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Rota-Baxter operators and Bernoulli polynomials

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Abstract. We develop the connection between Rota-Baxter operators arisen from algebra and mathematical physics and Bernoulli polynomials. We state that a trivial property of Rota-Baxter operators implies the symmetry of the power sum polynomials and Bernoulli polynomials. We show how Rota-Baxter operators equalities rewritten in terms of Bernoulli polynomials generate identities for the latter.

1 Introduction

Given an algebra A and a scalar $\lambda \in F$, where F is a ground field, a linear operator $R\colon A \to A$ is called a Rota-Baxter operator (RB-operator) on A of weight λ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \tag{1}$$

holds for all $x, y \in A$. The algebra A is called Rota-Baxter algebra. By algebra we mean a vector space endowed a bilinear not necessarily associative product.

The notion of Rota-Baxter operator was introduced by G. Baxter [6] in 1960 as formal generalization of integration by parts formula (when $\lambda = 0$) and then developed by G.-C. Rota [30] and others [5], [9].

In 1980s, the deep connection between constant solutions of the classical Yang--Baxter equation from mathematical physics and Rota-Baxter operators of weight zero on a semisimple finite-dimensional Lie algebra was discovered [7], [31]. Further, the connection of Rota-Baxter operators with the associative Yang-Baxter equation was found [4], [12], [28].

To the moment, applications of Rota-Baxter operators in symmetric polynomials, quantum field renormalization, Loday algebras, shuffle algebra etc. were found [4], [5], [10], [11], [17], [18], [19]. The notion of Rota-Baxter operator is useful in such branch of number theory as multiple zeta function [13], [35].

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In 1966, J. Miller found an interesting connection between Rota-Baxter operators and the power sum polynomials [25] over a field of characteristic zero. We start with an algebra A which is unital and power-associative (it means that every one-generated subalgebra is associative). Let R be a Rota-Baxter operator on A of weight -1. Denote by 1 the unit of A and put $a = R(1)$. For each $n \in \mathbb{N}$, define a polynomial $F_n(x) \in \mathbb{Q}[x]$ by the equalities

$$F_n(m) = \sum_{j=1}^{m} j^n .$$

Then $R(a^n) = F_n(a)$.

In 2010, O. Ogievetsky and V. Schechtman restated this connection to find a new proof of the Schlömilch-Ramanujan formula [29]. In 2017, the author reproved the connection formula to apply for Rota-Baxter operators of nonzero weight on the matrix algebra [16].

Our goal is to develop this connection. There exist several different proofs of the symmetry of the power sum polynomials

$$F_n(y) = (-1)^{n+1} F_n(-1-y)$$

and the symmetry of Bernoulli polynomials

$$B_n(x) = (-1)^n B_n(1-x)$$

involving infinite series, generating functions or some special identities [22], [26], [33]. In section 2, we prove that both symmetries follow from the trivial property of Rota-Baxter operators: let P be an RB-operator of weight -1, then the operator $(\mathrm{id} - P)$ is so.

In section 3, we show how identities concerned Rota-Baxter operators rewritten in terms of Bernoulli polynomials and Bernoulli numbers generate a plenty of identities for both of them. In particular, we find a symmetric expression for the product $B_i(x)B_j(x)B_k(x)$ and count the sum

$$\sum_{\substack{i+j+k=n \\ i,j,k>0}} \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x) ,$$

where $\mathcal{B}_s(x) = B_s(x)/s$ is the divided Bernoulli polynomial. The approach for counting the same sum for usual (not divided) Bernoulli polynomials was developed in [20]. About the products of Bernoulli polynomials and Bernoulli numbers see also [3], [8], [14].

2 Symmetry of the power sum polynomials

Statement 1 ([18]). *Let P be an RB-operator of weight λ. Then*

a) *the operator $-P - \lambda\,\mathrm{id}$ is an RB-operator of weight λ,*

b) *the operator $\lambda^{-1}P$ is an RB-operator of weight 1, provided $\lambda \neq 0$.*

Given an algebra A, let us define a map ϕ on the set of all RB-operators on A as $\phi(P) = -P - \lambda(P)\,\mathrm{id}$. It is clear that ϕ^2 coincides with the identity map.

Let $F_n(m) = \sum\limits_{j=1}^{m} j^n$ for natural n, m. Bernoulli polynomials $B_n(x)$ are connected with the power sum polynomials in the following way:

$$F_n(m) = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}\,. \tag{2}$$

Statement 2 ([16], [25], [29]). *Let A be a unital power-associative algebra, R be an RB-operator on A of weight λ, $a = R(1)$. Then $R(a^n) = (-\lambda)^{n+1}F_n(-a/\lambda)$ for all $n \in \mathbb{N}$. In particular, $R(a^n) = F_n(a)$ for all $n \in \mathbb{N}$ provided that $\lambda = -1$.*

Let us show how the trivial property of Rota-Baxter operators from Statement 1 a) implies the symmetry of the power sum polynomials and the symmetry of Bernoulli polynomials.

Lemma 1. *Let A be a unital power-associative algebra, R be an RB-operator on A of weight -1, $a = R(1)$ and $b = \phi(R)(1) = 1 - a$. For all positive natural n, we have*

$$R(a^n) - a^n = (-1)^{n+1}(\phi(R)(b^n) - b^n)\,. \tag{3}$$

Proof. From

$$(-1)^{n+1}(\phi(R)(b^n) - b^n) = (-1)^{n+1}(-R)(b^n) = R((-b)^n) = R((a-1)^n)\,,$$

we conclude that it is enough to state $R((a-1)^n) = R(a^n) - a^n$. We prove the last equality by induction on n. For $n = 1$, we get the true equality $\frac{a^2-a}{2} = \frac{a^2-a}{2}$. Suppose that this holds true for all natural numbers less than n. Now we rewrite $R((a-1)^{n+1})$ by (1) and the induction hypothesis

$$\begin{aligned} R((a-1)^{n+1}) &= R((a-1)^n(a-1)) = R((a-1)^nR(1)) - R((a-1)^n) \\ &= R((a-1)^n)R(1) - R(R((a-1)^n)) + R((a-1)^n) - R((a-1)^n) \\ &= (R(a^n) - a^n)a - R(R(a^n) - a^n)\,. \end{aligned} \tag{4}$$

Again by (1), we calculate

$$\begin{aligned} R(R(a^n)) &= R(R(a^n)\cdot 1) = R(a^n)R(1) - R(a^nR(1)) + R(a^n \cdot 1) \\ &= R(a^n)a - R(a^{n+1}) + R(a^n)\,. \end{aligned} \tag{5}$$

Substituting (5) in (4) gives us the proof of the inductive step. □

Theorem 1. *Let n be a positive natural number. Then*

a) *$F_n(y) = (-1)^{n+1}F_n(-1-y)$ for all y,*

b) *$B_n(x) = (-1)^nB_n(1-x)$ for all x.*

Proof. a) Let us consider a unital power-associative algebra A with a Rota-Baxter operator R on A of weight $\lambda = -1$. Put $a = R(1)$. We may consider the free (unital) associative RB-algebra of weight -1 generated by 1 [18] instead of A. Actually it is the polynomial algebra $F[x]$ with $x = a$. Define $Q = \phi(R) = \mathrm{id} - R$ and $b = Q(1) = 1 - a$. Applying Statement 2 to the formula (3), we get

$$F_n(a-1) = (-1)^{n+1} F_n(-a)\,,$$

which gives a).

b) It follows from a) via (2). □

3 Product of two Bernoulli polynomials

For any n,

$$F_n(m) = \frac{1}{n+1}\sum_{j=0}^{n}(-1)^j\binom{n+1}{j}B_j m^{n+1-j}\,, \tag{6}$$

where B_j is Bernoulli number.

Let us show how Rota-Baxter operators could generate a plenty of identities for Bernoulli numbers and Bernoulli polynomials. Let A be a power-associative algebra and R be a Rota-Baxter operator of weight -1 on A, $a = R(1)$. Consider the equality

$$R(a^n)R(a^m) = R(R(a^n)a^m + a^n R(a^m) - a^{n+m})\,, \quad n, m \in \mathbb{N}. \tag{7}$$

The left-hand side of (7) by (2) and Statement 2 is equal to

$$\begin{aligned} R(a^n)R(a^m) = \mathcal{B}_{n+1}(a+1)\mathcal{B}_{m+1}(a+1) - \mathcal{B}_{m+1}\mathcal{B}_{n+1}(a+1) \\ - \mathcal{B}_{n+1}\mathcal{B}_{m+1}(a+1) + \mathcal{B}_{n+1}\mathcal{B}_{m+1}\,, \end{aligned} \tag{8}$$

where $\mathcal{B}_n(x) = B_n(x)/n$ and $\mathcal{B}_n = B_n/n$.

Let us write down the right-hand side of (7) by (2), (6), and Statement 2,

$$\begin{aligned} & R(R(a^n)a^m + a^n R(a^m) - a^{n+m}) \\ & = \sum_{i=0}^{n}(-1)^{n-i}\frac{1}{n+1}\binom{n+1}{n-i}B_{n-i}(\mathcal{B}_{m+2+i}(a+1) - \mathcal{B}_{m+2+i}) \\ & \quad + \sum_{j=0}^{m}(-1)^{m-j}\frac{1}{m+1}\binom{m+1}{m-j}B_{m-j}(\mathcal{B}_{n+2+j}(a+1) - \mathcal{B}_{n+2+j}) \\ & \quad - \mathcal{B}_{n+m+1}(a+1) + \mathcal{B}_{n+m+1}\,. \end{aligned} \tag{9}$$

Comparing (8) and (9), we get the identity

$$\mathcal{B}_i(x)\mathcal{B}_j(x) - \mathcal{B}_i\mathcal{B}_j = \sum_{l\geq 0}\left(\frac{1}{i}\binom{i}{2l} + \frac{1}{j}\binom{j}{2l}\right)B_{2l}(\mathcal{B}_{i+j-2l}(x) - \mathcal{B}_{i+j-2l})\,. \tag{10}$$

Here $i = n+1 \geq 1$, $j = m+1 \geq 1$ and $x = a+1$.

Up to constant, the equality (10) coincides with the famous identity

$$\mathcal{B}_i(x)\mathcal{B}_j(x) = \sum_{l\geq 0}\left(\frac{1}{i}\binom{i}{2l} + \frac{1}{j}\binom{j}{2l}\right) B_{2l}\mathcal{B}_{i+j-2l}(x) + \frac{(-1)^{i-1}(i-1)!(j-1)!}{(i+j)!}B_{i+j} \quad (11)$$

known at least since 1923 [27].

Remark 1. Writing down (7) on the first power of a, we get the identity

$$B_{n+m} + \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{n-k}B_{n-k}B_{m+k+1} + \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{m-l}B_{m-l}B_{n+l+1} = 0 \quad (12)$$

discovered by T. Agoh in 1988 [1].

Remark 2. Let us sum (11) for $i+j=N\geq 2$ and $i=1,\dots,N-1$:

$$\begin{aligned}\sum_{\substack{i+j=N\\ i,j>0}}\mathcal{B}_i(x)\mathcal{B}_j(x) &= \sum_{i=1}^{N-1}\left(\frac{1}{i}+\frac{1}{N-i}\right)\mathcal{B}_N(x)\\
&+ \sum_{\substack{i+j=N\\ i,j>0}}\sum_{l>0}\left(\frac{1}{i}\binom{i}{2l}+\frac{1}{j}\binom{j}{2l}\right)B_{2l}\mathcal{B}_{N-2l}(x)\\
&+ \frac{B_N}{N(N-1)}\sum_{i=1}^{N-1}\frac{(-1)^{i-1}(i-1)!(N-1-i)!}{(N-2)!}\\
&= 2H_{N-1}\mathcal{B}_N(x) + 2\sum_{l=1}^{[\frac{N-1}{2}]}B_{2l}\mathcal{B}_{N-2l}(x)\frac{1}{2l}\sum_{i=1}^{N-1}\binom{i-1}{2l-1}\\
&+ \frac{B_N}{N(N-1)}\sum_{p=0}^{N-2}\frac{(-1)^p}{\binom{N-2}{p}}\\
&= 2H_{N-1}\mathcal{B}_N(x) + \frac{2}{N}\sum_{l=1}^{[\frac{N-1}{2}]}\binom{N}{2l}\mathcal{B}_{2l}B_{N-2l}(x) + \frac{2\mathcal{B}_N}{N}\\
&= 2H_{N-1}\mathcal{B}_N(x) + \frac{2}{N}\sum_{k=1}^{N}\binom{N}{k}\mathcal{B}_k B_{N-k}(x)\\
&+ B_{N-1}(x)\,, \end{aligned} \quad (13)$$

where $H_i = 1+1/2+\cdots+1/i$. We have used the equality (14) from [32]

$$\sum_{r=0}^{n}(-1)^r\Big/\binom{n}{r} = (1+(-1)^n)\frac{n+1}{n+2}\,. \quad (14)$$

Thus, we got in (13) the identity found by I. Gessel in 2005 [14] (see also [34]),

$$\frac{N}{2}\left(-B_{N-1}(x)+\sum_{k=1}^{N-1}\mathcal{B}_k(x)\mathcal{B}_{N-k}(x)\right) = \sum_{k=1}^{N}\binom{N}{k}\mathcal{B}_k B_{N-k}(x)+H_{N-1}B_N(x)\,. \quad (15)$$

By the same strategy, we can compute

$$\sum_{k=0}^{N} B_k(x)B_{N-k}(x)=\frac{2}{N+2}\sum_{t\geq 0}\binom{N+2}{2t+2}B_{2t}B_{N-2t}(x)\,, \quad (16)$$

which is the identity obtained by D. Kim et al. in 2012 [21] (see also [2]).

The case $x=0$ for (15) implies the famous identity of H. Miki [24] (1978)

$$\sum_{k=2}^{N-2}\mathcal{B}_k\mathcal{B}_{N-k}=\sum_{k=2}^{N-2}\binom{N}{k}\mathcal{B}_k\mathcal{B}_{N-k}+2H_N\mathcal{B}_N \quad (17)$$

and for (16) it implies the identity of Yu. Matiyasevich [23] (1997)

$$(N+2)\sum_{k=2}^{N-2}B_kB_{N-k}=2\sum_{k=2}^{N-2}\binom{N+2}{k}B_kB_{N-k}+N(N+1)B_N\,. \quad (18)$$

4 Product of three Bernoulli polynomials

We may also produce other identities involving the products of three, four etc. Bernoulli numbers. To do this, it is enough to consider the equality

$$R(a^n)R(a^m)R(a^l)=R\big(R(a^n)R(a^l)a^m+R(a^m)R(a^l)a^n+R(a^n)R(a^m)a^l - R(a^n)a^{m+l}-R(a^m)a^{n+l}-R(a^l)a^{n+m}+a^{n+m+l}\big) \quad (19)$$

and the same equalities for four, five etc. multipliers (see the formulas in [17]).

Let us derive the explicit identity which follows from (19).

Theorem 2. *The following identity holds for all $i,j,k>0$,*

$$\begin{aligned}\mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x)=\sum_{q,t\geq 0}B_{2q}B_{2t-2q}\Bigg[&\binom{i+j-2q}{2t-2q}\frac{1}{i+j-2q}\left(\frac{1}{i}\binom{i}{2q}+\frac{1}{j}\binom{j}{2q}\right)\\ &+\binom{i+k-2q}{2t-2q}\frac{1}{i+k-2q}\left(\frac{1}{i}\binom{i}{2q}+\frac{1}{k}\binom{k}{2q}\right)\\ &+\binom{j+k-2q}{2t-2q}\frac{1}{j+k-2q}\left(\frac{1}{j}\binom{j}{2q}+\frac{1}{k}\binom{k}{2q}\right)\Bigg]\\ &\times\mathcal{B}_{i+j+k-2t}(x)\\ &-\frac{(-1)^j}{ij\binom{i+j}{i}}B_{i+j}\mathcal{B}_k(x)-\frac{(-1)^k}{ik\binom{i+k}{i}}B_{i+k}\mathcal{B}_j(x)\\ &-\frac{(-1)^k}{jk\binom{j+k}{j}}B_{j+k}\mathcal{B}_i(x)-\frac{1}{2}\mathcal{B}_{i+j+k-2}(x)+\text{const}\,. \end{aligned} \quad (20)$$

Proof. Let $i = n+1 \geq 2$, $j = m+1 \geq 2$, $k = l+1 \geq 2$ and $x = a+1$. We calculate the left-hand side of (19) as

$$\begin{aligned}(\mathcal{B}_i(x) - \mathcal{B}_i)(\mathcal{B}_j(x) - \mathcal{B}_j)(\mathcal{B}_k(x) - \mathcal{B}_k) = {} & \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x) - (\mathcal{B}_i\mathcal{B}_j(x)\mathcal{B}_k(x) \\ & + \mathcal{B}_j\mathcal{B}_i(x)\mathcal{B}_k(x) + \mathcal{B}_k\mathcal{B}_i(x)\mathcal{B}_j(x)) \\ & + (\mathcal{B}_i\mathcal{B}_j\mathcal{B}_k(x) + \mathcal{B}_i\mathcal{B}_k\mathcal{B}_j(x) \\ & + \mathcal{B}_j\mathcal{B}_k\mathcal{B}_i(x)) - \mathcal{B}_i\mathcal{B}_j\mathcal{B}_k\,. \qquad (21)\end{aligned}$$

The last term on the right-hand side of (19) equals

$$\mathcal{B}_{i+j+k-2}(x) - \mathcal{B}_{i+j+k-2}\,. \qquad (22)$$

We also have

$$\begin{aligned}&-R\big(R(a^n)a^{m+l} + R(a^m)a^{n+l} + R(a^l)a^{n+m}\big) \\ &\quad = -\sum_{q\geq 0}\left(\frac{1}{i}\binom{i}{2q} + \frac{1}{j}\binom{j}{2q} + \frac{1}{k}\binom{k}{2q}\right) B_{2q}\mathcal{B}_{i+j+k-1-2q}(x) + \mathcal{B}_i\mathcal{B}_{j+k-1}(x) \\ &\qquad + \mathcal{B}_j\mathcal{B}_{i+k-1}(x) + \mathcal{B}_k\mathcal{B}_{i+j-1}(x) - \frac{3}{2}\mathcal{B}_{i+j+k-2}(x) + \text{const}\,. \qquad (23)\end{aligned}$$

We write down

$$\begin{aligned}R(R(R(a^n)a^m)a^l) = {} & \frac{1}{n+1}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+1}{n-p}B_{n-p}\frac{1}{m+p+2} \\ \times & \sum_{s=0}^{p+m+1}(-1)^{m+p+1-s}\binom{m+p+2}{m+p+1-s}B_{m+p+1-s}(\mathcal{B}_{l+s+2}(x) - \mathcal{B}_{l+s+2})\,. \qquad (24)\end{aligned}$$

We want to transform (24) to the form

$$\sum_{q\geq 0}\frac{1}{i}\binom{i}{2q}B_{2q}\sum_{t\geq 0}\frac{1}{i+j-2q}\binom{i+j-2q}{2t}B_{2t}\mathcal{B}_{i+j+k-2t-2q}(x) + \text{const}\,. \qquad (25)$$

By exchange $n - p = 2q$ in (24), we get the summand

$$\frac{1}{2}R(R(a^{n+m})a^l)\,. \qquad (26)$$

Further, by exchange $m+n-2q+1-s = 2t$, we have the additional summands

$$\begin{aligned}&\frac{1}{2}\sum_{q\geq 0}\frac{1}{i}\binom{i}{2q}B_{2q}(\mathcal{B}_{i+j+k-1-2q}(x) - \mathcal{B}_{i+j+k-1-2q}) \\ &\qquad\qquad - \frac{1}{2}\mathcal{B}_i(\mathcal{B}_{j+k-1}(x) - \mathcal{B}_{j+k-1})\,. \qquad (27)\end{aligned}$$

If we let $2q$ be equal i, up to a constant we get the summand

$$-\mathcal{B}_i\sum_{t=0}^{[(j-1)/2]}\frac{1}{j}\binom{j}{2t}B_{2t}\mathcal{B}_{j+k-2t}(x)$$
$$=-\mathcal{B}_i\sum_{t\geq 0}\frac{1}{j}\binom{j}{2t}B_{2t}\mathcal{B}_{j+k-2t}(x)+\mathcal{B}_i\mathcal{B}_j\mathcal{B}_k(x)\,. \quad (28)$$

Finally, letting $2t$ be equal j, we get (25) and the additional summand

$$(\mathcal{B}_k-\mathcal{B}_k(x))\sum_{q\geq 0}\frac{1}{i}\binom{i}{2q}B_{2q}\mathcal{B}_{i+j-2q}\,. \quad (29)$$

Applying the formula

$$R\big(R(a^n)R(a^m)a^l\big)=R\big(R(R(a^n)a^m)a^l\big)+R\big(R(R(a^m)a^n)a^l\big)-R\big(R(a^{n+m})a^l\big)\,,$$

summing all such six expressions, by the formulas (21)–(29) we prove the statement. We have rewritten the sum of (29) and the analogue of (29) for j by (11), the sum equals

$$(\mathcal{B}_k-\mathcal{B}_k(x))\left(\mathcal{B}_i\mathcal{B}_j+\frac{(-1)^i(i-1)!(j-1)!}{(i+j)!}B_{i+j}\right).$$

Theorem is proved. □

Corollary 1. *For all $i,j,k>0$, we have*

$$\int_0^x\mathcal{B}_i(y)\mathcal{B}_j(y)\mathcal{B}_k(y)\,dy$$
$$=\sum_{q,t\geq 0}B_{2q}B_{2t-2q}\Bigg[\binom{i+j-2q}{2t-2q}\frac{1}{i+j-2q}\left(\frac{1}{i}\binom{i}{2q}+\frac{1}{j}\binom{j}{2q}\right)$$
$$+\binom{i+k-2q}{2t-2q}\frac{1}{i+k-2q}\left(\frac{1}{i}\binom{i}{2q}+\frac{1}{k}\binom{k}{2q}\right)$$
$$+\binom{j+k-2q}{2t-2q}\frac{1}{j+k-2q}\left(\frac{1}{j}\binom{j}{2q}+\frac{1}{k}\binom{k}{2q}\right)\Bigg]\frac{\mathcal{B}_{i+j+k+1-2t}(x)}{i+j+k-2t}$$
$$-\frac{1}{ijk}\left(\frac{(-1)^j}{\binom{i+j}{i}}B_{i+j}\mathcal{B}_{k+1}(x)+\frac{(-1)^k}{\binom{i+k}{i}}B_{i+k}\mathcal{B}_{j+1}(x)+\frac{(-1)^k}{\binom{j+k}{j}}B_{j+k}\mathcal{B}_{i+1}(x)\right)$$
$$-\frac{\mathcal{B}_{i+j+k-1}(x)}{2(i+j+k-2)}+\text{const}\,. \quad (30)$$

Proof. It follows from (20) and the formula $\int_0^x B_n(y)\,dy=\mathcal{B}_{n+1}(x)-\mathcal{B}_{n+1}$. □

Let us denote by $H_{n,s}=1+1/2^s+\ldots+1/n^s$. So, $H_n=H_{n,1}$.

Corollary 2. *For all $N \geq 3$, we have*

$$\frac{1}{3!}\sum_{\substack{i+j+k=N\\ i,j,k>0}} \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x)$$
$$= \sum_{t>0}\binom{N-1}{2t}\mathcal{B}_{N-2t}(x)\left(\mathcal{B}_{2t}(H_{N-1}-H_{2t}) + \frac{1}{2!}\sum_{\substack{i+j=2t;\\ i,j>0}}\mathcal{B}_i\mathcal{B}_j\right)$$
$$-\frac{1}{12}\binom{N-1}{2}\mathcal{B}_{N-2}(x) + \frac{H_n^2 - H_{n,2}}{2}\mathcal{B}_N(x) + \text{const.} \tag{31}$$

Proof. We apply the formula (20) for all $i+j+k+N \geq 3$ and $i,j,k>0$. Let us compute the summation of the right-hand side of (20) divided by 6 for $q>0$:

$$A = \sum_{\substack{t\geq 0\\ q>0}} B_{2q}B_{2t-2q}\sum_{i=1}^{N-2}\frac{1}{i}\binom{i}{2q}\sum_{j=1}^{N-i-1}\frac{1}{i+j-2q}\binom{i+j-2q}{2t-2q}\mathcal{B}_{N-2t}(x)$$
$$= \sum_{t>q>0} B_{2q}\mathcal{B}_{2t-2q}\sum_{i=1}^{N-2}\frac{1}{i}\binom{i}{2q}\left(\binom{N-2q-1}{2t-2q} - \binom{i-2q}{2t-2q}\right)\mathcal{B}_{N-2t}(x)$$
$$+\sum_{t>0} B_{2t}\sum_{i=1}^{N-2}\frac{1}{i}\binom{i}{2t}(H_{N-1-2q} - H_{i-2t})\mathcal{B}_{N-2t}(x)$$
$$= \sum_{t>q>0}\mathcal{B}_{2q}\mathcal{B}_{2t-2q}\mathcal{B}_{N-2t}(x)\left(\binom{N-1-2q}{2t-2q}\binom{N-2}{2q} - \binom{2t-1}{2q-1}\binom{N-2}{2t}\right)$$
$$+\sum_{t>0}\mathcal{B}_{2t}\Bigg[\binom{N-2}{2t}H_{N-1-2t} - \binom{N-2}{2t}H_{N-2t-2}$$
$$+\frac{1}{2t}\binom{N-2}{2t} - \frac{1}{2t}\Bigg]\mathcal{B}_{N-2t}(x)$$
$$= \sum_{t>q>0}\binom{N-1}{2t}\binom{2t-1}{2q}\mathcal{B}_{2q}\mathcal{B}_{2t-2q}\mathcal{B}_{N-2t}(x)$$
$$+\sum_{t>0}\frac{1}{2t}\binom{N-1}{2t}\mathcal{B}_{2t}\mathcal{B}_{N-2t}(x) - \sum_{t>0}\frac{1}{2t}\mathcal{B}_{2t}\mathcal{B}_{N-2t}(x)$$
$$= \sum_{t>0}\binom{N-1}{2t}\mathcal{B}_{N-2t}(x)\left(\frac{\mathcal{B}_{2t}}{2t} + \sum_{t>q>0}\binom{2t-1}{2q}\mathcal{B}_{2q}\mathcal{B}_{2t-2q}\right)$$
$$-\sum_{t>0}\frac{1}{2t}\mathcal{B}_{2t}\mathcal{B}_{N-2t}(x)$$
$$= \sum_{t>0}\binom{N-1}{2t}\mathcal{B}_{N-2t}(x)\sum_{q>0}\frac{1}{2t}\binom{2t}{2q}\mathcal{B}_{2q}B_{2t-2q} - \sum_{t>0}\frac{1}{2t}\mathcal{B}_{2t}\mathcal{B}_{N-2t}(x)\,. \tag{32}$$

By the Miki identity (17), we may rewrite (32) as

$$A=\sum_{t>0}\binom{N-1}{2t}\mathcal{B}_{N-2t}(x)\left(-H_{2t-1}\mathcal{B}_{2t}+\frac{1}{2!}\sum_{\substack{i+j=2t,\\ i,j>0}}\mathcal{B}_i\mathcal{B}_j\right)$$
$$-\sum_{q>0}\frac{1}{2q}\mathcal{B}_{2q}\mathcal{B}_{N-2q}(x)\,. \quad (33)$$

Above we have used the equalities

$$\begin{aligned}\sum_{j=1}^{N-i-1}\frac{1}{i+j-2q}\binom{i+j-2q}{2t-2q}&=\frac{1}{2t-2q}\sum_{j=1}^{N-i-1}\binom{i+j-2q-1}{2t-2q-1}\\&=\frac{1}{2t-2q}\sum_{s=i-2q}^{N-2q-2}\binom{s}{2t-2q-1}\\&=\frac{1}{2t-2q}\left(\binom{N-2q-1}{2t-2q}-\binom{i-2q}{2t-2q}\right);\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{N-2}\binom{i-1}{2q-1}H_{i-2q}&=\sum_{j=1}^{N-2-2q}\frac{1}{j}\sum_{i=j+2q}^{N-2}\binom{i-1}{2q-1}\\&=\sum_{j=1}^{N-2-2q}\frac{1}{j}\left(\binom{N-2}{2q}-\binom{j+2q-1}{2q}\right)\\&=\binom{N-2}{2q}H_{N-2q-2}-\frac{1}{2q}\sum_{j=1}^{N-2-2q}\binom{j+2q-1}{2q-1}\\&=\binom{N-2}{2q}H_{N-2q-2}-\frac{1}{2q}\left(\binom{N-2}{2q}-1\right).\end{aligned}$$

The summation of the right-hand side of (20) for $q=0$ gives us

$$\begin{aligned}&3\sum_{t\geq0}B_{2t}\sum_{k=1}^{N-2}\sum_{\substack{i+j=N-k\\ i,j>0}}\binom{i+j}{2t}\frac{1}{i+j}\left(\frac{1}{i}+\frac{1}{j}\right)\mathcal{B}_{N-2t}(x)\\&\quad=3\sum_{t\geq0}B_{2t}\sum_{k=1}^{N-2}\binom{N-k}{2t}\sum_{i=1}^{N-k-1}\frac{1}{N-k}\left(\frac{1}{i}+\frac{1}{N-k-i}\right)\mathcal{B}_{N-2t}(x)\\&\quad=6\sum_{t>0}B_{2t}\sum_{k=1}^{N-2}\binom{N-k}{2t}\frac{H_{N-k-1}}{N-k}\mathcal{B}_{N-2t}(x)+6\mathcal{B}_N(x)\sum_{k=1}^{N-2}\frac{H_{N-k-1}}{N-k}\\&\quad=6\sum_{t>0}\mathcal{B}_{2t}\binom{N-1}{2t}\left(H_{N-1}-\frac{1}{2t}\right)\mathcal{B}_{N-2t}(x)\\&\qquad+3\big(H_{N-1}^2-H_{N-1,2}\big)\mathcal{B}_N(x)\,. \qquad (34)\end{aligned}$$

Here we have used the formulas (see [15, p. 279–280])

$$\sum_{s=1}^{n-1}\binom{s}{m}H_s=\binom{n}{m+1}\left(H_n-\frac{1}{m+1}\right),\quad \sum_{s=1}^{n}\frac{H_s}{s}=\frac{1}{2}\left(H_n^2+H_{n,2}\right).$$

The sum of the first three summands from the fourth line of (20) by (14) gives

$$\begin{aligned}-3\sum_{k=1}^{N-2}B_{N-k}\mathcal{B}_k(x)&\frac{1}{(N-k)(N-k-1)}\sum_{i=1}^{N-k-1}\frac{(-1)^i}{\binom{N-k-2}{i-1}}\\ &=3\sum_{k=1}^{N-2}\frac{(1+(-1)^{N-k})}{(N-k)(N-k-1)}\frac{N-k-1}{N-k}B_{N-k}\mathcal{B}_k(x)\\ &=6\sum_{0<t<N/2}\frac{1}{2t}\mathcal{B}_{2t}\mathcal{B}_{N-2t}(x)\,. \qquad (35)\end{aligned}$$

Finally, for the last non-constant term of (20) we have

$$\sum_{i=1}^{N-2}(N-1-i)\mathcal{B}_{N-2}(x)=\binom{N-1}{2}\mathcal{B}_{N-2}(x)\,. \qquad (36)$$

The formulas (33)–(36) imply (31). □

In [20], authors studied the expressions

$$S_{\geq 0}(r,n)=\sum_{i_1+\dots+i_r=n}B_{i_1}(x)\dots B_{i_r}(x)\,, \qquad (37)$$

where the sum runs over all nonnegative integers $i_1,\dots,i_r$ and $r\geq 1$. Moreover, in [20] it was shown that

$$\begin{aligned}S_{\geq 0}(r,n)=\sum_{k=1}^{n}B_k(x)\Bigg(\frac{\binom{n+r}{k}}{n+r}\sum_{i_1+\dots+i_r=n-k+1}&\big(B_{i_1}(1)\dots B_{i_r}(1)\\ &-B_{i_1}\dots B_{i_r}\big)\Bigg)+\text{const}\,, \qquad (38)\end{aligned}$$

and the formula for the last constant term was given. The simple idea lies behind such kind of decomposition: polynomials $1=B_0(x),B_1(x),\dots,B_n(x)$ form a linear basis of the vector space $\mathrm{Span}(1,x,x^2,\dots,x^n)$. Also, $(B_k(x))'=kB_{k-1}(x)$, the property which is not fulfilled for the polynomials $1,\mathcal{B}_1(x),\dots,\mathcal{B}_n(x)$, and so the approach from [20] could not be applied directly for calculating the analogues of $S_{\geq 0}(r,n)$ or $S_{>0}(r,n)$ for the sum of products $\mathcal{B}_{i_1}(x)\dots\mathcal{B}_{i_r}(x)$.

Let us decompose $S_{\geq 0}(r,n)=\sum\limits_{k=0}^{n}\alpha_k B_{n-k}(x)$. In [20], it was shown that $\alpha_1=0$. It is easy to see from (38) that all odd coefficients α_{2s+1} are zero.

Remark 3. The formula (38) applied for $r = 2$ up to a constant coincides with (16).

Due to (38), we have

$$S_{\geq 0}(3,n) = \sum_{k=1}^{n} B_k(x)\left(\frac{\binom{n+3}{k}}{n+3}\sum_{a+b+c=n-k+1} B_a(1)B_b(1)B_c(1) - B_aB_bB_c\right)$$
$$= -\frac{2}{n+3}\sum_{t\geq 0}\binom{n+3}{n-2t}\left(\sum_{a+b+c=2t+1} B_aB_bB_c\right)B_{n-2t}(x)$$
$$= \binom{n+2}{2}B_n(x) + \frac{1}{4}\binom{n+2}{4}B_{n-2}(x)$$
$$+ \frac{3}{n+3}\sum_{t\geq 2}\binom{n+3}{n-2t}B_{n-2t}(x)\sum_{q=0}^{t} B_{2q}B_{2t-2q}\,. \quad (39)$$

If we calculate the sum $S_{\geq 0}(3,n)$ due to the formula (20), we will get the same as (39) modulo the equality (16).

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