Rota-Baxter operators and Bernoulli polynomials

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Abstract. We develop the connection between Rota-Baxter operators arisen from algebra and mathematical physics and Bernoulli polynomials. We state that a trivial property of Rota-Baxter operators implies the symmetry of the power sum polynomials and Bernoulli polynomials. We show how Rota-Baxter operators equalities rewritten in terms of Bernoulli polynomials generate identities for the latter.

1 Introduction

Given an algebra $A$ and a scalar $\lambda \in F$, where $F$ is a ground field, a linear operator $R: A \to A$ is called a Rota-Baxter operator (RB-operator) on $A$ of weight $\lambda$ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

holds for all $x, y \in A$. The algebra $A$ is called Rota-Baxter algebra. By algebra we mean a vector space endowed a bilinear not necessarily associative product.

The notion of Rota-Baxter operator was introduced by G. Baxter [6] in 1960 as formal generalization of integration by parts formula (when $\lambda = 0$) and then developed by G.-C. Rota [30] and others [5], [9].

In 1980s, the deep connection between constant solutions of the classical Yang-Baxter equation from mathematical physics and Rota-Baxter operators of weight zero on a semisimple finite-dimensional Lie algebra was discovered [7], [31]. Further, the connection of Rota-Baxter operators with the associative Yang-Baxter equation was found [4], [12], [28].

To the moment, applications of Rota-Baxter operators in symmetric polynomials, quantum field renormalization, Loday algebras, shuffle algebra etc. were found [4], [5], [10], [11], [17], [18], [19]. The notion of Rota-Baxter operator is useful in such branch of number theory as multiple zeta function [13], [35].

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In 1966, J. Miller found an interesting connection between Rota-Baxter operators and the power sum polynomials [25] over a field of characteristic zero. We start with an algebra $A$ which is unital and power-associative (it means that every one-generated subalgebra is associative). Let $R$ be a Rota-Baxter operator on $A$ of weight $-1$. Denote by $1$ the unit of $A$ and put $a = R(1)$. For each $n \in \mathbb{N}$, define a polynomial $F_n(x) \in \mathbb{Q}[x]$ by the equalities

$$F_n(m) = \sum_{j=1}^{m} j^n.$$ 

Then $R(a^n) = F_n(a)$.

In 2010, O. Ogievetsky and V. Schechtman restated this connection to find a new proof of the Schlömilch-Ramanujan formula [29]. In 2017, the author reproved the connection formula to apply for Rota-Baxter operators of nonzero weight on the matrix algebra [16].

Our goal is to develop this connection. There exist several different proofs of the symmetry of the power sum polynomials

$$F_n(y) = (-1)^{n+1} F_n(-1 - y)$$

and the symmetry of Bernoulli polynomials

$$B_n(x) = (-1)^n B_n(1 - x)$$

involving infinite series, generating functions or some special identities [22], [26], [33]. In section 2, we prove that both symmetries follow from the trivial property of Rota-Baxter operators: let $P$ be an RB-operator of weight $-1$, then the operator $(\text{id} - P)$ is so.

In section 3, we show how identities concerned Rota-Baxter operators rewritten in terms of Bernoulli polynomials and Bernoulli numbers generate a plenty of identities for both of them. In particular, we find a symmetric expression for the product $B_i(x)B_j(x)B_k(x)$ and count the sum

$$\sum_{\substack{i+j+k=n \\ i,j,k>0}} B_i(x)B_j(x)B_k(x),$$

where $B_s(x) = B_s(x)/s$ is the divided Bernoulli polynomial. The approach for counting the same sum for usual (not divided) Bernoulli polynomials was developed in [20]. About the products of Bernoulli polynomials and Bernoulli numbers see also [3], [8], [14].

2 Symmetry of the power sum polynomials

Statement 1 ([18]). Let $P$ be an RB-operator of weight $\lambda$. Then

a) the operator $-P - \lambda \text{id}$ is an RB-operator of weight $\lambda$,

b) the operator $\lambda^{-1}P$ is an RB-operator of weight 1, provided $\lambda \neq 0$. 
Given an algebra $A$, let us define a map $\phi$ on the set of all RB-operators on $A$ as $\phi(P) = -P - \lambda(P) \text{id}$. It is clear that $\phi^2$ coincides with the identity map.

Let $F_n(m) = \sum_{j=1}^{m} j^n$ for natural $n, m$. Bernoulli polynomials $B_n(x)$ are connected with the power sum polynomials in the following way:

$$F_n(m) = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}.$$  

(2)

**Statement 2 ([16], [25], [29]).** Let $A$ be a unital power-associative algebra, $R$ be an RB-operator on $A$ of weight $\lambda$, $a = R(1)$. Then $R(a^n) = (-\lambda)^n F_n(-a/\lambda)$ for all $n \in \mathbb{N}$. In particular, $R(a^n) = F_n(a)$ for all $n \in \mathbb{N}$ provided that $\lambda = -1$.

Let us show how the trivial property of Rota-Baxter operators from Statement 1 a) implies the symmetry of the power sum polynomials and the symmetry of Bernoulli polynomials.

**Lemma 1.** Let $A$ be a unital power-associative algebra, $R$ be an RB-operator on $A$ of weight $-1$, $a = R(1)$ and $b = \phi(R)(1) = 1 - a$. For all positive natural $n$, we have

$$R(a^n) - a^n = (-1)^{n+1} (\phi(R)(b^n) - b^n).$$  

(3)

**Proof.** From

$$(-1)^{n+1} (\phi(R)(b^n) - b^n) = (-1)^{n+1} (-R(b^n)) = R((-b)^n) = R((a-1)^n),$$

we conclude that it is enough to state $R((a-1)^n) = R(a^n) - a^n$. We prove the last equality by induction on $n$. For $n = 1$, we get the true equality $\frac{a^2-a}{2} = \frac{a^2-a}{2}$. Suppose that this holds true for all natural numbers less than $n$. Now we rewrite $R((a-1)^{n+1})$ by (1) and the induction hypothesis

$$R((a-1)^{n+1}) = R((a-1)^n(a-1)) = R((a-1)^n R(1)) - R((a-1)^n)$$

$$= R((a-1)^n) R(1) - R((a-1)^n)) + R((a-1)^n) - R((a-1)^n)$$

$$= (R(a^n) - a^n)a - R(R(a^n) - a^n).$$  

(4)

Again by (1), we calculate

$$R(R(a^n)) = R(R(a^n) \cdot 1) = R(a^n) R(1) - R(a^n R(1)) + R(a^n \cdot 1)$$

$$= R(a^n) a - R(a^{n+1}) + R(a^n).$$  

(5)

Substituting (5) in (4) gives us the proof of the inductive step. □

**Theorem 1.** Let $n$ be a positive natural number. Then

a) $F_n(y) = (-1)^{n+1} F_n(-1 - y)$ for all $y$,

b) $B_n(x) = (-1)^n B_n(1 - x)$ for all $x$. 

Proof. a) Let us consider a unital power-associative algebra $A$ with a Rota-Baxter operator $R$ on $A$ of weight $\lambda = -1$. Put $a = R(1)$. We may consider the free (unital) associative RB-algebra of weight $-1$ generated by $1$ [18] instead of $A$. Actually it is the polynomial algebra $F[x]$ with $x = a$. Define $Q = \phi(R) = \text{id} - R$ and $b = Q(1) = 1 - a$. Applying Statement 2 to the formula (3), we get

$$F_n(a - 1) = (-1)^{n+1}F_n(-a),$$

which gives a).

b) It follows from a) via (2). □

3 Product of two Bernoulli polynomials

For any $n$,

$$F_n(m) = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} B_j m^{n+1-j},$$

where $B_j$ is Bernoulli number.

Let us show how Rota-Baxter operators could generate a plenty of identities for Bernoulli numbers and Bernoulli polynomials. Let $A$ be a power-associative algebra and $R$ be a Rota-Baxter operator of weight $-1$ on $A$, $a = R(1)$. Consider the equality

$$R(a^n)R(a^m) = R(R(a^n)a^m + a^n R(a^m) - a^{n+m}), \quad n, m \in \mathbb{N}.$$}

The left-hand side of (7) by (2) and Statement 2 is equal to

$$R(a^n)R(a^m) = B_{n+1}(a+1)B_{m+1}(a+1) - B_{m+1}B_{n+1}(a+1)$$

$$- B_{n+1}B_{m+1}(a+1) + B_{n+1}B_{m+1},$$

where $B_n(x) = B_n(x)/n$ and $B_n = B_n/n$.

Let us write down the right-hand side of (7) by (2), (6), and Statement 2,

$$R(R(a^n)a^m + a^n R(a^m) - a^{n+m})$$

$$= \sum_{i=0}^{n}(-1)^{n-i} \frac{1}{n+1} \binom{n+1}{n-i} B_{n-i}(B_{m+2+i}(a+1) - B_{m+2+i})$$

$$+ \sum_{j=0}^{m}(-1)^{m-j} \frac{1}{m+1} \binom{m+1}{m-j} B_{m-j}(B_{n+2+j}(a+1) - B_{n+2+j})$$

$$- B_{n+m+1}(a+1) + B_{n+m+1}.$$}

Comparing (8) and (9), we get the identity

$$B_i(x)B_j(x) - B_iB_j = \sum_{l \geq 0} \left( \frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l}(B_{i+j-2l}(x) - B_{i+j-2l}).$$

Here $i = n + 1 \geq 1$, $j = m + 1 \geq 1$ and $x = a + 1$. 
Up to constant, the equality (10) coincides with the famous identity

\begin{equation}
B_i(x)B_j(x) = \sum_{l \geq 0} \left( \frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l}B_{i+j-2l}(x) + \frac{(-1)^{i-1}(i-1)!(j-1)!}{(i+j)!} B_{i+j}
\end{equation}

known at least since 1923 [27].

**Remark 1.** Writing down (7) on the first power of $a$, we get the identity

\begin{equation}
B_{n+m} + \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{n-k} B_{n-k}B_{m+k+1}
+ \frac{1}{m+1} \sum_{l=0}^{m} \binom{m+1}{m-l} B_{m-l}B_{n+l+1} = 0
\end{equation}

discovered by T. Agoh in 1988 [1].

**Remark 2.** Let us sum (11) for $i + j = N \geq 2$ and $i = 1, \ldots, N-1$:

\begin{align*}
\sum_{i+j=N \atop i,j>0} B_i(x)B_j(x) &= \sum_{i=1}^{N-1} \left( \frac{1}{i} + \frac{1}{N-i} \right) B_N(x) \\
&+ \sum_{i+j=N \atop i,j>0} \sum_{l>0} \left( \frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l}B_{N-2l}(x) \\
&+ \frac{B_N}{N(N-1)} \sum_{i=1}^{N-1} \frac{(-1)^{i-1}(i-1)!(N-1-i)!}{(N-2)!} \\
&= 2H_{N-1}B_N(x) + 2 \sum_{i=1}^{N-1} B_{2l}B_{N-2l}(x) \frac{1}{2l} \sum_{i=1}^{N-1} \frac{i-1}{2l-1} \\
&+ \frac{B_N}{N(N-1)} \sum_{p=0}^{N-2} \frac{(-1)^p}{N-2-p} \\
&= 2H_{N-1}B_N(x) + 2 \sum_{i=1}^{N-1} \left( \frac{N+1}{2l} \right) B_{2l}B_{N-2l}(x) + \frac{2B_N}{N} \\
&= 2H_{N-1}B_N(x) + \frac{2}{N} \sum_{l=1}^{N-1} \left( \frac{N+1}{2l} \right) B_{2l}B_{N-2l}(x) + \frac{2B_N}{N} \\
&= 2H_{N-1}B_N(x) + \frac{2}{N} \sum_{k=1}^{N} \binom{N}{k} B_kB_{N-k}(x) \\
&+ B_{N-1}(x),
\end{align*}

(13)

where $H_i = 1 + 1/2 + \cdots + 1/i$. We have used the equality (14) from [32]

\begin{equation}
\sum_{r=0}^{n} (-1)^r \binom{n}{r} = (1 + (-1)^n) \frac{n+1}{n+2}.
\end{equation}

(14)
Thus, we got in (13) the identity found by I. Gessel in 2005 [14] (see also [34]),

\[
\frac{N}{2} \left( -B_{N-1}(x) + \sum_{k=1}^{N-1} B_k(x)B_{N-k}(x) \right)
= \sum_{k=1}^{N} \binom{N}{k} B_kB_{N-k}(x) + H_{N-1}B_N(x). \tag{15}
\]

By the same strategy, we can compute

\[
\sum_{k=0}^{N} B_k(x)B_{N-k}(x) = \frac{2}{N+2} \sum_{t\geq 0} \left( \frac{N+2}{2t+2} \right) B_{2t}B_{N-2t}(x), \tag{16}
\]

which is the identity obtained by D. Kim et al. in 2012 [21] (see also [2]).

The case \(x=0\) for (15) implies the famous identity of H. Miki [24] (1978)

\[
\sum_{k=2}^{N-2} B_kB_{N-k} = \sum_{k=2}^{N-2} \binom{N}{k} B_kB_{N-k} + 2H_NB_N \tag{17}
\]


\[
(N+2) \sum_{k=2}^{N-2} B_kB_{N-k} = 2 \sum_{k=2}^{N-2} \binom{N+2}{k} B_kB_{N-k} + N(N+1)B_N. \tag{18}
\]

## 4 Product of three Bernoulli polynomials

We may also produce other identities involving the products of three, four etc. Bernoulli numbers. To do this, it is enough to consider the equality

\[
R(a^n)R(a^m)R(a^l) = R(R(a^n)R(a^l)a^m + R(a^m)R(a^l)a^n + R(a^s)R(a^m)a^l
- R(a^n)a^m+l - R(a^m)a^{n+l} - R(a^l)a^{n+m} + a^{n+m+l}) \tag{19}
\]

and the same equalities for four, five etc. multipliers (see the formulas in [17]).

Let us derive the explicit identity which follows from (19).

**Theorem 2.** The following identity holds for all \(i, j, k > 0,\)

\[
B_i(x)B_j(x)B_k(x) = \sum_{q,t\geq 0} B_{2q}B_{2t-2q} \left[ \binom{i+j-2q}{2t-2q} \frac{1}{i+j-2q} \left( \frac{1}{i} \binom{i}{2q} + \frac{1}{j} \binom{j}{2q} \right) \\
+ \binom{i+k-2q}{2t-2q} \frac{1}{i+k-2q} \left( \frac{1}{i} \binom{i}{2q} + \frac{1}{k} \binom{k}{2q} \right) \\
+ \binom{j+k-2q}{2t-2q} \frac{1}{j+k-2q} \left( \frac{1}{j} \binom{j}{2q} + \frac{1}{k} \binom{k}{2q} \right) \right] \\
\times B_{i+j+k-2t}(x) - \frac{(-1)^j}{ij \binom{i+j}{i}} B_{i+j}B_k(x) - \frac{(-1)^k}{ik \binom{i+k}{i}} B_{i+k}B_j(x) \\
- \frac{(-1)^k}{jk \binom{j+k}{j}} B_{j+k}B_i(x) - \frac{1}{2} B_{i+j+k-2}(x) + \text{const}. \tag{20}
\]
Proof. Let \( i = n + 1 \geq 2, j = m + 1 \geq 2, k = l + 1 \geq 2 \) and \( x = a + 1 \). We calculate the left-hand side of (19) as

\[
(B_i(x) - B_i)(B_j(x) - B_j)(B_k(x) - B_k) = B_i(x)B_j(x)B_k(x) - (B_iB_j(x)B_k(x) + B_jB_iB_k(x) + B_kB_iB_j(x) + B_jB_kB_i(x)) - B_iB_jB_k(x) - B_iB_jB_k - B_iB_jB_i.
\]

The last term on the right-hand side of (19) equals

\[
B_{i+j+k-2}(x) - B_{i+j+k-2}.
\]

We also have

\[
-R(R(a^n)a^{m+l} + R(a^m)a^{n+l} + R(a^l)a^{n+m})
\]

\[
= -\sum_{q \geq 0} \left( \frac{1}{i} \left( \frac{i}{2q} \right) + \frac{1}{j} \left( \frac{j}{2q} \right) + \frac{1}{k} \left( \frac{k}{2q} \right) \right) B_{2q}B_{i+j+k-1-2q}(x) + B_{i}B_{j+k-1}(x)
\]

\[
+ B_{j}B_{i+k-1}(x) + B_{k}B_{i+j-1}(x) - \frac{3}{2} B_{i+j+k-2}(x) + \text{const}.
\]

We write down

\[
R(R(a^n)a^m)a^l = \frac{1}{n+1} \sum_{p=0}^{n} (-1)^{n-p} \binom{n+1}{n-p} B_{n-p} \frac{1}{m+p+2}
\]

\[
\times \sum_{s=0}^{p+m+1} (-1)^{m+p+1-s} \binom{m+p+2}{m+p+1-s} B_{m+p+1-s}(B_{t+s+2}(x) - B_{t+s+2}).
\]

We want to transform (24) to the form

\[
\sum_{q \geq 0} \frac{1}{i} \left( \frac{i}{2q} \right) B_{2q} \sum_{t \geq 0} \frac{1}{i+j-2q} \left( \frac{i+j-2q}{2t} \right) B_{2t}B_{i+j+k-2t-2q}(x) + \text{const}.
\]

By exchange \( n - p = 2q \) in (24), we get the summand

\[
\frac{1}{2} R(R(a^{n+m})a^l).
\]

Further, by exchange \( m + n - 2q + 1 - s = 2t \), we have the additional summands

\[
\frac{1}{2} \sum_{q \geq 0} \frac{1}{i} \left( \frac{i}{2q} \right) B_{2q}(B_{i+j+k-1-2q}(x) - B_{i+j+k-1-2q})
\]

\[
- \frac{1}{2} B_i(B_{j+k-1}(x) - B_{j+k-1}).
\]
If we let $2q$ be equal $i$, up to a constant we get the summand

$$-B_i \sum_{t=0}^{[\frac{(j-1)/2}]} \frac{1}{j} \left( \frac{j}{2t} \right) B_{2t}B_{j+k-2t}(x)$$

$$= -B_i \sum_{t=0}^{j} \frac{1}{j} \left( \frac{j}{2t} \right) B_{2t}B_{j+k-2t}(x) + B_iB_jB_k(x). \quad (28)$$

Finally, letting $2t$ be equal $j$, we get (25) and the additional summand

$$(B_k - B_k(x)) \sum_{q \geq 0} \frac{1}{i} \left( \frac{i}{2q} \right) B_{2q}B_{i+j-2q}. \quad (29)$$

Applying the formula

$$R(R(a^n)a^m) = R(R(a^n)a^m) + R(R(a^m)a^n) - R(a^{n+m})a^j,$$

summing all such six expressions, by the formulas (21)–(29) we prove the statement. We have rewritten the sum of (29) and the analogue of (29) for $j$ by (11), the sum equals

$$(B_k - B_k(x)) \left( B_iB_j + \frac{(-1)^i(i-1)!(j-1)!}{(i+j)!}B_{i+j} \right).$$

Theorem is proved. □

**Corollary 1.** For all $i, j, k > 0$, we have

$$\int_0^x B_i(y)B_j(y)B_k(y) \, dy$$

$$= \sum_{q,t \geq 0} B_{2q}B_{2t-2q} \left[ \left( \frac{i+j-2q}{2t-2q} \right) \frac{1}{i+j-2q} \left( \frac{1}{i} \left( \frac{i}{2q} \right) + \frac{1}{j} \left( \frac{j}{2q} \right) \right) \right.$$

$$+ \left( \frac{i+k-2q}{2t-2q} \right) \frac{1}{i+k-2q} \left( \frac{1}{i} \left( \frac{i}{2q} \right) + \frac{1}{k} \left( \frac{k}{2q} \right) \right) \right.$$

$$+ \left( \frac{j+k-2q}{2t-2q} \right) \frac{1}{j+k-2q} \left( \frac{1}{j} \left( \frac{j}{2q} \right) + \frac{1}{k} \left( \frac{k}{2q} \right) \right) \left] \frac{B_{i+j+k+1-2t}(x)}{i+j+k-2t} \right.$$

$$- \frac{1}{ijk} \left( \frac{(-1)^j}{i+j+k} \right) B_{i+j}B_{k+1}(x) + \frac{(-1)^k}{i+k} B_{i+k}B_{j+1}(x) + \frac{(-1)^k}{j+k} B_{j+k}B_{i+1}(x) \right.$$

$$- \frac{B_{i+j+k-1}(x)}{2(i+j+k-2)} + \text{const}. \quad (30)$$

**Proof.** It follows from (20) and the formula $\int_0^x B_n(y) \, dy = B_{n+1}(x) - B_{n+1}$. □

Let us denote by $H_{n,s} = 1 + 1/2^s + \ldots + 1/n^s$. So, $H_n = H_{n,1}$.
Corollary 2. For all $N \geq 3$, we have
\[
\frac{1}{3!} \sum_{i+j+k=N, i,j,k>0} B_i(x)B_j(x)B_k(x) = \sum_{t>0} \left( \frac{N-1}{2t} \right) B_{N-2t}(x) \left( B_{2t}(H_{N-1} - H_{2t}) + \frac{1}{2t} \sum_{i,j=2t, i,j>0} B_iB_j \right) \\
- \frac{1}{12} \left( \frac{N-1}{2} \right) B_{N-2}(x) + \frac{H^2_n - H_{n,2}}{2} B_N(x) + \text{const.} \quad (31)
\]

Proof. We apply the formula (20) for all $i+j+k+N \geq 3$ and $i,j,k>0$. Let us compute the summation of the right-hand side of (20) divided by 6 for $q>0$:

\[
A = \sum_{t \geq 0 \land q>0} B_{2q}B_{2t-2q} \sum_{i=1}^{N-2} \frac{1}{i} \left( \frac{N-1}{2q} \right) \sum_{j=1}^{N-i-1} \frac{1}{i+j-2q} \left( \frac{i+j-2q}{2t-2q} \right) B_{N-2t}(x) \\
= \sum_{t>q>0} B_{2q}B_{2t-2q} \sum_{i=1}^{N-2} \frac{1}{i} \left( \frac{N-1}{2q} \right) \left( \left( \frac{N-2q-1}{2t} \right) - \left( \frac{N-2q}{2t} \right) \right) B_{N-2t}(x) \\
+ \sum_{t>0} B_{2t} \sum_{i=1}^{N-2} \frac{1}{i} \left( \frac{N-1}{2q} \right) \left( H_{N-1-2q} - H_{i-2t} \right) B_{N-2t}(x) \\
= \sum_{t>q>0} B_{2q}B_{2t-2q}B_{N-2t}(x) \left( \left( \frac{N-1-2q}{2t} \right) \left( \frac{N-2}{2q} \right) - \left( \frac{2t-1}{2q-1} \right) \left( \frac{N-2}{2t} \right) \right) \\
+ \sum_{t>0} B_{2t} \left[ \left( \frac{N-2}{2t} \right) H_{N-1-2t} - \left( \frac{N-2}{2t} \right) H_{N-2t-2} \right. \\
\left. + \frac{1}{2t} \left( \frac{N-2}{2t} \right) - \frac{1}{2t} \right] B_{N-2t}(x) \\
= \sum_{t>q>0} \left( \frac{N-1}{2t} \right) \left( \frac{2t-1}{2q} \right) B_{2q}B_{2t-2q}B_{N-2t}(x) \\
+ \sum_{t>0} \frac{1}{2t} \left( \frac{N-1}{2t} \right) B_{2t}B_{N-2t}(x) - \sum_{t>0} \frac{1}{2t} B_{2t}B_{N-2t}(x) \\
= \sum_{t>0} \left( \frac{N-1}{2t} \right) B_{N-2t}(x) \left( \frac{B_{2t}}{2t} + \sum_{t>q>0} \left( \frac{2t-1}{2q} \right) B_{2q}B_{2t-2q} \right) \\
- \sum_{t>0} \frac{1}{2t} B_{2t}B_{N-2t}(x) \\
= \sum_{t>0} \left( \frac{N-1}{2t} \right) B_{N-2t}(x) \sum_{q>0} \frac{1}{2t} \left( \frac{2t}{2q} \right) B_{2q}B_{2t-2q} - \sum_{t>0} \frac{1}{2t} B_{2t}B_{N-2t}(x). \quad (32)
\]
By the Miki identity (17), we may rewrite (32) as

$$A = \sum_{t \geq 0} \binom{N - 1}{2t} B_{N-2t}(x) \left( -H_{2t-1}B_{2t} + \frac{1}{2!} \sum_{i+j=2t, \ i,j>0} B_iB_j \right)$$

$$- \sum_{q>0} \frac{1}{2q} B_{2q}B_{N-2q}(x). \quad (33)$$

Above we have used the equalities

$$\sum_{j=1}^{N-i-1} \frac{1}{j} \binom{i+j-2q}{2t-2q} = \frac{1}{2t-2q} \sum_{j=1}^{N-i-1} \binom{i+j-2q-1}{2t-2q-1}$$

$$= \frac{1}{2t-2q} \sum_{s=i-2q}^{N-2q-2} \binom{s}{2t-2q-1}$$

$$= \frac{1}{2t-2q} \left( \binom{N-2q-1}{2t-2q} - \binom{i-2q}{2t-2q} \right);$$

$$\sum_{i=1}^{N-2} \binom{i-1}{2q-1} H_{i-2q} = \sum_{j=1}^{N-2q} \frac{1}{j} \sum_{i=j+2q}^{N-2} \binom{i-1}{2q-1}$$

$$= \sum_{j=1}^{N-2q} \frac{1}{j} \left( \binom{N-2}{2q} - \binom{j+2q-1}{2q} \right)$$

$$= \binom{N-2}{2q} H_{N-2q-2} - \frac{1}{2q} \sum_{j=1}^{N-2q} \binom{j+2q-1}{2q-1}$$

$$= \binom{N-2}{2q} H_{N-2q-2} - \frac{1}{2q} \left( \binom{N-2}{2q} - 1 \right).$$

The summation of the right-hand side of (20) for $q = 0$ gives us

$$3 \sum_{t \geq 0} B_{2t} \sum_{k=1}^{N-2} \sum_{i+j=N-k \atop i,j>0} \binom{i+j}{2t} \frac{1}{i+j} \left( \frac{1}{i} + \frac{1}{j} \right) B_{N-2t}(x)$$

$$= 3 \sum_{t \geq 0} B_{2t} \sum_{k=1}^{N-2} \binom{N-k}{2t} \sum_{i=1}^{N-k-1} \frac{1}{N-k} \left( \frac{1}{i} + \frac{1}{N-k-i} \right) B_{N-2t}(x)$$

$$= 6 \sum_{t \geq 0} B_{2t} \sum_{k=1}^{N-2} \binom{N-k}{2t} \frac{H_{N-k-1}}{N-k} B_{N-2t}(x) + 6B_N(x) \sum_{k=1}^{N-2} \frac{H_{N-k-1}}{N-k}$$

$$= 6 \sum_{t \geq 0} B_{2t} \left( \binom{N-1}{2t} \left( H_{N-1} - \frac{1}{2t} \right) B_{N-2t}(x)\right)$$

$$+ 3 \left( H_{N-1}^2 - H_{N-1,2} \right) B_N(x). \quad (34)$$
Here we have used the formulas (see [15, p. 279–280])

\[
\sum_{s=1}^{n-1} \binom{s}{m} H_s = \binom{n}{m+1} \left( H_n - \frac{1}{m+1} \right), \quad \sum_{s=1}^{n} \frac{H_s}{s} = \frac{1}{2} (H_n^2 + H_{n,2}).
\]

The sum of the first three summands from the fourth line of (20) by (14) gives

\[
-3 \sum_{k=1}^{N-2} B_{N-k} B_k(x) \frac{1}{(N-k)(N-k-1)} \sum_{i=1}^{N-k-1} \frac{(-1)^i}{(N-i-2)}
= 3 \sum_{k=1}^{N-2} \frac{(1 + (-1)^{N-k})}{(N-k)(N-k-1)} \frac{N-k-1}{N-k} B_{N-k} B_k(x)
= 6 \sum_{0 < t < N/2} \frac{1}{2t} B_{2t} B_{N-2t}(x).
\number{35}
\]

Finally, for the last non-constant term of (20) we have

\[
\sum_{i=1}^{N-2} (N-1-i) B_{N-2}(x) = \binom{N-1}{2} B_{N-2}(x).
\number{36}
\]

The formulas (33)–(36) imply (31).

In [20], authors studied the expressions

\[
S_{\geq 0}(r, n) = \sum_{i_1 + \cdots + i_r = n} B_{i_1}(x) \cdots B_{i_r}(x),
\number{37}
\]

where the sum runs over all nonnegative integers \(i_1, \ldots, i_r\) and \(r \geq 1\). Moreover, in [20] it was shown that

\[
S_{\geq 0}(r, n) = \sum_{k=1}^{n} B_k(x) \left( \frac{\binom{n+r}{k}}{n+r} \sum_{i_1 + \cdots + i_r = n-k+1} (B_{i_1}(1) \cdots B_{i_r}(1)}
- B_{i_1} \cdots B_{i_r} \right) \right) + \text{const},
\number{38}
\]

and the formula for the last constant term was given. The simple idea lies behind such kind of decomposition: polynomials \(1 = B_0(x), B_1(x), \ldots, B_n(x)\) form a linear basis of the vector space \(\text{Span}(1, x, x^2, \ldots, x^n)\). Also, \(B_k(x)' = kB_{k-1}(x)\), the property which is not fulfilled for the polynomials \(1, B_1(x), \ldots, B_n(x)\), and so the approach from [20] could not be applied directly for calculating the analogues of \(S_{\geq 0}(r, n)\) or \(S_{>0}(r, n)\) for the sum of products \(B_{i_1}(x) \cdots B_{i_r}(x)\).

Let us decompose \(S_{\geq 0}(r, n) = \sum_{k=0}^{n} \alpha_k B_{n-k}(x)\). In [20], it was shown that \(\alpha_1 = 0\).

It is easy to see from (38) that all odd coefficients \(\alpha_{2s+1}\) are zero.
Remark 3. The formula (38) applied for \( r = 2 \) up to a constant coincides with (16). Due to (38), we have

\[
S_{\geq 0}(3, n) = \sum_{k=1}^{n} B_k(x) \left( \sum_{a+b+c=n-k+1} B_a(1)B_b(1)B_c(1) - B_aB_bB_c \right)
\]

\[
= -\frac{2}{n+3} \sum_{t\geq 0} \left( \frac{n+3}{n-2t} \left( \sum_{a+b+c=2t+1} B_aB_bB_c \right) B_{n-2t}(x) \right)
\]

\[
= \left( \frac{n+2}{2} \right) B_n(x) + \frac{1}{4} \left( \frac{n+2}{4} \right) B_{n-2}(x)
\]

\[
+ \frac{3}{n+3} \sum_{t\geq 2} \left( \frac{n+3}{n-2t} B_{n-2t}(x) \sum_{q=0}^{t} B_{2q}B_{2t-2q} \right).
\]

(39)

If we calculate the sum \( S_{\geq 0}(3, n) \) due to the formula (20), we will get the same as (39) modulo the equality (16).

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