alpha)f^\rho(t)g^\sigma(t)g^{\nabla}(t)}{g(t)g^\sigma(t)g^\rho(t)}\,.$$

Definition 3 ([19]). Let $a, t \in \mathbb{T}$ and $h\colon \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s\,, \qquad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 6 ([19]). *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a,b]_{\mathbb{T}}$. Then*

(*i*) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$,

(*ii*) $\int_a^t cf(s) \diamond_\alpha s = c\int_a^t f(s) \diamond_\alpha s$,

(*iii*) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$,

(iv) $\int_a^t f(s)\diamond_\alpha s = \int_a^b f(s)\diamond_\alpha s + \int_b^t f(s)\diamond_\alpha s$,

(v) $\int_a^a f(s)\diamond_\alpha s = 0$.

We need the following result.

Theorem 7 ([17]). *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x), g(x) \neq 0$, $\forall x \in [a,b]_{\mathbb{T}}$. If $\beta \geq 0$ and $\gamma \geq 1$, then*

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} \leq \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}}\diamond_\alpha x\,. \tag{5}$$

The sign of equality holds in (5) if and only if $f(x) = cg(x)$, where c is a real constant.

3 Main Results

In order to present our main results, first we give an extension of dynamic Radon's Inequality on time scales.

Theorem 8. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x)$, $g(x) \neq 0$, $\forall x \in [a,b]_{\mathbb{T}}$ and $a < c < b$, where $c \in [a,b]_{\mathbb{T}}$. If $\beta \geq 0$ and $\gamma \geq 1$, then*

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} \leq \frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} + \int_c^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}}\diamond_\alpha x\,. \tag{6}$$

Proof. From the generalized Radon's Inequality (5), we have that

$$\frac{\left(\int_c^b |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_c^b |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} \leq \int_c^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}}\diamond_\alpha x\,. \tag{7}$$

By adding $\frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}}$ in both sides of (7), we can write

$$\frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} + \frac{\left(\int_c^b |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_c^b |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} \leq \frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} + \int_c^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}}\diamond_\alpha x\,. \tag{8}$$

By applying the classical Radon's Inequality on the left-hand side of (8), we get

$$\begin{aligned} &\frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x + \int_c^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma} \diamond_\alpha x + \int_c^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}} \\ &\qquad \leq \frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}} + \int_c^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x\,, \end{aligned} \tag{9}$$

we have the desired inequality from (9) and hence, the proof is complete. □

Remark 1. If we take $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $c = m+1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in \mathbb{R}$, $g(k) = y_k \in (0,+\infty)$, $k \in \{1,2,\dots,n\}$, $m,n \in \mathbb{N}$, $\beta = 1$ and $\gamma = 1$, then (6) reduces to

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} y_k} \leq \frac{\left(\sum_{k=1}^{m} x_k\right)^2}{\sum_{k=1}^{m} y_k} + \sum_{k=m+1}^{n} \frac{x_k^2}{y_k}\,. \tag{10}$$

If $m = 1$, then (10) reduces to (1).

Remark 2. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $c = m+1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in [0,+\infty)$, $g(k) = y_k \in (0,+\infty)$, $k \in \{1,2,\dots,n\}$, $m,n \in \mathbb{N}$ and $\gamma = 1$, then (6) takes the form

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^{\beta+1}}{\left(\sum_{k=1}^{n} y_k\right)^{\beta}} \leq \frac{\left(\sum_{k=1}^{m} x_k\right)^{\beta+1}}{\left(\sum_{k=1}^{m} y_k\right)^{\beta}} + \sum_{k=m+1}^{n} \frac{x_k^{\beta+1}}{y_k^{\beta}}\,. \tag{11}$$

If $m = 1$, then (11) reduces to (2).

Remark 3. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $c = m+1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in [0,+\infty)$ and $g(k) = y_k \in (0,+\infty)$ for $k \in \{1,2,\dots,n\}$, $m,n \in \mathbb{N}$, then (6) reduces to (4). If $m = 1$, then (4) reduces to (3).

Corollary 1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions and $a < c < b$, where $c \in [a,b]_{\mathbb{T}}$. If $\beta \geq 0$ and $\gamma \geq 1$, then*

$$\begin{aligned} &\frac{\left(\int_a^b |w(x)||f(x)|^{\gamma}|g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||f(x)g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}} \\ &\qquad \leq \frac{\left(\int_a^c |w(x)||f(x)|^{\gamma}|g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||f(x)g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}} + \int_c^b \frac{|w(x)||f(x)|^{\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x\,. \end{aligned} \tag{12}$$

Proof. Letting $|g(x)|$ be replaced by $|f(x)g(x)|$ in (6), we get our claim. □

Remark 4. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $c = m+1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ and $g(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \ldots, n\}$, $m, n \in \mathbb{N}$, then (12) takes the form

$$\frac{\left(\sum\limits_{k=1}^{n} x_k^{\gamma} y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{n} x_k^{\gamma} y_k^{\gamma}\right)^{\beta+\gamma-1}} \leq \frac{\left(\sum\limits_{k=1}^{m} x_k^{\gamma} y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{m} x_k^{\gamma} y_k^{\gamma}\right)^{\beta+\gamma-1}} + \sum_{k=m+1}^{n} \frac{x_k^{\gamma}}{y_k^{\beta}} . \tag{13}$$

If $m = 1$, then (13) takes the form

$$\frac{\left(\sum\limits_{k=1}^{n} x_k^{\gamma} y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{n} x_k^{\gamma} y_k^{\gamma}\right)^{\beta+\gamma-1}} \leq \sum_{k=1}^{n} \frac{x_k^{\gamma}}{y_k^{\beta}} , \tag{14}$$

as given in [10].

Corollary 2. *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, $w(x) \neq 0$ with $\int_a^b |w(x)| \diamond_\alpha x = 1$ and $a < c < b$, where $c \in [a,b]_{\mathbb{T}}$. If $\eta_2 \geq \eta_1 > 0$, then*

$$\left(\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \leq \left(\frac{\left(\int_a^c |w(x)||f(x)|^{\eta_1} \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}}{\left(\int_a^c |w(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}-1}} + \int_c^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x\right)^{\frac{1}{\eta_2}} . \tag{15}$$

Proof. Putting $\beta + \gamma = \frac{\eta_2}{\eta_1} \geq 1$ and $g \equiv 1$ in (6), we have that

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}}{\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}-1}} \leq \frac{\left(\int_a^c |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}}{\left(\int_a^c |w(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}-1}} + \int_c^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}} \diamond_\alpha x . \tag{16}$$

Using the fact that $\int_a^b |w(x)| \diamond_\alpha x = 1$, the inequality (16) becomes

$$\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}} \leq \frac{\left(\int_a^c |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}}{\left(\int_a^c |w(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}-1}} + \int_c^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}} \diamond_\alpha x . \tag{17}$$

Letting $|f(x)|$ be replaced by $|f(x)|^{\eta_1}$ and taking power $\frac{1}{\eta_2}$ on both sides of (17), we get our desired result. □

Remark 5. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $c = 2$, $b = n+1$, $w \equiv \frac{1}{n}$, $f(k) = x_k \in [0, +\infty)$ for $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$ and $\eta_1 < \eta_2$, then inequality (15) takes the form

$$\left(\frac{1}{n}\sum_{k=1}^{n} x_k^{\eta_1}\right)^{\frac{1}{\eta_1}} < \left(\frac{1}{n}\sum_{k=1}^{n} x_k^{\eta_2}\right)^{\frac{1}{\eta_2}}, \tag{18}$$

unless the x_k for $k \in \mathbb{N}$ are all equal.

The inequality from (18) is called Schlömilch's Inequality in literature as given in [12].

Corollary 3. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, neither $w \equiv 0$ nor $g \equiv 0$, and $a < c < b$, where $c \in [a,b]_{\mathbb{T}}$. If $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{\int_a^b |w(x)||f(x)||g(x)| \diamond_\alpha x}{\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}} \leq \left[\left\{\frac{\int_a^c |w(x)||f(x)||g(x)| \diamond_\alpha x}{\left(\int_a^c |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}}\right\}^p + \int_c^b |w(x)||f(x)|^p \diamond_\alpha x\right]^{\frac{1}{p}}. \tag{19}$$

Proof. If $\beta > 0$, $\gamma = 1$ and $\beta + 1 = p > 1$, then (6) takes the form

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^p}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{p-1}} \leq \frac{\left(\int_a^c |w(x)||f(x)| \diamond_\alpha x\right)^p}{\left(\int_a^c |w(x)||g(x)| \diamond_\alpha x\right)^{p-1}} + \int_c^b \frac{|w(x)||f(x)|^p}{|g(x)|^{p-1}} \diamond_\alpha x. \tag{20}$$

Replacing $|w(x)|$ by $|w(x)||g(x)|^{p-1}$ in (20), we get

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{p-1} \diamond_\alpha x\right)^p}{\left(\int_a^b |w(x)||g(x)|^p \diamond_\alpha x\right)^{p-1}} \leq \frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{p-1} \diamond_\alpha x\right)^p}{\left(\int_a^c |w(x)||g(x)|^p \diamond_\alpha x\right)^{p-1}} + \int_c^b |w(x)||f(x)|^p \diamond_\alpha x. \tag{21}$$

Replacing $|g(x)|$ by $|g(x)|^{\frac{q}{p}}$, taking power $\frac{1}{p} > 0$ and using the fact that $\frac{1}{p} + \frac{1}{q} = 1$, we obtain the desired result. □

Remark 6. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $c = 2$, $b = n+1$ and $w \equiv 1$. If $f(k) = x_k \in (0, +\infty)$ and $g(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, then (19) takes the form

$$\sum_{k=1}^{n} x_k y_k \leq \left(\sum_{k=1}^{n} x_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q \right)^{\frac{1}{q}} . \tag{22}$$

The inequality from (22) is called Rogers-Hölder's Inequality in literature as given in [14].

Theorem 9. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x)$, $g(x) \neq 0$, $\forall x \in [a,b]_{\mathbb{T}}$ and $a < c < b$, where $c \in [a,b]_{\mathbb{T}}$. If $\beta \geq 0$ and $\gamma \geq 1$, then the following inequality holds true:*

$$\Lambda(c,a) \leq \Lambda(b,a), \tag{23}$$

where

$$\Lambda(t,s) = \int_s^t \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x - \frac{\left(\int_s^t |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x \right)^{\beta+\gamma}}{\left(\int_s^t |w(x)||g(x)|^{\gamma} \diamond_\alpha x \right)^{\beta+\gamma-1}},$$

$\forall s, t \in [a,b]_{\mathbb{T}}$.

Proof. Adding $\int_a^c \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x$ in both sides of (6), we obtain

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x \right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x \right)^{\beta+\gamma-1}} + \int_a^c \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x$$
$$\leq \frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x \right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma} \diamond_\alpha x \right)^{\beta+\gamma-1}} + \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x . \tag{24}$$

Therefore

$$\int_a^c \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x - \frac{\left(\int_a^c |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x \right)^{\beta+\gamma}}{\left(\int_a^c |w(x)||g(x)|^{\gamma} \diamond_\alpha x \right)^{\beta+\gamma-1}}$$
$$\leq \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x - \frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x \right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x \right)^{\beta+\gamma-1}} . \tag{25}$$

Thus, the proof of Theorem 9 is complete. □

Remark 7. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $c = m+1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in [0, +\infty)$ and $g(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$, $m, n \in \mathbb{N}$, then (23) takes the form

$$\Lambda_m \leq \Lambda_n , \tag{26}$$

where

$$\Lambda_i = \sum_{k=1}^{i} \frac{x_k^{\beta+\gamma}}{y_k^{\beta}} - \frac{\left(\sum\limits_{k=1}^{i} x_k y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{i} y_k^{\gamma}\right)^{\beta+\gamma-1}}, \qquad i \in \mathbb{N}, i \leq n.$$

Thus, we conclude that

$$0 = \Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_{n-1} \leq \Lambda_n \,. \tag{27}$$

Inequalities (26) and (27) are proved in [11].

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