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Finite groups with prime graphs of diameter 5

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Abstract. In this paper we consider a prime graph of finite groups. In particular, we expect finite groups with prime graphs of maximal diameter.

1 Introduction

In this paper, all groups are finite. The prime graph $\Gamma(G)$ of a group G is defined as follows. The vertex set is $\pi(G)$ the set of prime divisors of the order of G, and two distinct primes $r, s \in \pi(G)$ considered as vertices of the graph are adjacent if and only if there exists an element $g \in G$ such that $|g| = rs$.

We say that the graph Γ is isomorphic to the prime graph of the group G if the vertices of the graph Γ can be marked of primes so that the resulting graph Γ' is equal to the graph $\Gamma(G)$.

The structure of a finite group G such that $\Gamma(G)$ is disconnected has been determined by Williams [6]. All simple groups G such that $\Gamma(G)$ is disconnected have been described in [2] and [6].

The complement of the prime graph of a solvable group does not contain triangles that have been proved by Lucido [3]. In [1] were completely classified of a prime graphs of solvable groups. In particular, the graph Γ is isomorphic to $\Gamma(G)$, where G is a solvable group if and only if the complement of Γ is without triangles and 3-colorable. However, the general question remains open.

Question 1. *When a graph Γ is isomorphic to a prime graph of some group?*

In this paper we consider the diameter of $\Gamma(G)$. We write $d(p, q)$ to denote the distance between elements p and q if they are in the same connected component of $\Gamma(G)$. We define the diameter of $\Gamma(G)$ as follows:

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$$\mathrm{diam}(\Gamma(G)) = \max\{d(p,q)|p,q \text{ in the same connected component of } \Gamma(G)\}.$$

Lucido [3] proved that $\mathrm{diam}(\Gamma(G)) \leq 5$. Moreover, she classifies all almost simple groups with property that the diameter of its prime graph is equal to 5.

Theorem 1. *[3, Proposition 12] Let G be an almost simple group such that $\mathrm{diam}(\Gamma(G)) = 5$. Then G has the following structure:*

$G \simeq E_8(q) \cdot H$, where $q = p^n$, $q \equiv 0, 1$ or 4 mod 5, $n = 3^s 5^t$ where $s, t > 0$ and $H = \langle\gamma\rangle$, where γ is a field automorphism of order n.

The main goal of this paper is to describe groups with a prime graph of diameter 5.

A chain graph is a graph without any cycles. Denote by $s(G)$ the number of connected components of $\Gamma(G)$ and by $\pi_i(G), i = 1, \ldots, s(G)$, its ith connected component. If G has even order, then put $2 \in \pi_1(G)$.

Theorem 2. *Let G be a group such that $\mathrm{diam}(\Gamma(G)) = 5$ and a, b_1, b_2, b_3, b_4, c is a chain in $\Gamma(G)$. Then $G = K.S.A$ where K is a nilpotent subgroup, S is a simple group and $A \leq \mathrm{Out}(S)$, $\{a, b_1, b_4, c\} \subseteq \pi(S) \setminus (\pi(K) \cup \pi(A))$. Furthermore one of the following statements holds:*

1. *$\mathrm{diam}(\Gamma(G) \setminus \Gamma(K)) = 5$. In particular, G/K is an almost simple group satisfying Lucido theorem.*

2. *If $b_2 \in \pi(K)$, then $b_3 \in \pi(K) \cup \pi(A)$. The graph $\Gamma(S)$ contains more than 2 connected components and $\{a, b_1, b_4, c\} \cap \pi_1(S) = \varnothing$.*

Remark 1. Note that, when G satisfies the statement 1 of Theorem 2, then G/K satisfies the Lucido theorem. Hence, $G/K \simeq E_8(q).H$, where $q = p^n$, $q \equiv 0, 1$ or 4 mod 5, $n = 3^s 5^t$ where $s, t > 0$ and $H = \langle\gamma\rangle$, where γ is a field automorphism of order n. The existence of groups satisfying the statement 2 of Theorem 2 depends on the validity of the following conjecture.

Conjecture 1. *Let S be a finite simple group with disconnected prime graph, $a, b \in \pi_2(S)$ be different primes and $x, y \in S$ be elements of order a and b respectively. If there exists an irreducible S-module M such that $C_M(x) > 1$, then $C_M(y) > 1$.*

It is easy to see that if the conjecture is true, then statement 2 of Theorem 2 is impossible.

Williams [6] and Kondratev [2] described simple groups G such that $\Gamma(G)$ is disconnected. In particular, from this description follows that S is isomorphic $A_1(q)$, where $3 < q \equiv \pm 1(4)$, or simple group of exceptional Lie type.

2 Definitions and preliminary results

Lemma 1. *[1, Theorem 2]An unlabeled graph F is isomorphic to the prime graph of some solvable group if and only if its complement $\overline{F}$ is 3-colorable and triangle-free.*

Lemma 2. *[3, Proposition 1] Let G be a finite solvable group. If p, q, r are three distinct primes of $\pi(G)$, then G contains an element of order the product of two of these primes.*

Lemma 3. *[3, Lemma 5] Let G be a finite simple group. If $p \in \pi_1(G)$, then $d(2,p) \leq 2$.*

For graph Γ we will denote $\overline{\Gamma}$ the complement of Γ. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. In graph theory this number is called the independence number of a graph. By analogy we denote by $t(2,G)$ the maximal number of vertices in the independent sets of $\Gamma(G)$ containing 2. Let $\omega(G)$ be the spectrum of G; i.e., the set of element orders of G. Denote by $\omega_i(G)$ the set of $n \in \omega(G)$ such that every prime divisor of n belongs to $\pi_i(G)$.

Lemma 4. *[4] Let G be a finite group satisfying the two conditions:*

(a) there exist three primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$;

(b) there exists an odd prime in $\pi(G)$ nonadjacent in $\Gamma(G)$ to the prime 2.

Then there exists a non-abelian simple group S such that $S \leq G = G/K \leq \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$, and one of the following statements holds:

(1) $S \simeq A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2,S) = 3$.

(2) For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2,S) \geq t(2,G)$.

Lemma 5. [5, Lemma 3.6] *Let s and p be distinct primes, a group H be a semidirect product of a normal p-subgroup T and a cyclic subgroup $C = \langle g \rangle$ of order s, and let $[T,g] \neq 1$. Suppose that H acts faithfully on a vector space V of positive characteristic t not equal to p. If the minimal polynomial of g on V does not equal $x^s - 1$, then*

(*i*) *$C_T(g) \neq 1$;*

(*ii*) *T is non-abelian;*

(*iii*) *$p = 2$ and $s = 2^{2^\delta} + 1$ is a Fermat prime.*

Lemma 6. [7, Lemma 2.12] *Let S be a finite simple group of Lie type. If $r, s, t \in \pi(S)$ and $rt, st \in \omega(S)$, but $rs \notin \omega(S)$, then a Sylow t-subgroup of S is not cyclic.*

3 Proof of Theorem 2

Let G be such that $\mathrm{diam}(\Gamma(G)) = 5$, $a, b_1, b_2, b_3, b_4, c \in V(\Gamma(G))$ be a path of length 5. The numbers a, b_2, c is a triangle in $\overline{\Gamma(G)}$. Lemma 2 implies that G is non-solvable. Therefore, $2 \in \pi(G)$. Moreover, since $\mathrm{diam}(\Gamma(G)) > 2$ it follows that $t(2,G) \geq 3$. Using Lemma 4, we get $G \simeq K.S.A$, where K is a solvable radical of G, S is a non-abelian simple group and $A \leq \mathrm{Out}(S)$.

Lemma 7. *$S \not\simeq L_2(q)$ for odd q.*

Proof. Notice, that $\Gamma(L_2(q))$ is a graph with three connected components every of which is complete. In particular, $\Gamma(L_2(q))$ does not contains chains.

The proof is by contradiction. Let $S \simeq L_2(q)$ for some $q = p^n$ where p is odd prime. Assume that $a \in \pi(K)$. Lemma 4 implies that $b_2, b_3, b_4, c \in \pi(S)\setminus\pi(|K||A|)$. Therefore, $\Gamma(S)$ contains a chain of length 4; a contradiction with the fact that $\Gamma(S)$ does not contain chain. By analogy, we can show that $\pi(K)$ does not contain b_1, b_4, c.

Assume that $b_2 \in \pi(K)$. Therefore, $a, b_1 \in \pi((q-1)/2)$ or $c, b_4 \in \pi((q-1)/2)$. Let $r \in \{a, c\} \cap \pi((q-1)/2)$. We have that $L_2(q)$ includes a Frobenius group F with kernel Q of order q and complement K of order $(q-1)/2$. Let $\overline{G} = T \lhd G$ such that $T < K$ and K/T includes a normal Sylow b_2-subgroup H. Let P be a Sylow p-subgroup of $\overline{G}$. Assume that $P < C_{\overline{G}}(H)$. Since H is a normal subgroup of $\overline{G}$ we have that $X = \langle P^G \rangle < C_{\overline{G}}(H)$. We have that $X\overline{K}/\overline{K}$ is a normal subgroup of S. Therefore, $X\overline{K}/\overline{K} = S$ and $r \in \pi(C_{\overline{G}}(H))$; a contradiction. Therefore, P acts no trivial on H. From Frattini's argument it follows that $N_{\overline{G}}(P)$ includes a r-subgroup T such that $T\overline{K}/\overline{K}$ is a Sylow r-subgroup of S. From Lemma 5 it follows that $C_{\overline{G}}(H)$ contains an element of order b_2r; a contradiction. By analogy, we can show that $\pi(K)$ does not contain b_3. Thus $a, b_1, b_2, b_3, b_4, c \in \pi(S.A)$; a contradiction with Theorem 1. □

Lemma 8. *$S \not\simeq Alt_7$.*

Proof. Assume that $S \simeq Alt_7$. Then a Sylow 3-subgroup of S is an elementary abelian group of order 9. Therefore, for every $p \in \pi(K)$ we have $3p \in \omega(G)$. Using this fact it is easy show that $\mathrm{diam}(G) \leq 3$.

□

Lemma 9. *If $t \in \pi(K)$, then $2t \in \omega(G)$.*

Proof. Assume that there exists $t \in \pi(K)$ such that $2t \notin \omega(G)$. Let $R \lhd G$ be such that $\overline{K} = K/R$ includes a normal t-subgroup T. Let $P \in Syl_2(K.S/R)$. We have that P is cyclic or generalized quaternion group. Since PR/R is a Sylow subgroup of S, we see that PR/R is a dihedral group. A finite simple group has a dihedral Sylow 2-subgroup only in the case isomorphic to Alt_7 or $L_2(q)$. From Lemmas 7 and 8 we have that S does not isomorphic $L_2(q)$ and Alt_7. □

Lemma 10. *$b_1, b_4, c, a \notin \pi(K)$.*

Proof. Assume that $c \in \pi(K)$. Lemma 4 implies that $a, b_1, b_2, b_3 \in \pi(S) \setminus (\pi(A) \cap \pi(K))$. Since $\pi_n(S)$, where $n > 1$, is a full component of $\Gamma(S)$, we see that $a, b_1, b_2, b_3 \in \pi_1(S)$. From Lemma 9 it follows that $2c \in \omega(G)$. Therefore, $\omega(G)$ does not contain an element of order $2t$ for $t \in \{a, b_1, b_2\}$. Since $a, b_1, b_2, b_3 \in \pi_1(S)$, we obtain $d_S(2, a) > 3$. It is a contradiction with Lemma 3.

Assume that $b_1 \in \pi(K)$. From Lemma 4 it follows that $b_3, c \in \pi(S)$. If $b_4 \in \pi(S)$, then similar as above, we can get a contradiction. Let $b_4 \in \pi(K)$. From the Frattini argument we have that automorphisms groups of Sylow b_1- and

b_4-subgroups are non-solvable. Therefore, Sylow b_1- and b_2-subgroups are not cyclic, in particular $b_1b_4 \in \omega(K)$; a contradiction. Assume that $b_4 \in \pi(A)$. Hence, $2b_4 \in \omega(S.A)$. Since $S \not\simeq Alt_7$ and $S \not\simeq L_2(q)$, we see that $2b_1 \in \omega(G)$. Thus $a, b_1, 2, b_4, c$ is a chain in $\Gamma(G)$; a contradiction. Therefore, $b_1 \in \pi(K)$. Similarly, it can be shown that $a, b_4 \notin \pi(K)$. □

Lemma 11. $\{b_1, b_4, c, a\} \cap \pi(A) = \varnothing$.

Proof. Similar to Lemma 10. □

Lemma 12. *K is nilpotent.*

Proof. Assume that $\pi(K)$ includes p, q such that a Hall $\{p, q\}$-subgroup H of K does not nilpotent. From the Frattini argument it follows that $N_G(H)$ includes a subgroup T such that $T/N_K(H) \simeq S$. Let $X \lhd N_G(H)$ be such that a Hall $\{p, q\}$-subgroup of $N_K(H)/X$ is not nilpotent but for every $X < Y \lhd N_G(H)$ a Hall $\{p, q\}$-subgroup of $N_K(H)/Y$ is nilpotent. Denote by $\widetilde{\ } : N_G(H) \to N_G(H)/X$ the natural homomorphism. Let $N = N_G(H)$. From definition follows that $\widetilde{N}$ includes a normal t-subgroup T, where $t \in \{p, q\}$. Let $r \in \{p, q\} \setminus \{t\}$, $R \in Syl_r(\widetilde{N})$. From definition of $\widetilde{N}$ follows that R acts no trivial on T. The Frattini argument implies that $N_{\widetilde{G}}(R)/N_{\widetilde{K}(R)}$ includes a subgroup isomorphic S. Let $x, y \in N_{\widetilde{G}}(R)$ be elements of order a and c respectively. Assume that $Z(R) < C_{\widetilde{G}}(x)$. Therefore, $\langle x^{N_{\tilde{G}}(R)} \rangle < C_{\widetilde{G}}(Z(R))$. We know that $\langle x^{N_{\tilde{G}}(R)} \rangle / \langle x^{N_{\tilde{G}}}(R) \rangle \cap N_{\widetilde{K}}(R) \simeq S$. Therefore, $y \in C_{\widetilde{G}}(Z(R))$ and $d(a, c) = 2$; a contradiction. We have $x \notin C_{\widetilde{G}}(Z(R))$. Similar we can show that $x \notin C_{\widetilde{G}}(Z(R))$. Applying Lemma 5 for groups $T.Z(R).\langle x \rangle$ and $T.Z(R).\langle y \rangle$ we obtain $\widetilde{G}$ contains elements of order ta and tc, in particular, $d(a, c) = 2$; a contradiction.

We show that for every $p, q \in \pi(K)$ a Hall $\{p, q\}$-subgroup of K is nilpotent. Therefore, Sylow p- and q-subgroups are commute. Hence, K is nilpotent. □

Lemma 13. *If $b_2 \in \pi(K) \setminus \pi(G/K)$, then $b_3 \in \pi(K) \cup \pi(A)$. The graph $\Gamma(S)$ contains more than 2 connected components and $a, b_1, b_4, c \notin \pi_1(S)$.*

Proof. From Lemmas 10 and 11 follow that $a, b_1, b_4, c \in \pi(S)$. Assume that $c, b_4 \in \pi_1(S)$. Since $d(2, c) = 2$ there exists $h \in \pi(S)$ such that $2h, ch \in \omega(S)$. From Lemma 6 follows that a Sylow h-subgroup of S is not cyclic. Consequently $hb_2 \in \omega(G)$. Thus a, b_1, b_2, h, c is a chain in $\Gamma(G)$, in particular $d(a, c) < 5$; a contradiction. We have $c, b_4 \in \pi_2(S)$. Since $\pi_2(G)$ is a full component of $\Gamma(S)$, we see that $b_3 \in \pi(K) \cup \pi(A)$. Assume that $a, b_1 \in \pi_1(S)$. If $b_3 \in \pi(K)$, then similar as above we can show that $a, c \notin \pi_1(S)$; a contradiction. Therefore, $b_3 \in \pi(A)$. In this case we know that $2b_3 \in \omega(S.A)$. Hence, $a, b_1, 2, b_3, b_4, c$ is a chain of $\Gamma(A.S)$. From Lucido Theorem 1 follows that $a, b_1 \notin \pi_1(S)$; a contradiction. □

The main assertion of the theorem follows from Lemmas 10, 11 and 12. The statements 1) and 2) of Theorem follows from Lemmas 1 and 13.

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