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A simple construction of basic polynomials invariant under the Weyl group of the simple finite-dimensional complex Lie algebra

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Abstract. For every simple finite-dimensional complex Lie algebra, I give a simple construction of all (except for the Pfaffian) basic polynomials invariant under the Weyl group. The answer is given in terms of the two basic polynomials of smallest degree.

1 Basics

For necessary information on Lie algebras and Lie groups, see [5]. Recall the notation: let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra of rank l, let R_+ (resp. R_-) be the set of its positive (resp. negative) roots, and $\{\alpha_1, \dots, \alpha_l\}$ the set of simple roots. Let the Weyl group $W_{\mathfrak{g}}$ of the root system R act in the space $V = \mathbb{R}^l$, let $(-,-)$ be the non-degenerate $W_{\mathfrak{g}}$-invariant bilinear form on V, such that $(x,y) = \sum_{j=1}^{l} x_j y_j$ for any $x, y \in V$; let h be the Coxeter number; $\delta = \sum_{j=1}^{l} b_j \alpha_j$ the highest root; $b = \max b_j$.

As is known (by abuse of notation we denote the Lie algebra $\mathfrak{sl}(n+1)$ by the symbol of its root system A_n in Cartan's notation, and similarly for the other

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simple Lie algebras),

$$b = \begin{cases} 1 & \text{for } A_l, \\ 2 & \text{for } B_l,\, C_l \text{ and } D_l, \\ 3 & \text{for } G_2 \text{ and } E_6, \\ 4 & \text{for } F_4 \text{ and } E_7, \\ 6 & \text{for } E_8; \end{cases} \qquad h = \begin{cases} l+1 & \text{for } A_l, \\ 2l & \text{for } B_l, \text{ and } C_l, \\ 2l-2 & \text{for } D_l, \\ 12 & \text{for } E_6 \text{ and } F_4, \\ 18 & \text{for } E_7, \\ 30 & \text{for } E_8, \\ 6 & \text{for } G_2. \end{cases} \tag{1}$$

If $\alpha = \sum_{j=1}^{l} n_j \alpha_j \in R_+$, then we define the *height* of α to be $ht(\alpha) = \sum_{j=1}^{l} n_j$.

As it was shown in [1], the algebra of invariant polynomials is generated by l basic homogeneous polynomials of degrees d_j for $j = 1, \ldots, l$ such that $d_1 = 2$, $d_j \leq d_{j+1}$, $d_l = h$.

Note that

$$d_j = \begin{cases} j+1 & \text{for } A_l \\ 2j & \text{for } B_l \text{ and } C_l, \end{cases} \tag{2}$$

$$\{d_j \mid j = 1, \ldots, l-1, l\} = \{2, 4, \ldots, 2(l-1),\ l\}, \text{ as ordered sets, for } D_l. \tag{3}$$

For the exceptional simple Lie algebras, the d_j were first found in [6] using results of [7]:

$$\{d_1, \ldots, d_l\} = \begin{cases} \{2, 6\} & \text{for } G_2; \\ \{2, 6, 8, 12\} & \text{for } F_4; \\ \{2, 5, 6, 8, 9, 12\} & \text{for } E_6; \\ \{2, 6, 8, 10, 12, 14, 18\} & \text{for } E_7; \\ \{2, 8, 12, 14, 18, 20, 24, 30\} & \text{for } E_8. \end{cases} \tag{4}$$

Note that the degrees satisfy duality relations

$$d_j + d_{l+1-j} = h + 2. \tag{5}$$

Denote by I_j the invariant polynomial of degree d_j in eq. (2), (3), and (4).

2 Degrees and exponents

Let us remind a characterization of *exponents*, i.e., numbers $m_j := d_j - 1$, given in [2]. Let n_k be the number of roots of height k. Then $n_k - n_{k+1}$ is the number of times k occurs as an exponent of $\mathfrak{g}$.

Note that d_1=2 for all simple Lie algebras. Recall the values of b, see (1). The values of d_2, d_3 and d_4 are given by

Theorem 1. *We have*

$$\begin{aligned} d_2 &= b + 2 && \text{for all } \mathfrak{g}, \text{ except } G_2, \\ d_3 &= 2b && \text{for all exceptional } \mathfrak{g}, \text{ except } G_2, \\ d_4 &= 2b + 2 && \text{for } E_6, E_7 \text{ and } E_8. \end{aligned}$$

The other quantities d_j can be obtained from duality.

The proof follows easily from Table (4) and duality eq. (5) borrowed from any sufficiently thick book (e.g., [5]) and papers [6], [7]. Below we give an independent proof.

Proof. We give the proof only for the most complicated case $\mathfrak{g}= E_8$. For all other cases the proof is analogous.

We enumerate the simple roots of E_8 first along the Dynkin diagram, starting from the longest end of the branch, as the simple roots of A_7, and set

$$\begin{aligned}&(\alpha_8,\alpha_5)=-1,\\ &(\alpha_8,\alpha_k)=0 \qquad \text{for } k\neq 5,8,\\ &(\alpha_j,\alpha_j)=2 \qquad \text{for } j=1,\dots,8.\end{aligned}$$

Then, as it is well-known [5], the highest root δ satisfies the conditions:

$$\begin{aligned}&(\alpha_1,\delta)=1,\\ &(\alpha_j,\delta)=0 \quad \text{for } j=2,\dots,8,\end{aligned}$$

and hence it has the form

$$\delta=2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7+3\alpha_8, \qquad ht(\delta)=29=h-1.$$

Let us consider the set of positive roots decreasing in height. We denote

$$\beta_j:=\alpha_1+\alpha_2+\cdots+\alpha_j \quad \text{for } j=1,2,\dots,8.$$

Then we see that

(1) One root for the each height in the interval $[h-1,h-b]$=$[29,24]$, namely,

$$\delta,\ \delta-\beta_1,\ \delta-\beta_2,\ \delta-\beta_3,\ \delta-\beta_4, \text{ and } \delta-\beta_5.$$

(2) Two roots for the each height in the interval $[h-b-1,h-2\,b+2]$=$[23,20]$, namely,

$$\begin{aligned}&\delta-\beta_6\,, && \delta-(\beta_5+\alpha_8)\,;\\ &\delta-\beta_7\,, && \delta-(\beta_6+\alpha_8)\,;\\ &\delta-\beta_8\,, && \delta-(\beta_6+\alpha_5+\alpha_8)\,;\\ &\delta-(\beta_8+\alpha_5)\,, && \delta-(\beta_6+\alpha_4+\alpha_5+\alpha_8)\,.\end{aligned}$$

Note that this is due to the fact that node 5 in the Dynkin diagram for E_8 is the branch node, and $b=6$.

(3) Three roots for height $h-2b+1=19$, namely,

$$\delta-(\beta_8+\alpha_5+\alpha_6),\quad \delta-(\beta_8+\alpha_4+\alpha_5),\quad \delta-(\beta_6+\alpha_3+\alpha_4+\alpha_5+\alpha_8).$$

Hence, from Kostant's characterization of exponents [2] we have

$$m_7=h-1-b=23,\quad m_6=h-2\,b+1=19,$$

and from the duality we have

$$m_2 = b + 1 = 7, \quad m_3 = 2b - 1 = 11.$$

Note that $m_1 = 1$ and $m_8 = 29 = h - 1$. So,

$$d_2 = m_2 + 1 = b + 2 = 8, \quad d_3 = m_3 + 1 = 2b = 12.$$

We analogously obtain $d_4 = 2b + 2 = 14$. □

3 Main Theorem

For an explicit construction of $W_\mathfrak{g}$-invariant polynomials, see the papers by Mehta [4] and by Macdonald [3].

Let $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$, and $AI = (\nabla I_2, \nabla I)$ for any polynomial I, where

$$\nabla := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_l})\,;$$

clearly, $\deg AI = \deg I + b$.

Theorem 2. *All basic $W_\mathfrak{g}$-invariant polynomials for all algebras $\mathfrak{g}$, except for D_l and G_2, can be obtained by applying A^k for some k to polynomials I_1 and I_3, where $I_3 = \Delta(AI_2)$.*

More precisely, for the root systems A_l, B_l, C_l, and D_l, we have

$$I_{j+1} = \begin{cases} A^j I_1 & \text{for } j = 0, \dots, l-1 \quad \text{for the root systems } A_l,\ B_l, \text{ and } C_l, \\ A^j I_1 & \text{for } j = 0, \dots, l-2 \quad \text{for the root system } D_l, \end{cases}$$

and we obtain all basic polynomials, except for the Pfaffian for D_l.

For the root systems F_4, E_6, E_7 and E_8 the basic invariant polynomials have the form: $A^j I_1$ and $A^k I_3$, where

$$\begin{array}{lll} j = 0, 1 & \text{and } k = 0, 1 & \text{for } F_4; \\ j = 0, 1, 2 & \text{and } k = 0, 1, 2 & \text{for } E_6; \\ j = 0, 1, 2, 3, 4 & \text{and } k = 0, 1 & \text{for } E_7; \\ j = 0, 1, 2, 3 & \text{and } k = 0, 1, 2, 3 & \text{for } E_8. \end{array}$$

Proof. Consider the set of polynomials $A^j I_1$ and $A^k I_3$ for fixed Lie algebra $\mathfrak{g}$ of rank l from the list of Theorem 2. We obtain all l first $W_\mathfrak{g}$-invariant polynomials I_l, except for the Pfaffian for D_l. The explicit formulas in the paper [4] imply that these l polynomials are algebraically independent. So we may take these polynomials, and the Pfaffian for a basis of $W_\mathfrak{g}$-invariant polynomials. □

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