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Crystallographic actions on Lie groups and post-Lie algebra structures

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Abstract. This survey on crystallographic groups, geometric structures on Lie groups and associated algebraic structures is based on a lecture given in the Ostrava research seminar in 2017.

1 Introduction

Crystallographic groups and crystallographic actions already have a long history. They were studied more than hundred years ago as the symmetry groups of crystals in three-dimensional Euclidean space and as wallpaper groups in two-dimensional Euclidean space. Such groups are discrete and cocompact subgroups of the group of isometries of a Euclidean space. After Hilbert asked in 1900 about Euclidean crystallographic groups in his 18th problem, Bieberbach solved this question in 1910. Since then Euclidean crystallographic structures are quite well understood, and several other types of crystallographic structures have been considered, such as almost-crystallographic and affine crystallographic structures. For the affine case it was expected that the results from the Euclidean case should generalize in a straightforward manner. This, however, turned out to be not the case. The Bieberbach theorems do not hold. In particular, groups admitting an affine crystallographic action need not be virtually abelian. However, it was conjectured that such groups are virtually polycyclic. This became known as Auslander's conjecture. J. Milnor proved several results on affine crystallographic actions in his fundamental paper [47] in 1977. See also the discussion in [41] by W. M. Goldman. This resulted in an active research on affine crystallographic actions and more generally on nil-affine crystallographic actions until today. We want to give a survey on these developments and state some results which we have obtained in this context. An

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important step here is to be able to formulate the problems on the level of Lie algebras and in terms of pre-Lie algebra and post-Lie algebra structures.

This survey is by no means complete and there are several other interesting results in this area, which we do not mention.

2 Euclidean crystallographic actions

Let $E(n)$ denote the isometry group of the Euclidean space $\mathbb{R}^n$. This group is given by matrices as follows,

$$E(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in O_n(\mathbb{R}), v \in \mathbb{R}^n \right\}.$$

The multiplication is the usual matrix multiplication

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & Aw+v \\ 0 & 1 \end{pmatrix}.$$

The translations form a normal subgroup of $E(n)$, given by

$$T(n) = \left\{ \begin{pmatrix} E_n & v \\ 0 & 1 \end{pmatrix} \mid v \in \mathbb{R}^n \right\}.$$

In particular we have $E(n) \cong O_n(\mathbb{R}) \ltimes T(n) \cong O_n(\mathbb{R}) \ltimes \mathbb{R}^n$. The group $E(n)$ acts on $\mathbb{R}^n$ by

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Av+b \\ 1 \end{pmatrix}.$$

More generally, the *affine group of the Euclidean space* $\mathbb{R}^n$, denoted by $A(n)$, is given as follows

$$A(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in GL_n(\mathbb{R}), v \in \mathbb{R}^n \right\}.$$

We will use the following definition of an Euclidean crystallographic group (ECG), which can be found in [4], section 1.

Definition 1. A *Euclidean crystallographic group* Γ is a subgroup of $E(n)$ which is discrete and cocompact, i.e., has compact quotient $\mathbb{R}^n/\Gamma$.

Let Γ be an *ECG*. Then Γ acts *properly discontinuously* on $\mathbb{R}^n$, i.e., for all compact sets $K \subseteq \mathbb{R}^n$ the set of returns

$$\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$$

is finite.

Definition 2. Let Γ be a group acting on $\mathbb{R}^n$ via a homomorphism

$$\rho \colon \Gamma \to E(n).$$

The action is called a *crystallographic action*, if Γ acts properly discontinuously on $\mathbb{R}^n$ and the orbit space $\mathbb{R}^n/\Gamma$ is compact, i.e., Γ acts cocompactly.

Example 1. The group $\Gamma = \mathbb{Z}^n$ acts crystallographically on $\mathbb{R}^n$ by translations.

A homomorphism $\rho\colon \Gamma \to E(n)$ determines a crystallographic action if and only if the kernel of ρ is finite, and the image of ρ is a crystallographic group.

As already said, the study of ECGs has a long history. ECGs in dimension 2 are the 17 wallpaper groups, which have been known for several centuries. However the proof that the list was complete was only given in 1891 by Fedorov, after the much more difficult classification in dimension 3 had been completed by Fedorov and independently by Schönflies in 1891. There are 219 distinct ECGs in dimension 3. All of them are realized as symmetry groups of genuine crystals. Hilbert published in 1900 his famous 23 problems [44]. In the first part of the 18th problem he asked, whether there are only finitely many different crystallographic groups in any dimension. This was answered affirmatively by Bieberbach [9], [10] in 1910. His theorems are usually stated as follows.

Proposition 1 (Bieberbach 1). *Let $\Gamma \leq E(n)$ be an Euclidean crystallographic group. Then Γ contains the translation subgroup $\mathbb{Z}^n$ as a normal subgroup with finite quotient $F = \Gamma/\mathbb{Z}^n$.*

Proposition 2 (Bieberbach 2). *Two Euclidean crystallographic groups in dimension n are isomorphic if and only if they are conjugated in the affine group $A(n) \cong GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$.*

Proposition 3 (Bieberbach 3). *In each dimension there are only finitely many Euclidean crystallographic groups.*

Let $\Gamma \leq E(n)$ be an ECG. By the first Bieberbach Theorem, the translation subgroup is an abelian subgroup of finite index, isomorphic to the full lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Hence we have a short exact sequence

$$1 \to \mathbb{Z}^n \to \Gamma \to F \to 1$$

with a finite group $F \cong \Gamma/\mathbb{Z}^n$ acting by conjugation of $\mathbb{Z}^n$. We obtain a faithful representation

$$F \hookrightarrow \operatorname{Aut}(\mathbb{Z}^n) = GL_n(\mathbb{Z}),$$

so that we may consider F as a finite subgroup in $GL_n(\mathbb{Z})$ up to conjugation. Now $GL_n(\mathbb{Z})$ has only finitely many conjugacy classes of finite subgroups. This was first shown by Jordan, then by Zassenhaus and more generally later by Harish-Chandra for arithmetic groups. Hence we have only finitely many such groups F up to isomorphism with given action of F on $\mathbb{Z}^n$. Furthermore there are only finitely many inequivalent extensions $1 \to \mathbb{Z}^n \to \Gamma \to F \to 1$, since the extension classes are classified by the group

$$H^2(F, \mathbb{Z}^n) \cong H^1(F, \mathbb{Q}^n/\mathbb{Z}^n),$$

which is discrete and compact, hence finite. This yields Bieberbach's third Theorem. Zassenhaus [55] showed that every ECG arises as an exact sequence as above and gave an algorithm yielding the n-dimensional ECGs up to affine equivalence, given the finitely many conjugacy classes of finite subgroups F in $GL_n(\mathbb{Z})$ together with their normalizers.

Proposition 4. *For a given finite group $F \leq GL_n(\mathbb{Z})$ the isomorphism classes of crystallographic groups Γ with conjugacy class represented by F are in bijection with the orbits of the normalizer $N_{GL_n(\mathbb{Z})}(F)$ on the finite group $H^2(F, \mathbb{Z}^n)$.*

In 1978 the classification in dimension 4 was achieved in [12].

Proposition 5 (Zassenhaus et al. 1978). *There are exactly* 4783 *different crystallographic groups in four-dimensional space* $\mathbb{R}^4$.

In 2000 Plesken and Schulz [48] classified all ECGs in dimension 5 and 6. There are 222018 different ECGs in dimension 5 and 28927922 different ECGs in dimension 6.

Example 2. The group $GL_2(\mathbb{Z})$ has exactly 13 different conjugacy classes of finite subgroups, called arithmetic ornament classes. Zassenhaus' algorithm yields 17 ECGs up to isomorphism.

It is easy to see that the 13 arithmetic ornament classes are given as follows:

$$C_1 \cong \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle, \quad C_2 \cong \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle, \quad C_3 \cong \left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle,$$
$$C_4 \cong \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \quad C_6 \cong \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle, \quad D_1 \cong \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle,$$
$$D_1 \cong \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad D_2 \cong \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle,$$
$$D_2 \cong \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle, \quad D_3 \cong \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle,$$
$$D_3 \cong \left\langle \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle, \quad D_4 \cong \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle,$$
$$D_6 \cong \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle.$$

The groups are isomorphic to one of the cyclic groups C_1, C_2, C_3, C_4, C_6, or one of the dihedral groups D_1, D_2, D_3, D_4, D_6. The conjugacy classes are finer than the isomorphism classes. The indices $1, 2, 3, 4, 6$ are no coincidence here. An element $A \in GL_2(\mathbb{Z})$ of finite order has one of the orders $1, 2, 3, 4, 6$. Indeed, if there is an element $A \in GL_2(\mathbb{Z})$ of order n, then $\varphi(n)$, the degree of the irreducible cyclotomic polynomial Φ_n divides 2 by Cayley-Hamilton. But $\varphi(n) \mid 2$ is equivalent to $n = 1, 2, 3, 4, 6$. The wallpaper groups Γ arise from these 13 arithmetic ornament classes by equivalence classes of extensions $1 \to \mathbb{Z}^2 \to \Gamma \to F \to 1$, determined by $H^2(F, \mathbb{Z}^2)$. For each of these classes we can compute this group. In case that $H^2(F, \mathbb{Z}^2) = 0$ the ornament class just yields one extension. This happens in 10 cases. In the other three cases $H^2(F, \mathbb{Z}^2)$ is isomorphic to $\mathbb{Z}/2$, $\mathbb{Z}/2$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$ respectively. This yields $2 + 2 + 4 = 8$ further extensions. Altogether we obtain 18 inequivalent extensions leading to 17 different groups.

Remark 1. Denote by $c(n)$ the number of different ECGs in $E(n)$. Peter Buser [36] showed in 1985, using Gromov's work on almost flat manifolds, the estimate

$$c(n) \leq e^{e^{4n^2}}.$$

This bound seems to be not yet optimal, but I haven't found better estimates. Schwarzenberger [49] has shown that $c(n)$ grows at least as fast as 2^{n^2} and conjectured that this is the exact asymptotic result. This seems to be still open.

By Bieberbach's first Theorem the translation group of an ECG is an abelian subgroup of finite index. Hence every ECG is virtually abelian. We can reformulate the structure results by Bieberbach as follows.

Proposition 6. *The groups admitting a Euclidean crystallographic action are precisely the finitely generated virtually abelian groups. For a given group the crystallographic action is unique up to affine conjugation.*

Definition 3. An Euclidean crystallographic group $\Gamma \leq E(n)$ is called a *Bieberbach group*, if it is torsionfree, i.e., if it acts freely on $\mathbb{R}^n$.

If $M = \mathbb{R}^n/\Gamma$ is a compact complete connected flat Riemannian manifold, then its fundamental group $\pi_1(M) \cong \Gamma$ is a Bieberbach group. Conversely, every flat complete Riemannian manifold M is the quotient $\mathbb{R}^n/\Gamma$ for a subgroup $\Gamma \leq E(n)$ acting freely and properly discontinuously on $\mathbb{R}^n$. This shows the geometric importance of Bieberbach groups. Among the 17 wallpaper groups, there are just 2 Bieberbach groups, namely the fundamental groups of the torus and of the Klein bottle. Among the 219 space groups there are only 10 Bieberbach groups. In dimension $4, 5, 6$ we have $74, 1060, 38746$ Bieberbach groups respectively, so also these numbers grow rapidly.

3 Hyperbolic and spherical crystallographic actions

ECGs have been generalized to non-Euclidean crystallographic groups, namely to spherical and hyperbolic crystallographic groups. We will shortly explain the notions and give a few examples, but we will not attempt to give a survey. Let X be a *space of constant curvature* κ, i.e., a simply-connected complete Riemannian manifold of constant curvature κ up to scaling, together with its isometry group $G = \mathrm{Iso}(X)$. Any space of constant curvature is isomorphic to either the Euclidean space $(\mathbb{E}^n, O_n(\mathbb{R}) \ltimes \mathbb{R}^n)$ with $\kappa = 0$, or to the sphere $(\mathbb{S}^n, O_{n+1}(\mathbb{R}))$ with $\kappa = 1$, or to the hyperbolic space $(\mathbb{H}^n, O^+(n,1))$ with $\kappa = -1$. Here $O^+(n,1)$ is the index 2 subgroup of $O(n,1)$ preserving the two connected components of $\{A \in \mathbb{R}^{n+1} \mid \langle A, A\rangle = -1\}$, where $\langle A, B\rangle$ denotes the standard Lorentzian form on $\mathbb{R}^{n+1}$. Then Definition 1 is generalized as follows.

Definition 4. Let (X, G) be a space of constant curvature. A subgroup $\Gamma \leq G = \mathrm{Iso}(X)$ is called a *crystallographic group*, or CG, if Γ is discrete and X/Γ has finite volume.

Any discrete subgroup $\Gamma \leq \mathrm{Iso}(X)$ has a convex fundamental domain. So for an ECG any fundamental domain is bounded since any unbounded convex domain in Euclidean space has infinite volume. Hence any CG in $\mathbb{E}^n$ is cocompact and thus an ECG. This shows that both definitions coincide for ECGs. Any CG in $\mathbb{S}^n$ is a discrete subgroup of a compact group $O_{n+1}(\mathbb{R})$ and hence finite. So spherical CRs are finite subgroups of $O_{n+1}(\mathbb{R})$. For small n, all finite subgroups of $O_{n+1}(\mathbb{R})$ are classified. For example, any finite subgroup of $O_2(\mathbb{R})$ is either cyclic or dihedral. For higher n this is not the case. A special case of the Margulis lemma implies that for each n, there is a positive integer $m(n)$ such that any finite subgroup of $O_n(\mathbb{R})$ has an abelian subgroup of index $m(n)$, see Corollary 4.2.4 of Thurston's book [51]. The most interesting case of non-Euclidean CGs is the hyperbolic case. Already in dimension 2 there is a continuum of CGs, even of cocompact ones. The latter arise as fundamental groups of closed surfaces of genus $g > 1$. Their totality can be described via Teichmüller theory.

Example 3. Let $X = \mathbb{H}^2$ be the upper half-plane, $G = \mathrm{Iso}(X) \cong \mathrm{PSL}_2(\mathbb{R})$ and Γ be the modular group consisting of transformations of the form

$$z \mapsto \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Then Γ is a non-cocompact hyperbolic CG with $\mathrm{vol}(\mathbb{H}^2/\Gamma) = \frac{\pi}{3}$.

Another example is given by Bianchi groups.

Example 4. Let d be a positive squarefree integer and $K = \mathbb{Q}(\sqrt{-d})$ an imaginary-quadratic number field. Denote by $\mathcal{O}_d$ its ring of integers in K. Let $\Gamma(d) = \mathrm{PSL}_2(\mathcal{O}_d) \subset \mathrm{PSL}_2(\mathbb{C})$. Then $\Gamma(d)$ is a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^3)$, called a Bianchi group. It is a non-compact hyperbolic CG.

In fact, the covolume of $\Gamma(d)$ is given by

$$\mathrm{vol}(\mathbb{H}^3/\Gamma(d)) = \frac{|d_K|^{3/2}}{4\pi^2}\zeta_K(2),$$

where d_K denotes the discriminant of K and $\zeta_K(s)$ denotes the Dedekind zeta function of the base field $K = \mathbb{Q}(\sqrt{-d})$.

There is a general method of constructing arithmetic discrete subgroups of semisimple Lie groups due to Margulis. On the other hand, there exist also non-arithmetic hyperbolic CGs in any dimension [42]. By *Mostow rigidity*, any isomorphism of hyperbolic CGs in $\mathbb{H}^n$ for $n \geq 3$ is induced by a conjugation in the group $\mathrm{Iso}(\mathbb{H}^n)$. This is far from being true for $n = 2$, see above. An important numerical invariant of a hyperbolic CG is its covolume

$$v(\Gamma) = \mathrm{vol}(\mathbb{H}^n/\Gamma).$$

For $n \geq 4$ the set of covolumes is discrete and for $n = 3$ it is a non-discrete closed well-ordered set of order-type ω^ω, where each point has finite multiplicity. The covolumes are bounded from below by a positive constant depending only on n.

There is much more to say for this section, in particular we should mention the classical work of E. Vinberg concerning hyperbolic CGs and hyperbolic reflections groups. The references can be found in the book [54]. We will finish this section by referring to related topics such as *lattices in Lie groups*, *Fuchsian groups* and *Kleinian groups*. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ and a *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

4 Affine and nil-affine crystallographic actions

We are mainly interested in this survey in another generalization of Euclidean crystallographic groups, namely in affine and nil-affine crystallographic groups. Let X be a locally compact topological Hausdorff space and G the group of homeomorphisms of X.

Definition 5. A subgroup Γ of G is called *crystallographic*, if G acts properly discontinuously and cocompactly on X. A continuous action of a group Γ on X is called a *crystallographic action*, if it is properly discontinuous and cocompact.

For X being the affine space $\mathbb{A}^n$ and $G = A(n)$, a crystallographic group Γ is called an ACG, an *affine crystallographic group*. For $X = \mathbb{E}^n$, a group $\Gamma \leq G = E(n)$ acts properly discontinuously on X if and only if Γ is discrete. In general acting properly discontinuously is stronger than being discrete. The group $A(n)$ is a generalization of the Euclidean isometry group $E(n)$ as we have seen in section 2 in the context of the Bieberbach theorems. As in the Euclidean case, torisonfree ACGs arise as fundamental groups of flat manifolds, i.e., of complete compact affinely flat manifolds. A natural question in this context is whether the Bieberbach theorems hold for ACGs. Looking at some examples it is clear that this is not the case.

Example 5. Let k be a fixed integer. The group

$$\Gamma_k = \left\{ \left(\begin{array}{ccc|c} 1 & kc & 0 & ka \\ 0 & 1 & 0 & kb \\ 0 & 0 & 1 & kc \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \;\middle|\; a, b, c \in \mathbb{Z} \right\} \leq A(3)$$

is a 3-dimensional ACG, which is not virtually abelian. Because of

$$\Gamma_k/[\Gamma_k, \Gamma_k] \cong \mathbb{Z}^2 \oplus \mathbb{Z}/k\mathbb{Z}$$

two groups Γ_k and $\Gamma_{k'}$ are isomorphic if and only if $k = k'$. Hence there are infinitely many different ACGs in dimension 3.

Indeed, every abelian subgroup of finite index in Γ_k would necessarily be isomorphic to $\mathbb{Z}^3$, but is is easy to see that Γ_k does not contain such a subgroup. So ACGs need not be virtually abelian. On the other hand, all such examples are virtually solvable and then, being a discrete solvable subgroup of a Lie group with finitely many components, already virtually *polycyclic*. Is this a possible generalization of Bieberbach's First Theorem, i.e., is it true that every ACG is virtually polycyclic?

In other words, is the fundamental group of every complete compact affine manifold virtually polycylic? L. Auslander studied this problem and published a paper [3] in 1964 stating an even more general result, namely that the fundamental group of every complete affine manifold is virtually polycyclic, without the compactness assumption. Unfortunately his proof was in error. Nevertheless the statement later on became widely known as the Auslander conjecture:

Conjecture 1 (Auslander). *Every ACG is virtually polycyclic.*

The history of this conjecture is as follows. In 1977 J. Milnor studied the fundamental groups of flat affine manifolds in his famous paper [47]. He proved that every torsion-free virtually polycyclic group can be realized as the fundamental group of some complete flat affine manifold. Then he conjectured also the converse, namely Auslander's statement that the fundamental group of every complete flat affine manifold is virtually polycyclic. However, Margulis [46] found a counterexample in dimension 3.

Proposition 7 (Margulis). *There exists non-compact complete affine manifolds in dimension* 3 *with a free non-abelian fundamental group of rank* 2*.*

A free non-abelian group cannot be virtually polycyclic. So it is clear that one needs the compactness assumption and Auslander's original claim cannot hold. This led to the formulation of the Auslander conjecture in terms of affine crystallographic groups. Auslander's conjecture is still open, although many special cases are known. Fried and Goldman proved the conjecture in 1983 in dimension $n \leq 3$ [40]. Tomanov [52] proved it in 2016 for $n \leq 5$ and Abels, Margulis and Soifer [1] have worked on several cases for many years. In 2005 they showed that every crystallographic subgroup $\Gamma \leq A(n+2)$ with linear part contained in $O(n,2)$ is virtually polycyclic [2]. They had a proof for Auslander's conjecture in dimension 6 available on the arXiv. However, they have withdrawn it now. In his paper [47] Milnor also asked the following important question.

Question 1 (Milnor 1977). *Does every virtually polycyclic group admit an affine crystallographic action?*

Actually, the original question uses the terminology of left-invariant affine structures on Lie groups, see section 6. Because of some positive evidence this question was also sometimes called the *Milnor conjecture*. A positive answer for both Milnor and Auslander would give a very nice algebraic description of the class of groups admitting an affine crystallographic action. In fact, then this class would be precisely the class of virtually polycyclic groups and we would have a perfect analogue to Proposition 6 concerning the Euclidean case. Moreover, it is known that an affine crystallographic action of a virtually polycyclic group is unique up to conjugation with a polynomial diffeomorphism of $\mathbb{R}^n$.

However, Y. Benoist found a counterexample to Milnor's conjecture in [11] and we provided families of counterexamples in [14], [16]. The counterexamples in [16] are torsion-free nilpotent groups of Hirsch length 11 and nilpotency class 10 not admitting an affine crystallographic action. Hence one needs to replace $A(n)$ by a larger group for such a correspondence to hold. Indeed, other alternatives have been proposed. First it was shown that the group $P(n)$ of polynomial diffeomorphisms of $\mathbb{R}^n$ is a possible alternative. In [37] it was shown that any virtually polycyclic group admits a polynomial crystallographic action of bounded degree. However, this group appears to be too large and does not have such a geometric meaning as $E(n)$ and $A(n)$ have. A more natural generalization of $A(n) = \mathrm{Aff}(\mathbb{R}^n)$ turned out to be the group $\mathrm{Aff}(N) = \mathrm{Aut}(N) \ltimes N$, the group of *nil-affine transformations*, in this context. Here N denotes a connected and simply-connected nilpotent Lie group. For the abelian Lie group $N = \mathbb{R}^n$ we recover the group $A(n)$. We repeat the definition of a crystallographic action for $\mathrm{Aut}(N)$.

Definition 6. A *nil-affine crystallographic action* consists of a representation $\rho : \Gamma \to \mathrm{Aff}(N)$ for some connected and simply connected nilpotent Lie group N letting ρ act properly discontinuously and cocompactly on N. The image $\rho(\Gamma)$ of such an nil-affine crystallographic action will be referred to as an *nil-affine crystallographic group*.

In 2003 K. Dekimpe showed the following result in [38].

Proposition 8. *Every virtually polycyclic group Γ admits a nil-affine crystallographic action $\rho\colon \Gamma \to \mathrm{Aff}(N)$. This action is unique up to conjugation inside of $\mathrm{Aff}(N)$.*

It is now also natural to ask for the converse, i.e., to ask for the Auslander conjecture for nil-affine crystallographic groups.

Conjecture 2 (Generalized Auslander). *Let N be a connected and simply connected nilpotent Lie group and let $\Gamma \subseteq \mathrm{Aff}(N)$ be a group acting crystallographically on N. Then Γ is virtually polycyclic.*

If this conjecture has a positive answer, then we have an analogue of Proposition 6 for nil-affine crystallographic actions. Then the groups admitting a nil-affine crystallographic action would be precisely the virtually polycyclic groups. We have shown in [18] that the generalized Auslander conjecture is true for $n \leq 5$ and that it can be reduced to the ordinary Auslander conjecture in case N is 2-step nilpotent.

5 Simply transitive groups of affine and nil-affine transformations

Affine and nil-affine crystallographic actions of discrete groups are closely related to simply transitive actions by affine and nil-affine transformations of Lie groups.

Definition 7. A group G acts *simply transitively* on $\mathbb{R}^n$ by affine transformations if there is a homomorphism $\rho\colon G \to A(n)$ letting G act on $\mathbb{R}^n$, such that for all $y_1, y_2 \in \mathbb{R}^n$ there is a unique $g \in G$ such that $\rho(g)(y_1) = y_2$.

Such groups are connected and simply connected n-dimensional Lie groups which are homeomorphic to $\mathbb{R}^n$. L. Auslander named such groups *simply transitive groups of affine motions*. He proved that such groups are solvable [5]. We mention the following generalization of this result.

Proposition 9. *Let G be a Lie group which is homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. If G admits a faithful linear representation then G is solvable.*

Proof. Let G be a connected Lie group. By a theorem of Malcev and Iwasawa, G is homeomorphic to $C \times \mathbb{R}^k$ for some k, where C is the maximal compact subgroup of G. If we assume that G is homeomorphic to $\mathbb{R}^n$ then it follows that G has no nontrivial compact subgroup.

Since G has a faithful linear representation, it is the semidirect product $B \ltimes H$ with a reductive group H and a simply connected solvable group B, which is normal in G. This reduces the proof to the case where G is reductive. We have to show that our group is trivial then.

A reductive group G having a faithful linear representation has a compact center Z with semisimple quotient G/Z. So we may assume that G is semisimple and has trivial center. A semisimple group G with trivial center is analytically isomorphic to its adjoint group and hence has a non-trivial compact subgroup unless G is trivial. But since our G has no nontrivial compact subgroup it is trivial. □

Let us explain the connection between crystallographic actions of a discrete group and simply transitive actions of a Lie group. If G is a solvable Lie group admitting a simply transitively action by affine transformations on $\mathbb{R}^n$, then a *cocompact lattice* Γ in G admits an affine crystallographic action. Conversely, if a torsionfree nilpotent group Γ admits an affine crystallographic action via $\rho\colon \Gamma \to A(n)$, then $\rho(\Gamma)$ is unipotent. Hence its *Malcev completion* G_Γ is inside $A(n)$, and acts simply transitively by affine transformations. We have the following result.

Proposition 10. *There is a bijective correspondence between affine crystallographic actions of a finitely generated torsionfree nilpotent group Γ and simply transitive actions by affine transformations of its Malcev completion G_Γ.*

This generalizes to nil-affine crystallographic actions. We say that G admits a simply transitively action by *nil-affine transformations* on N, if there is a homomorphism $\rho\colon G \to \mathrm{Aff}(N)$ letting G act simply transitively on N.

6 Left-invariant affine structures on Lie groups

Milnor formulated his question 1 in terms of left-invariant affine structures on Lie groups. We follow here his article [47].

Definition 8. An *affine* structure (or affinely flat structure) on an n-dimensional manifold M is a collection of coordinate homeomorphisms

$$f_\alpha\colon U_\alpha \to V_\alpha \subseteq \mathbb{R}^n,$$

where the U_α are open sets covering M, and the V_α are open subsets of $\mathbb{R}^n$; whenever $U_\alpha \cap U_\beta \neq \emptyset$, it is required that the change of coordinate homeomorphism

$$f_\beta f_\alpha^{-1}\colon f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

extends to an affine transformation in $A(n) = \mathrm{Aff}(\mathbb{R}^n)$. We call M together with this structure an *affine* manifold, or an *affinely flat manifold*.

A special case of affinely flat manifolds are *Riemannian-flat manifolds*, where the coordinate changes extend to isometries in $E(n)$, i.e., to affine transformations $x \mapsto Ax + b$ with $A \in O_n(\mathbb{R})$.

For surfaces we have the following result by Benzecri [8].

Proposition 11. *A closed surface admits an affine structure if and only if its Euler characteristic vanishes.*

In particular, a closed surface different from the 2-torus or the Klein bottle does not admit any affine structure.

Definition 9. An affine structure on a Lie group G is called *left-invariant* if each left-multiplication map $L(g)\colon G \to G$ is an affine diffeomorphism.

Definition 10. An affine structure on G is called *complete*, if the universal covering $\widetilde{G}$ is affinely diffeomorphic to $\mathbb{R}^n$.

Proposition 12. *There is a canonical bijection between left-invariant complete affine structures on G and simply transitive actions of G on $\mathbb{R}^n$ by affine motions.*

If G admits a left-invariant complete affine structure, then for any discrete group Γ the coset space G/Γ is a complete affinely flat manifold with fundamental group isomorphic to Γ.

Here is Milnor's question in the original context.

Question 2 (Milnor 1977). *Does every solvable n-dimensional Lie group G admit a complete left-invariant affine structure, or equivalently, does the universal covering group $\widetilde{G}$ act simply transitively by affine transformations on $\mathbb{R}^n$?*

Milnor remarked that the answer is positive for 2-step nilpotent and 3-step nilpotent Lie groups and for Lie groups whose Lie algebra admits a non-singular derivation. Such Lie algebras are necessarily nilpotent. Furthermore the answer is positive for all connected and simply connected complex nilpotent Lie groups of dimension $n \leq 7$. However, as we have mentioned above, Benoist gave a counterexample in [11] and we gave families of counterexamples in [14], [16]. The Lie algebras of all such counterexamples here are *filiform nilpotent* Lie algebras. One can verify that all connected and simply-connected filiform nilpotent Lie groups of dimension $n \leq 9$ admit a complete left-invariant affine structure. Hence the minimal dimension for this kind of counterexamples is 10. The result in [16] is the following.

Proposition 13. *There exist families of nilpotent Lie groups of dimension* 10 *and nilpotency class* 9 *not admitting any left-invariant affine structure.*

There are also families of such counterexamples in dimension 11, 12 and 13, but a general result for all dimensions $n \geq 10$ is only conjectured, see [21], but not known.

7 Pre-Lie algebra and post-Lie algebra structures

Several statements from the previous sections can be formulated on the level of Lie algebras in terms of certain compatible algebraic structures on the Lie algebra of the corresponding Lie group. In particular, Milnor's question can be reduced to the level of Lie algebras, namely to pre-Lie algebra structures on Lie algebras and to faithful finite-dimensional representations of Lie algebras. All bijective correspondence mentioned are understood up to suitable equivalence of the structures involved.

Definition 11. A *pre-Lie algebra* $(V, \cdot)$ is a vector space V equipped with a binary operation $(x, y) \mapsto x \cdot y$ such that for all $x, y, z \in V$

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z).$$

If $(V, \cdot)$ is a pre-Lie algebra, then for $x, y \in V$ the binary operation

$$[x, y] := x \cdot y - y \cdot x$$

defines a Lie algebra.

Definition 12. A bilinear product $x \cdot y$ on $\mathfrak{g} \times \mathfrak{g}$ is called a *pre-Lie algebra structure on* $\mathfrak{g}$, if it satisfies

$$\begin{aligned} [x, y] &= x \cdot y - y \cdot x, \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z), \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$. A Lie algebra $\mathfrak{g}$ over a field K is said to *admit a pre-Lie algebra structure*, if there exists a pre-Lie algebra structure on $\mathfrak{g}$.

Example 6. The Heisenberg Lie algebra $\mathfrak{n}_3(K)$ of dimension 3 with basis $\{e_1, e_2, e_3\}$ and Lie bracket $[e_1, e_2] = e_3$ admits a pre-Lie algebra structure, given by

$$e_1 \cdot e_2 = \frac{1}{2} e_3, \quad e_2 \cdot e_1 = -\frac{1}{2} e_3.$$

Denote by $L(x)$ the left multiplication operator given by $L(x)(y) = x \cdot y$. Then the second identity becomes

$$L([x, y]) = [L(x), L(y)].$$

for all $x, y \in \mathfrak{g}$. Hence the left multiplication operators define a $\mathfrak{g}$-module $\mathfrak{g}_L$ by

$$L \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto L(x).$$

Denote by $I \colon \mathfrak{g} \to \mathfrak{g}_L$ the identity map. Then the first identity becomes

$$I([x, y]) = I(x) \cdot y - I(y) \cdot x.$$

Hence the identity map is a 1-cocycle, i.e., $I \in Z^1(\mathfrak{g}, \mathfrak{g}_L)$. We have the following result [13].

Proposition 14. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra. Then $\mathfrak{g}$ admits a pre-Lie algebra structure if and only if there is a n-dimensional $\mathfrak{g}$-module M with nonsingular 1-cocycle in $Z^1(\mathfrak{g}, M)$.*

Example 7. Let $\mathfrak{g}$ be a Lie algebra admitting a nonsingular derivation D. Then $\mathfrak{g}$ admits a pre-Lie algebra structure, given by

$$x \cdot y = D^{-1}([x, D(y)])$$

for all $x, y \in \mathfrak{g}$.

Jacobson [45] proved the following result in 1955.

Proposition 15. *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero admitting a nonsingular derivation. Then $\mathfrak{g}$ is nilpotent.*

This result does not hold for fields of prime characteristic $p > 0$. There are even simple modular Lie algebras of nonclassical type admitting nonsingular derivations, see [7]. This is of interest in the theory of pro-p groups of finite coclass. In general a given Lie algebra need not admit a pre-Lie algebra structure.

Example 8. The Lie algebra $\mathfrak{sl}_2(K)$ over a field K of characteristic zero does not admit a pre-Lie algebra structure.

More generally, we have the following result.

Proposition 16. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field of characteristic zero. Then $\mathfrak{g}$ does not admit a pre-Lie algebra structure.*

Proof. Let $\mathfrak{g}$ be n-dimensional. Suppose that $\mathfrak{g}$ admits a pre-Lie algebra structure. Then we have $I \in Z^1(\mathfrak{g}, \mathfrak{g}_L)$. By Whitehead's first Lemma, $I \in B^1(\mathfrak{g}, \mathfrak{g}_L)$. Hence there exists an $e \in \mathfrak{g}$ with $R(e) = I$, where $R(x)$ denotes the right multiplication operator. The adjoint operators $\operatorname{ad}(x) = L(x) - R(x)$ have trace zero, since $\mathfrak{g}$ is perfect. So do all $L(x)$ and hence all $R(x)$. Then we obtain $n = \operatorname{tr}(I) = \operatorname{tr}(R(e)) = 0$, a contradiction. $\square$

Helmstetter [43] proved more generally that if $\mathfrak{g}$ is *perfect*, i.e., if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}$ does not admit a pre-Lie algebra structure. We have the following canonical bijections (up to suitable equivalence).

Proposition 17. *There is a canonical bijection between left-invariant affine structures on G and pre-Lie algebra structures on $\mathfrak{g}$.*

Proposition 18. *There is a canonical bijection between simply transitive affine actions of G and complete pre-Lie algebra structures on $\mathfrak{g}$.*

Here a pre-Lie algebra structure on $\mathfrak{g}$ is *complete*, if all right multiplications $R(x)$ in $\operatorname{End}(\mathfrak{g})$ are nilpotent. A left-invariant affine structure on G is complete if and only if the corresponding pre-Lie algebra structure on the Lie algebra of G is complete, see [50]. Hence the algebraic analogue of Milnor's question is as follows.

Question 3 (Milnor 1977). *Does every solvable Lie algebra over a field of characteristic zero admit a (complete) pre-Lie algebra structure?*

In the nilpotent case the different versions of Milnor's question are equivalent. By Proposition 10 we obtain also a correspondence to affine crystallographic actions. The counterexamples to Milnor's question are given by n-dimensional nilpotent Lie algebras not admitting a faithful linear representation of degree $n+1$. This is based on the following important observation, see [11].

Proposition 19. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra over a field K of characteristic zero. Suppose that $\mathfrak{g}$ admits a pre-Lie algebra structure. Then $\mathfrak{g}$ admits a faithful linear Lie algebra representation $\varphi\colon \mathfrak{g} \to \mathfrak{gl}_{n+1}(K)$ of degree $n+1$.*

This motivates to study a refinement of Ado's theorem.

Definition 13. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of dimension n. Denote by $\mu(\mathfrak{g})$ the minimal dimension of a faithful linear representation of $\mathfrak{g}$.

By the Ado-Iwasawa theorem, $\mu(\mathfrak{g})$ is always finite. However the proofs for Ado's theorem do not give good upper bounds for $\mu(\mathfrak{g})$. The following result was proved in [17]. Here $p(n)$ denotes the partition function.

Theorem 1. *Let $\mathfrak{g}$ be a k-step nilpotent Lie algebra of dimension n over a field of characteristic zero. Then we have*

$$\mu(\mathfrak{g}) \leq \sum_{j=0}^{k} \binom{n-j}{k-j} p(j) < 3 \cdot \frac{2^n}{\sqrt{n}}.$$

In the general case we have the following result [23].

Theorem 2. *Let $\mathfrak{g}$ be a Lie algebra with r-dimensional solvable radical and nilradical $\mathfrak{n}$ over an algebraically closed field of characteristic zero. Then we have*

$$\mu(\mathfrak{g}) \leq \mu(\mathfrak{g}/\mathfrak{n}) + 3 \cdot \frac{2^r}{\sqrt{r}}.$$

For the 10-dimensional counterexamples to Milnor's question we proved that $12 \leq \mu(\mathfrak{g}) \leq 18$, but we do not know the exact value in all cases. The canonical bijections of Proposition 17 and Proposition 18 can be generalized to nil-affine transformations and post-Lie algebra structures, see [24].

Theorem 3. *Let G and N be connected and simply connected nilpotent Lie groups. Then there exists a simply transitive action by nil-affine transformations of G on N if and only if the corresponding pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ admits a complete post-Lie algebra structure.*

In the classical case $N = \mathbb{R}^n$ a complete post-Lie algebra structure on $(\mathfrak{g}, \mathbb{R}^n)$ is just a complete *pre-Lie algebra structure* on $\mathfrak{g}$. In the other extreme case $G = \mathbb{R}^n$ a complete post-Lie algebra structure on $(\mathbb{R}^n, \mathfrak{n})$ is a complete *LR-structure* on $\mathfrak{n}$, see [22].

The definitions of a post-Lie algebra and a post-Lie algebra structure are as follows [53], [24].

Definition 14. A *post-Lie algebra* $(V, \cdot, \{\,,\})$ is a vector space V over a field K equipped with two K-bilinear operations $x \cdot y$ and $\{x, y\}$, such that $\mathfrak{g} = (V, \{\,,\})$ is a Lie algebra, and

$$\begin{aligned} \{x, y\} \cdot z &= (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z) \\ x \cdot \{y, z\} &= \{x \cdot y, z\} + \{y, x \cdot z\} \end{aligned}$$

for all $x, y, z \in V$.

Note that if $\mathfrak{g}$ is abelian then $(V, \cdot)$ is a pre-Lie algebra. We can associate to a post-Lie algebra $(V, \cdot, \{,\})$ a second Lie algebra $\mathfrak{n} = (V, [\,,])$ via

$$[x, y] := x \cdot y - y \cdot x + \{x, y\}.$$

This Lie bracket satisfies the following identity

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z),$$

i.e., the post-Lie algebra is a *left module* over the Lie algebra $\mathfrak{n}$.

Definition 15. Let $(\mathfrak{g}, [x, y])$, $(\mathfrak{n}, \{x, y\})$ be two Lie brackets on a vector space V. A *post-Lie algebra structure on the pair* $(\mathfrak{g}, \mathfrak{n})$ is a K-bilinear product $x \cdot y$ satisfying the identities

$$\begin{aligned} x \cdot y - y \cdot x &= [x, y] - \{x, y\} \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ x \cdot \{y, z\} &= \{x \cdot y, z\} + \{y, x \cdot z\} \end{aligned}$$

for all $x, y, z \in V$.

These identities imply the identities given before, so that $(V, \cdot, [\,,])$ is a post-Lie algebra with associated Lie algebra $\mathfrak{n}$. If $\mathfrak{n}$ is abelian then the conditions of a post-Lie algebra structure reduce to the conditions

$$\begin{aligned} [x, y] &= x \cdot y - y \cdot x, \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z), \end{aligned}$$

so that $x \cdot y$ is a pre-Lie algebra structure on $\mathfrak{g}$. On the other hand, if $\mathfrak{g}$ is abelian then the conditions reduce to

$$\begin{aligned} x \cdot y - y \cdot x &= -\{x, y\} \\ x \cdot (y \cdot z) &= y \cdot (x \cdot z), \\ (x \cdot y) \cdot z &= (x \cdot z) \cdot y, \end{aligned}$$

so that $-x \cdot y$ is an LR-structure on $\mathfrak{n}$.

8 Milnor's question for nil-affine transformations

Milnor's question 2 and the algebraic version 3 can be asked more generally for nil-affine transformations and post-Lie algebra structures. So we may ask the following existence question.

Question 4. *Exactly which pairs of Lie algebras* $(\mathfrak{g}, \mathfrak{n})$ *over a given vector space* V *over a field of characteristic zero admit a post-Lie algebra structure?*

For the correspondence to nil-affine transformations we would need to consider *complete* post-Lie algebra structures, see [24], but we would like to ask more generally for all post-Lie algebra structures. Of course this question is very ambitious and it is not clear how a complete answer should look like. It seems reasonable to study here first certain algebraic properties of $\mathfrak{g}$ and $\mathfrak{n}$, such as being abelian, nilpotent, solvable, simple, semisimple, reductive and complete as the most basic ones.

If $\mathfrak{n}$ is abelian we are back to Milnor's original question and we ask exactly which Lie algebras $\mathfrak{g}$ admit a pre-Lie algebra structure. This is as we already know a difficult question and there are only partial answers. For example, if $\mathfrak{g}$ is semisimple or more generally perfect, then $\mathfrak{g}$ does not admit any pre-Lie algebra structure, see Proposition 16 and [43]. If $\mathfrak{g}$ is reductive, the question is already open. Certainly $\mathfrak{gl}_n(K)$ does admit a pre-Lie algebra structure, but on the other hand, there are several restrictions. For example, we have the following result, see [15].

Proposition 20. *Let* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ *be a reductive Lie algebra, where* $\mathfrak{s}$ *is simple and* $\mathfrak{a}$ *is the center of* $\mathfrak{g}$ *with* $\dim(\mathfrak{a}) = 1$. *Then* $\mathfrak{g}$ *admits a pre-Lie algebra structure if and only if* $\mathfrak{s} \cong \mathfrak{sl}_n(K)$ *for some* $n \geq 2$.

For more results and details concerning the reductive case and étale affine representations of reductive groups see [6], [15], [28], [29]. On the other hand, if $\mathfrak{g}$ is abelian, then we ask which Lie algebras exactly admit an LR-structure. This question is more accessible and we have obtained several results, see [20].

Proposition 21. *Let* $\mathfrak{n}$ *be a Lie algebra admitting an LR-structure. Then* $\mathfrak{n}$ *is two-step solvable.*

However, not every two-step solvable Lie algebra admits an LR-structure.

Proposition 22. *There are* 3*-step nilpotent Lie algebras with* 4 *generators of dimension* $n \geq 13$ *not admitting any LR-structure.*

There are no such examples with less than 4 generators.

Proposition 23. *Let* $\mathfrak{n}$ *be a* 2*-step nilpotent Lie algebra or a* 3*-step nilpotent Lie algebra with at most* 3 *generators. Then* $\mathfrak{g}$ *admits a complete LR-structure.*

For further results we refer to [20], [22].

For the general case concerning post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ we also have several results, see [24], [25], [26], [31], [32], [39]. Let us explain some of them.

Proposition 24. *Suppose that $(\mathfrak{g},\mathfrak{n})$ admits a post-Lie algebra structure, where $\mathfrak{g}$ is nilpotent. Then $\mathfrak{n}$ is solvable. If $\mathfrak{g}$ is nilpotent with $H^0(\mathfrak{g},\mathfrak{n})=0$, then $\mathfrak{n}$ is nilpotent.*

In case one of the Lie algebras is semisimple, but the other Lie algebra not, we have the following result.

Proposition 25. *Let $(\mathfrak{g},\mathfrak{n})$ be a pair of Lie algebras, where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is solvable. Then $(\mathfrak{g},\mathfrak{n})$ does not admit a post-Lie algebra structure.*

The situation is not symmetric in $\mathfrak{g}$ and $\mathfrak{n}$.

Proposition 26. *Let $(\mathfrak{g},\mathfrak{n})$ be a pair of Lie algebras, where $\mathfrak{n}$ is semisimple and $\mathfrak{g}$ is solvable and unimodular. Then $(\mathfrak{g},\mathfrak{n})$ does not admit a post-Lie algebra structure.*

The unimodularity assumption is essential here. Otherwise any triangular decomposition of $\mathfrak{n}$ induces an obvious post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{g}$ is solvable but not unimodular.

In case one of the Lie algebras $\mathfrak{g}$, $\mathfrak{n}$ is simple we have the following results.

Proposition 27. *Suppose that $(\mathfrak{g},\mathfrak{n})$ admits a post-Lie algebra structure, where $\mathfrak{g}$ is simple. Then $\mathfrak{n}$ is simple and isomorphic to $\mathfrak{g}$. The post-Lie algebra product then is either $x\cdot y=0$ with $[x,y]=\{x,y\}$, or $x\cdot y=[x,y]$ with $[x,y]=-\{x,y\}$.*

If we interchange the roles of $\mathfrak{g}$ and $\mathfrak{n}$ we only can prove the following result, see [35].

Proposition 28. *Suppose that $(\mathfrak{g},\mathfrak{n})$ admits a post-Lie algebra structure, where $\mathfrak{n}$ is simple and $\mathfrak{g}$ is reductive. Then $\mathfrak{g}$ is simple and isomorphic to $\mathfrak{n}$.*

In case both $\mathfrak{g}$ and $\mathfrak{n}$ are semisimple, but not simple, we can have many interesting post-Lie algebra structures.

Example 9. Let $\mathfrak{g}$ and $\mathfrak{n}$ both isomorphic to $\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$. Then there exist nontrivial post-Lie algebra structures on $(\mathfrak{g},\mathfrak{n})$. If $\mathfrak{n}=\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$ has the basis $(e_1,f_1,h_1,e_2,f_2,h_2)$ with Lie brackets

$$\begin{aligned}
&\{e_1,f_1\}=h_1, && \{e_2,f_2\}=h_2,\\
&\{e_1,h_1\}=-2e_1, && \{e_2,h_2\}=-2e_2,\\
&\{f_1,h_1\}=2f_1, && \{f_2,h_2\}=2f_2,
\end{aligned}$$

then the following product defines a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$:

$$\begin{aligned}
&e_1\cdot e_2=-4e_2+h_2, && f_1\cdot e_2=2e_2-h_2, && h_1\cdot e_2=6e_2-2h_2,\\
&e_1\cdot f_2=4f_2+4h_2, && f_1\cdot f_2=-2f_2-h_2, && h_1\cdot f_2=-6f_2-4h_2,\\
&e_1\cdot h_2=-8e_2-2f_2, && f_1\cdot h_2=2e_2+2f_2, && h_1\cdot h_2=8e_2+4f_2.
\end{aligned}$$

Here the Lie brackets of $\mathfrak{g}$ are given by

$$\begin{array}{lll}
[e_1,f_1]=h_1, & [f_1,h_1]=2f_1, & [h_1,f_2]=-6f_2-4h_2,\\
[e_1,h_1]=-2e_1, & [f_1,e_2]=2e_2-h_2, & [h_1,h_2]=8e_2+4f_2,\\
[e_1,e_2]=-4e_2+h_2, & [f_1,f_2]=-2f_2-h_2, & [e_2,f_2]=h_2,\\
[e_1,f_2]=4f_2+4h_2, & [f_1,h_2]=2e_2+2f_2, & [e_2,h_2]=-2e_2,\\
[e_1,h_2]=-8e_2-2f_2, & [h_1,e_2]=6e_2-2h_2, & [f_2,h_2]=2f_2.
\end{array}$$

It is easy to see that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$.

The following table shows, what we know about the existence of post-Lie algebra structures on pairs $(\mathfrak{g},\mathfrak{n})$, with respect to the seven different classes of Lie algebras given below. So more precisely the classes are abelian, nilpotent non-abelian, solvable non-nilpotent, simple, semisimple non-simple, reductive non-semisimple, non-abelian and complete non-semisimple Lie algebras.

$(\mathfrak{g},\mathfrak{n})$	$\mathfrak{n}$ abe	$\mathfrak{n}$ nil	$\mathfrak{n}$ sol	$\mathfrak{n}$ sim	$\mathfrak{n}$ sem	$\mathfrak{n}$ red	$\mathfrak{n}$ com
$\mathfrak{g}$ abelian	✓	✓	✓	–	–	–	✓
$\mathfrak{g}$ nilpotent	✓	✓	✓	–	–	–	✓
$\mathfrak{g}$ solvable	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ simple	–	–	–	✓	–	–	–
$\mathfrak{g}$ semisimple	–	–	–	–	✓	?	–
$\mathfrak{g}$ reductive	✓	?	?	–	?	✓	✓
$\mathfrak{g}$ complete	✓	✓	✓	?	?	✓	✓

Note that a checkmark only means that there is *some* pair $(\mathfrak{g},\mathfrak{n})$ with the given algebraic properties admitting a post-Lie algebra structure. It does not imply that all such pairs admit a post-Lie algebra structure.

Besides existence of post-Lie algebra structures it is also interesting to obtain classification results. For the general case such results are difficult to obtain. There are only some classifications in low dimensions. We refer to [31] for a classification of post-Lie algebra structures on $(\mathfrak{g},\mathfrak{n})$, where both $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic to the 3-dimensional Heisenberg Lie algebra. We have much better classification results for *commutative* post-Lie algebra structures, which will be discussed in the next section.

9 Commutative post-Lie algebra structures

A post-Lie algebra structure $(V,\cdot)$ on a pair $(\mathfrak{g},\mathfrak{n})$ is called *commutative*, if the algebra product is commutative, i.e., if $x\cdot y=y\cdot x$ for all $x,y\in V$. This implies that $[x,y]=\{x,y\}$, so that the Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$ are equal. We only write $\mathfrak{g}$ instead of the pair $(\mathfrak{g},\mathfrak{g})$.

Definition 16. A *commutative post-Lie algebra structure*, or *CPA-structure* on a Lie algebra $\mathfrak{g}$ is a K-bilinear product $x \cdot y$ satisfying the identities:

$$\begin{aligned} x \cdot y &= y \cdot x \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ x \cdot [y, z] &= [x \cdot y, z] + [y, x \cdot z] \end{aligned}$$

for all $x, y, z \in V$.

There is always the *trivial* CPA-structure on $\mathfrak{g}$, given by $x \cdot y = 0$ for all $x, y \in \mathfrak{g}$. Any CPA-structure on a semisimple Lie algebra over a field of characteristic zero is trivial, see [26]. This was generalized in [27] as follows.

Proposition 29. *Any CPA-structure on a perfect Lie algebra of characteristic zero is trivial.*

For complete Lie algebras one can classify all CPA-structures. A Lie algebra $\mathfrak{g}$ is called *complete*, if $Z(\mathfrak{g}) = 0$ and $\mathrm{Der}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. This is equivalent to the cohomological conditions $H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = 0$. A complete Lie algebra is called *simply-complete*, if $\mathfrak{g}$ does not have a non-trivial complete ideal. Every complete Lie algebra can be written as the direct sum of simply-complete Lie algebras. We have the following result [27].

Theorem 4. *Let $\mathfrak{g}$ be a complex simply-complete Lie algebra with nilradical $\mathfrak{n}$. Suppose that $\mathfrak{g}$ is not metabelian and that $\mathfrak{n} = [\mathfrak{g}, \mathfrak{n}]$. Then there is a bijective correspondence between CPA-structures on $\mathfrak{g}$ and elements $z \in Z([\mathfrak{g}, \mathfrak{g}])$, given by*

$$x \cdot y = [[z, x], y].$$

We believe that the condition $\mathfrak{n} = [\mathfrak{g}, \mathfrak{n}]$ is automatically satisfied for complete Lie algebras. However, we could not find this statement with a proof in the literature. The only simply-complete metabelian Lie algebra is the 2-dimensional non-abelian Lie algebra $\mathfrak{r}_2(\mathbb{C})$, where we can classify all CPA-structures directly. There are also classification results concerning CPA-structures on nilpotent Lie algebras. An important fact here is the following, see [30].

Theorem 5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over a field of characteristic zero satisfying $Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$. Then every CPA-structure on $\mathfrak{g}$ is complete, i.e., all left multiplications $L(x)$ are nilpotent.*

In this case we have

$$L(Z(\mathfrak{g}))^{\lceil \frac{\dim Z(\mathfrak{g})+1}{2} \rceil}(\mathfrak{g}) = 0.$$

Definition 17. A CPA-structure $(V, \cdot)$ on $\mathfrak{g}$ is called *associative* if $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$. It is called *central* if $\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g})$.

The first part of the definition is justified by the following lemma [34].

Lemma 1. *Let $(V, \cdot)$ be a CPA-structure on a Lie algebra $\mathfrak{g}$. Then we have $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ if and only if the algebra $(V, \cdot)$ is associative.*

It is easy to see that every central CPA-structure on $\mathfrak{g}$ is associative and conversely that every associative CPA-structure on $\mathfrak{g}$ satisfies $\mathfrak{g} \cdot \mathfrak{g} \subseteq Z([\mathfrak{g}, \mathfrak{g}])$. Also, every central CPA-structure on $\mathfrak{g}$ satisfies $\mathfrak{g} \cdot Z(\mathfrak{g}) = 0$. If $\dim Z(\mathfrak{g}) = 1$ then the formula after Theorem 5 yields the following corollary.

Corollary 1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over a field of characteristic zero satisfying $Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$ and $\dim Z(\mathfrak{g}) = 1$. Then every CPA-structure on $\mathfrak{g}$ satisfies $\mathfrak{g} \cdot Z(\mathfrak{g}) = 0$.*

In particular, every CPA-structure on a filiform nilpotent Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{g} \cdot Z(\mathfrak{g}) = 0$. On the other hand, not all CPA-structures on a filiform nilpotent Lie algebra are central or associative. But we have shown the following result in [34].

Theorem 6. *Let $\mathfrak{g}$ be a complex filiform Lie algebra of solvability class $d \geq 3$. Then every CPA-structure $(V, \cdot)$ on $\mathfrak{g}$ is associative and the algebra $(V, \cdot)$ is Poisson-admissible.*

For certain families of filiform nilpotent Lie algebras a classification of all CPA-structures is possible [34], [39]. As an example let us consider the Witt Lie algebra.

Definition 18. The Witt Lie algebra W_n for $n \geq 5$ over a field of characteristic zero is defined by the Lie brackets

$$
\begin{aligned}
&[e_1, e_j] = e_{j+1}, \quad 2 \leq j \leq n-1, \\
&[e_i, e_j] = \frac{6(j-i)}{j(j-1)\binom{j+i-2}{i-2}} e_{i+j}, \quad 2 \leq i \leq \frac{n-1}{2},\ i+1 \leq j \leq n-i,
\end{aligned}
$$

where $(e_1, \ldots, e_n)$ is an *adapted basis* for W_n.

To give a CPA-structure $(V, \cdot)$ on $\mathfrak{g}$ explicitly it is enough to list the non-zero products $e_i \cdot e_j$ for all $1 \leq i \leq j \leq n$.

Proposition 30. *Every CPA-structure on the complex Witt algebra W_n for $n \geq 7$ with respect to an adapted basis $(e_1, \ldots, e_n)$ is given as follows,*

$$
\begin{aligned}
e_1 \cdot e_1 &= \alpha e_{n-2} + \beta e_{n-1} + \gamma e_n, \\
e_1 \cdot e_2 &= \frac{6(n-4)}{(n-2)(n-3)} \alpha e_{n-1} + \delta e_n, \\
e_2 \cdot e_2 &= \varepsilon e_n,
\end{aligned}
$$

where $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{C}$ are arbitrary parameters.

Note that all CPA-structures on the Witt algebra are associative but not necessarily central. We also have a result concerning CPA-structures on free-nilpotent Lie algebras $F_{g,c}$ with $g \geq 2$ generators and nilpotency class $c \geq 2$, see [30].

Theorem 7. *All CPA-structures on $F_{3,c}$ with $c \geq 3$ are central.*

The result is not true for $F_{3,2}$. We believe that all CPA-structures on $F_{g,c}$ with $g \geq 2$ and $c \geq 3$ are central. However, we could only prove a part of it so far, see [30]. Finally we have determined the CPA-structures on certain infinite-dimensional Lie algebras, e.g., on Kac-Moody algebras [33]. For the infinite-dimensional Witt algebra $\mathcal{W}$ in characteristic zero with a set of basis vectors $\{e_i\}$ and Lie brackets

$$[e_i, e_j] = (j-i)e_{i+j}$$

we have that all CPA-structures on $\mathcal{W}$ are trivial. Note that in case the basis is finite, $\mathcal{W}$ is isomorphic to W_n for some n.

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