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Jets and the variational calculus

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Abstract. We review the approach to the calculus of variations using Ehresmann's theory of jets. We describe different types of jet manifold, different types of variational problem and different cohomological structures associated with such problems.

1 Introduction

Given any suitably differentiable map f between two differentiable manifolds M and N, where f might be defined only on an open subset $U \subset M$, a *jet* of f is an equivalence class of maps between M and N, again perhaps defined only on open subsets of M, such that all the maps in the equivalence class have the same value and derivatives (perhaps only up to some given order k) at some specific point $p \in U$. We would typically denote such a jet by $j^k_p f$.

Jets were introduced by Charles Ehresmann in a series of papers in *Comptes Rendus* in 1951–52 [9]–[13], and expositions of the properties of manifolds of jets may be found in [16], [30]. One of the significant applications of this theory has been to the calculus of variations: this is not surprising, given that in the simplest Euler–Lagrange equation for the extremals of a Lagrangian,

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}},$$

the derivative $\dot{x}$ is effectively regarded as a coordinate on a manifold, in this case a tangent manifold, one of the simplest types of manifold of jets.

In this paper I review an approach to the calculus of variations using jets, along the lines of some talks given to the Ostrava Seminar. Instead of concentrating directly on a single theme, I have taken the opportunity to describe different approaches to the problem involving jets of sections (finite jets v. infinite jets),

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different types of problem (parametric v. non-parametric problems), and indeed mentioning different approaches to the construction of jets themselves (geometric v. algebraic). Although there are many similarities, there are also occasional differences, and this seems to me to be of interest. Some of the material in this paper has been considered in greater depth in [32].

I should like to express my thanks to the University of Ostrava, where I have held a visiting position for some years, and in particular to the late Olga Rossi, formerly Head of the Department of Mathematics, and to Pasha Zusmanovich, who has organised the Ostrava Seminars and who invited me to offer this contribution.

2 Jets

2.1 Tangent vectors

The most elementary type of jet is a tangent vector: a 'vector' attached to a point of a differentiable manifold. The original concept of a tangent vector did not, though, involve the idea of a jet, and indeed tensor calculus, based upon vector spaces of tangent vectors and their duals and tensor products, predated the theory of jets by several decades.

The concept of a tangent vector was originally based upon the idea of a list of numbers 'transforming as a vector' under a change of coordinates. If the tangent vector v was represented by the numbers v^i with respect to the coordinates x^i, and by the numbers $\hat{v}^j$ with respect to the coordinates $\hat{x}^j$, then the transformation rule would be given by the formula

$$\hat{v}^j = \frac{\partial \hat{x}^j}{\partial x^i} v^i$$

(here and subsequently we adopt the convention of an implied sum over repeated lower-case indices). Formally, given a manifold N of class C^k where $k \geq 1$ and with $\dim N = n$, a tangent vector v at $p \in N$ is an equivalence class

$$v = \big[\big((v^i), x\big)\big] \in T_p N$$

where $(v^i) \in \mathbb{R}^n$ is a list of numbers, (U, x) is a coordinate chart on N with $p \in U$, and the equivalence relation is given by

$$\big((\hat{v}^j), \hat{x}\big) \sim \big((v^i), x\big) \quad \text{if} \quad \hat{v}^j = \left.\frac{\partial \hat{x}^j}{\partial x^i}\right|_p v^i \,.$$

We say that $T_p N$ is the *tangent space to N at p*.

Ehresmann's jet approach to a tangent vector starts, instead, with a curve $\gamma : \mathbb{R} \to N$ with $\gamma(0) = p$. In this approach, a tangent vector is an equivalence class $[\gamma]$ where $\tilde{\gamma} \sim \gamma$ if

$$\tilde{\gamma}(0) = \gamma(0) = p$$

and

$$(f \circ \tilde{\gamma})'(0) = (f \circ \gamma)'(0) \tag{1}$$

for every $f \in C^1(N)$. A tangent vector defined in this way may be identified with a tangent vector defined as a list of numbers transforming as a vector, by fixing a coordinate chart (U, x^i) and setting $\xi^i = (\gamma^i)'(0)$; this is independent of the choice of chart by virtue of the chain rule. Each curve in the equivalence class specifies the 'direction' of the tangent vector.

At around the same time as Ehresmann's work on jets, André Weil was developing an algebraic approach to these problems using the idea of 'near points' [39]. For a tangent vector, this would use the algebra $\mathbb{R}(\epsilon)$, $\epsilon^2 = 0$, of 'dual numbers' (that is, the quotient of the polynomial algebra $\mathbb{R}[x]$ by the ideal generated by x^2). In the case of a C^∞ manifold N, an 'algebraic tangent vector' ξ at $p \in N$ would then be an algebra homomorphism $C^\infty(N) \to \mathbb{R}(\epsilon)$ satisfying

$$\xi(f) = f(p) \mod \epsilon \qquad (f \in C^\infty(N))$$

so that the map $\delta_\xi : C^\infty(N) \to \mathbb{R}$ given by

$$\delta_\xi f \cdot \epsilon = \xi(f) - f(p)$$

is a derivation: $\delta_\xi(fg) = g(p)\delta_\xi f + f(p)\delta_\xi(g)$. Our specification that N be of class C^∞ is significant, for in that case it is possible to identify such algebra homomorphisms with tangent vectors as defined earlier, whereas in the C^k case with k finite there are algebraic tangent vectors, and hence derivations, which do not correspond to tangent vectors defined by jets. From now on we shall assume that all manifolds and maps are of class C^∞, and that the equivalence relation defining each jet (such as equation (1) above) uses functions of class C^∞.

In this paper we concentrate on Ehresmann's jet approach, and consider objects more general than tangent vectors, involving maps with higher-dimensional domains and equivalence relations involving higher-order derivatives. We should, though, mention that the algebraic approach can provide a further generalisation, to Weil bundles (see [18] for a survey of these ideas).

2.2 Manifolds of jets

From now on we shall denote the typical codomain of maps giving rise to jets by E rather than N; E will either be a stand-alone manifold, or else the total space of a fibred manifold $\pi : E \to M$.

A tangent vector to the manifold E is an example of a *first-order velocity*; for each $q \in E$ the tangent space T_qE of tangent vectors at q is a vector space, and the disjoint union of all the tangent spaces forms the tangent bundle $TE \to E$, a vector bundle.

More general first-order velocities are given by jets of maps $\gamma : B \to E$ where $B \subset \mathbb{R}^m$ is an open ball around the origin of some positive radius ε. We write $j^1_0\gamma$ for the equivalence class containing γ, where $\tilde{\gamma} \sim \gamma$ if

$$\tilde{\gamma}(0) = \gamma(0) = q$$

and

$$(f \circ \tilde{\gamma})'(0) = (f \circ \gamma)'(0)$$

for every $f \in C^\infty(E)$. We put $T^1_{m|q}E$ for the set of all such velocities at q; then $T^1_{m|q}E$ is a vector space which we may consider either as the direct sum $\bigoplus^m T_qE$, or else as the tensor product $T_qE \otimes \mathbb{R}^{m*}$. We call the disjoint union $T^1_m E$ of all the spaces $T^1_{m|q}E$ the *bundle of m-dimensional 1-velocities* on E; it is, again, a vector bundle over E, and we write $\tau_{mE} : T^1_m E \to E$ for the projection map.

We say that a velocity is *regular* if it is the jet of an immersion $\gamma : B \to E$; any equivalent map $\tilde{\gamma} \sim \gamma$ is also an immersion in some neighbourhood of zero. We write $\mathring{T}^1_m E \subset T^1_m E$ for the open submanifold of regular m-dimensional 1-velocities on E.

Related to these vector bundles of velocities are bundles of contact elements and bundles of jets of sections.

Let $\kappa : \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism satisfying the condition $\kappa(0) = 0$. We write $j^1_0\kappa$ for the equivalence class containing κ, where $\tilde{\kappa} \sim \kappa$ if $D\tilde{\kappa}|_0 = D\kappa|_0$, and we write L^1_m for the set of all such equivalence classes. It is clear that we may identify $j^1_0\kappa$ with the $m \times m$ matrix $D\kappa|_0$ and that L^1_m may be identified with the general linear group $\mathsf{GL}(m)$, where the group product is given by $(j^1_0\kappa_1)(j^1_0\kappa_2) = j^1_0(\kappa_1 \circ \kappa_2)$. We call L^1_m a *first-order jet group*; it has a subgroup L^{1+}_m called an *oriented jet group*, containing those $j^1_0\kappa$ satisfying $\det D\kappa|_0 > 0$.

There is a natural right action of L^1_m on $\mathring{T}^1_{m|q}E$ given by

$$(j^1_0\gamma, j^1_0\kappa) \mapsto j^1_0(\gamma \circ \kappa) \,,$$

and this induces a smooth right action of L^1_m on the regular velocity manifold $\mathring{T}^1_m E$. The quotient by this action is a smooth (Hausdorff) manifold $J^1_m E$, known variously as the manifold of *contact elements*, the manifold of *jets of immersions*, the manifold of *jets of submanifolds*, or the *Grassmannian manifold*; it is a bundle over E. These names suggest that there might also be different ways of constructing $J^1_m E$, by taking as elements

- equivalence classes of immersed m-dimensional submanifolds;
- m-dimensional subspaces of tangent spaces; or
- equivalence classes of nonzero decomposable tangent m-vectors.

There is also an oriented version of the construction using the action of the subgroup L^{1+}_m on $\mathring{T}^1_m E$, giving the manifold $J^{1+}_m E$ of *oriented contact elements*. Caution—it need not be the case that the manifold of oriented contact elements is an orientable manifold.

The basic example of a manifold of contact elements arises when $m = 1$; the bundle $J^1_1 E$ is just the projective tangent bundle of E.

In the case of both velocities and contact elements, we start with a manifold E without further structure. If instead we start with a manifold E which is fibred over another manifold M by a projection π, we may also consider jets of local sections of π. If $p \in M$ and ϕ is a local section with p in its domain then we write $j^1_p\phi$ for the equivalence class containing ϕ, where $\tilde{\phi} \sim \phi$ if

$$\tilde{\phi}(p) = \phi(p)$$

and

$$(f \circ \tilde{\phi} \circ \gamma)'(0) = (f \circ \phi \circ \gamma)'(0)$$

for every $f \in C^\infty(E)$ and every smooth curve γ in M with $\gamma(0) = p$. If $q \in E$ with $\pi(q) = p$ we put $J^1_q\pi$ for the set of all such jets obtained by considering local sections ϕ satisfying $\phi(p) = q$; then $J^1_q\pi$ is an affine space, modelled on the vector space $V_q\pi \otimes T^*_p M$, where $V_q\pi$ is the vector space of tangent vectors to E at q which are vertical over π. We call the disjoint union $J^1\pi$ of all the spaces $J^1_q\pi$ the *first order jet manifold of* π; it is an affine bundle over E with projection $\pi_{1,0} : J^1\pi \to E$ called the *target map*. We also put $\pi_1 = \pi \circ \pi_{1,0} : J^1\pi \to M$ and call it the *source map*.

There is a local relationship between $J^1\pi$ and $T^1_m E$. If $B \subset \mathbb{R}^m$ is an open ball around the origin, a chart $(x^{-1}(B); x)$ on M defines, for each local section $\phi : x^{-1}(B) \to E$, a map $\gamma = \phi \circ x^{-1} : B \to E$, so that we obtain a local correspondence $j^1_p\phi \mapsto j^1_0\gamma$. This depends on the choice of chart, but by acting with the jet group on $T^1_m E$ to give $J^1_m E$ we may glue together the projected local correspondences to give a global injection $J^1\pi \subset J^1_m E$ as an open submanifold. This global correspondence regards $j^1_p\phi \in J^1\pi$ as the jet of the submanifold $\operatorname{im}\phi \subset E$ at $\phi(p)$.

All three constructions correspond to covariant functors, on the category of manifolds and smooth maps in the first two cases, and on the category of fibred manifolds and fibred maps projecting to diffeomorphisms in the third case. In the case of velocities and a map $f : E \to F$ we have $T^1_m f : T^1_m E \to T^1_m F$ given by $T^1_m(f)(j^1_0\gamma) = j^1_0(f \circ \gamma)$, and this map passes to the quotient to induce a map of contact elements $J^1_m f : J^1_m E \to J^1_m F$. In the case of fibred manifolds $\pi : E \to M$ and $\rho : F \to N$ with a fibred map $f : E \to F$ projecting to the diffeomorphism $\bar{f} : M \to N$ we have $J^1 f : J^1\pi \to J^1\rho$ with $J^1 f(j^1_p\phi) = j^1_{\bar{f}(p)}(f \circ \phi \circ \bar{f}^{-1})$.

We frequently carry out calculations in local coordinates. Given a chart $(U; u^a)$ on E, we put $u^a_i(j^1_0\gamma) = D_i(u^a \circ \gamma)$ and then, putting $U^1 = \tau_m^{-1}(U)$, we obtain a chart $(U^1; u^a, u^a_i)$ on $T^1_m E$. If instead $\pi : E \to M$ is a fibred manifold and $(U; x^i, u^\alpha)$ is a fibred chart on E, we put

$$u^\alpha_i(j^1_p\phi) = \left.\frac{\partial\phi^\alpha}{\partial x^i}\right|_p$$

where $\phi(p) \in U$. Now putting $U^1 = \pi_{1,0}^{-1}(U)$ we obtain a chart $(U^1; x^i, u^\alpha, u^\alpha_i)$ on $J^1\pi$. It is worth noting that the subscript of the coordinate u^α_i on $J^1\pi$ depends on the choice of coordinates x^i on M, whereas the subscript of the coordinate u^a_i on $T^1_m E$ is just a number, corresponding to a particular component of $\mathbb{R}^m$. In order to obtain a chart on $J^1_m E$ in a neighbourhood of a given contact element, we pretend that locally E is fibred over $\mathbb{R}^m$ transversely to the submanifold generating the contact element, and choose a fibred chart $(U; u^i, u^\alpha)$ (often called called a *split chart*) corresponding to this fibration; we then construct the chart $(U^1; u^i, u^\alpha, u^\alpha_i)$ as in the fibred manifold case.

Most of the time, we shall concentrate on the case of jets of local sections of fibred manifolds. The other two case are similar, but there are sometimes subtle differences.

2.3 Repeated jets and higher-order jets

It is clear that if $\pi : E \to M$ is a fibred manifold then so is $\pi_1 : J^1\pi \to M$; indeed fibred manifolds are characterised by admitting local sections through each point of the total space, and one local section of π_1 through an arbitrary point $j^1_p\phi \in J^1\pi$ is the *prolongation* $j^1\phi$, defined by $j^1\phi(\tilde{p}) = j^1_{\tilde{p}}\phi$. We may therefore consider the *repeated jet manifold* $J^1\pi_1$. (Some authors label jet manifolds by the total space of the original fibred manifold rather than by the projection map, so that they would write J^1E instead of $J^1\pi$, and J^1J^1E instead of $J^1\pi_1$.)

There is, however, a distinguished submanifold of $J^1\pi_1$, arising because, as simple examples show, not every local section of π_1 is a prolongation $j^1\phi$. If ψ is a general local section of π_1 with coordinate expression $(\psi^\alpha, \psi^\alpha_i)$ then ψ is a prolongation if, and only if, the coordinates ψ^α_i are the derivatives of the coordinates ψ^α, and in that case ψ is the prolongation of $\pi_{1,0} \circ \psi$. Caution—the equivalence class $j^1_p\psi$ of local sections of π_1 might contain a prolongation $j^1\phi$, where ϕ is a local section of π, even if ψ itself is not a prolongation.

If we put $J^2\pi \subset J^1\pi_1$ for the submanifold of jets of prolongations then the elements of $J^2\pi$ may be written as $j^1_p(j^1\phi)$ where ϕ is a local section of π. In this way we may identify elements of $J^2\pi$ with 2-jets $j^2_p\phi$ of local sections ϕ, where two such local sections are 2-equivalent if their values, first and second derivatives are equal at p. We say that $J^2\pi$ is the *holonomic* submanifold of $J^1\pi_1$; the term was introduced by Ehresmann as an analogy with holonomic mechanics (see [27]).

We may see the distinction in local coordinates. The coordinates on $J^1\pi$ are $(x^i, u^\alpha, u^\alpha_i)$, so that the fibre coordinates (as a fibred manifold over M) are (u^α, u^α_i). The coordinates on $J^1\pi_1$ are therefore $(x^i, u^\alpha, u^\alpha_i, u^\alpha_{\cdot;j}, u^\alpha_{ij})$ where

$$u^\alpha_{\cdot;j}(j^1_p\psi) = \left.\frac{\partial\psi^\alpha}{\partial x^j}\right|_p , \qquad u^\alpha_{ij}(j^1_p\psi) = \left.\frac{\partial\psi^\alpha_i}{\partial x^j}\right|_p .$$

If $\psi = j^1\phi$ is a prolongation then $\psi^\alpha = \phi^\alpha$ so that

$$u^\alpha_{\cdot;j}(j^1_p(j^1\phi)) = u^\alpha_j(j^1_p(j^1\phi)) ,$$

and in addition

$$u^\alpha_{ij}(j^1_p(j^1\phi)) = u^\alpha_{ji}(j^1_p(j^1\phi))$$

because $(j^1\phi)^\alpha_i = \partial\phi^\alpha/\partial x^i$ and partial derivatives commute. We may therefore use coordinates $(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij})$ on the second jet manifold $J^2\pi$, where now the second derivative coordinates u^α_{ij} are symmetric in their lower indices.

The two different constraints $u^\alpha_{\cdot;j} = u^\alpha_j$ and $u^\alpha_{ij} = u^\alpha_{ji}$ describing the holonomic submanifold $J^2\pi \subset J^1\pi_1$ suggest that there might be an intermediate submanifold, and this is indeed the case. Recall that if the local section ψ of π_1 is a prolongation then $\psi^\alpha_i = \partial\psi^\alpha/\partial x^i$ throughout the domain of ψ; if instead this condition holds at a point p in the domain of ψ (so that $\psi(p) = j^1_p(\pi_{1,0} \circ \psi)$, whereas this need not be the case for other points $\tilde{p} \neq p$) then we say that ψ is *adapted at* p, and we say that $j^1_p\psi \in J^1\pi_1$ is a *semiholonomic jet*. This property is well-defined, and the set of all semiholonomic jets forms a submanifold $\hat{J}^2\pi \subset J^1\pi_1$ satisfying the first coordinate constraint $u^\alpha_{\cdot;j} = u^\alpha_j$ but not necessarily the second constraint $u^\alpha_{ij} = u^\alpha_{ji}$. We may also describe $\hat{J}^2\pi$ as the submanifold of $J^1\pi_1$ where the two maps

$$(\pi_1)_{1,0} : J^1\pi_1 \to J^1\pi , \qquad J^1\pi_{1,0} : J^1\pi_1 \to J^1\pi$$

are equal, where $(\pi_1)_{1,0}$ is the source map corresponding to π_1, and $J^1\pi_{1,0}$ arises by considering $\pi_{1,0}$ as a fibred map over the identity between the two fibred manifolds π_1 and π.

We therefore have $J^2\pi \subset \hat{J}^2\pi \subset J^1\pi_1$, and indeed each of these manifolds is the total space of an affine bundle over $J^1\pi$. Note that there are no canonical projections $J^1\pi_1 \to J^2\pi$ or $J^1\pi_1 \to \hat{J}^2\pi$; but there is a canonical projection $\hat{J}^2\pi \to J^2\pi$; in fact

$$\hat{J}^2\pi = J^2\pi \oplus_{J^1\pi} \left(V\pi \otimes {\textstyle\bigwedge}^2 T^*M\right) \tag{2}$$

so that any semiholonomic jet may be written as the sum of a holonomic jet and a vector-valued 2-form expressing its *curvature*.

All this may be extended to higher orders. For instance $J^1\pi_1$, $\hat{J}^2\pi$ and $J^2\pi$ are all fibred manifolds over M, and so we may consider their first jet manifolds and various holonomic and semiholonomic submanifolds; further details may be found in [26]. The most important such manifolds are the higher-order holonomic jet manifolds $J^k\pi$, whose elements are k-jets $j^k_p\phi$ of local sections ϕ of π, with equivalence of values and derivatives of order up to k at $p \in M$, and with target map $\pi_k : J^k\pi \to M$; and the resulting first-order jet manifolds $J^1\pi_k$ with submanifolds $J^{k+1}\pi \subset \hat{J}^{k+1}\pi \subset J^1\pi_k$.

The use of coordinates on these higher-order jet manifolds gives rise to some complications. This can already be seen in the case of $J^2\pi$; the symmetry of the second derivative coordinates u^α_{ij} means that, with the usual summation convention for repeated lower-case indices, we must write the formula for the differential of a function f on $J^2\pi$ as

$$df = \frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial u^\alpha}du^\alpha + \frac{\partial f}{\partial u^\alpha_i}du^\alpha + \frac{1}{\#(ij)}\frac{\partial f}{\partial u^\alpha_{ij}}du^\alpha_{ij}$$

where

$$\#(ij) = \begin{cases} 1 & (i = j) \\ 2 & (i \neq j)\,. \end{cases}$$

There are similar complications with higher-order jets, and so three options are commonly used to deal with the problem:

- use numerical coefficients with the summation convention, as above;
- use non-decreasing indices and explicit sums; or
- use vector multi-indices u^α_I with $I \in \mathbb{N}^{\dim M}$, and with $I(i)$ denoting the i-th component of the multi-index I.

The present author prefers the third option, using vector multi-indices, and we shall note later that this option permits a geometric proof of an interesting result in the calculus of variations. Using that notation, we write $|I| = \sum_{i=1}^m I(i)$ for the *length* of the multi-index I, and $I! = \prod_{i=1}^m I(i)!$ for its *factorial*. We put 1_j for the multi-index with 1 in position j and zero everywhere else, and a symbol such as $I + 1_j$ is just the ordinary addition of vectors of natural numbers. We do not use a summation convention with multi-indices (always indicated by upper-case letters); sums over repeated multi-indices will be shown explicitly.

This description of the construction of manifolds of repeated or higher-order jets of sections may be applied in much the same way to construct repeated or higher-order velocity manifolds. For instance we may consider $T^1_m T^1_m E$, whose elements are $j^1_0 \zeta$ where $\zeta : B \to T^1_m E$, $B \subset \mathbb{R}^m$. This has submanifolds $T^2_m E \subset \hat{T}^2_m E \subset T^1_m T^1_m E$. We need to use a slightly different version of a prolongation in this case, for if $\gamma : B \to E$ then we define $\bar{\jmath}^1\gamma : B \to T^1_m E$ by $\bar{\jmath}^1\gamma(t) = j^1_0(\gamma \circ \mathrm{T}_t)$ where $\mathrm{T}_t : \mathbb{R}^m \to \mathbb{R}^m$ is the translation map $\mathrm{T}_t(s) = s + t$. Then $T^2_m E$ is the holonomic submanifold of 1-jets of prolongations $j^1_0(\bar{\jmath}^1\gamma)$, which we identify with 2-jets $j^2_0\gamma$, and $\hat{T}^2_m E$ is the semiholonomic submanifold of 1-jets of adapted maps $\zeta : B \to T^1_m E$, where ζ is adapted if $\zeta(0) = \bar{\jmath}^1(\tau_{mE} \circ \zeta)$. Just as in the case of jets of sections, we find that $\hat{T}^2_m E$ is the submanifold where the two maps

$$\tau_{m, T^1_m E} : T^1_m T^1_m E \to T^1_m E\,, \qquad T^1_m \tau_{mE} : T^1_m T^1_m E \to T^1_m E$$

are equal. There is now also an ‘exchange map’ $e : T^1_m T^1_m E \to T^1_m T^1_m E$ which generalises the canonical involution of the double tangent manifold TTE, and $T^2_m E \subset T^1_m T^1_m E$ is the fixed point set of e.

The construction of higher-order contact manifolds $J^k_m E$ is straightforward, but the relationship with repeated velocities and repeated contact manifolds is rather more complicated; see, for example, [19], [35].

For the rest of this paper we shall consider only the holonomic structures, typically $J^k\pi$, but sometimes also $T^k E$. The semiholonomic and nonholonomic structures are, nevertheless, important. As an example, we can define a general *k-th order differential equation* to be a closed fibred submanifold $R \subset J^k\pi$. Such an equation may fail to have solutions (local sections ϕ of π whose prolongations $j^k\phi$ take values in R) because integrability conditions are not satisfied, where these integrability conditions arise from repeatedly ‘differentiating’ the equation. Letting $\pi_R : R \to M$ be the restriction of $\pi_k : J^k\pi \to M$, a single such differentiation gives a submanifold $J^1\pi_R \subset J^1\pi_k$, and a first set of integrability conditions would arise if $J^1\pi_R \cap J^{k+1}\pi$ did not project surjectively to $J^k\pi$.

In the general case, repeated differentiations may be necessary to obtain a (potentially infinite) family of integrability conditions, and sophisticated tools such as Spencer cohomology may then be needed to test whether all such integrability conditions are satisfied: that is to say, whether there is a formal Taylor series solution at any point of M. There are, however, special cases where only a single differentiation is needed, and one such special case arises when the fibred manifold $R \subset J^1\pi$ is the image of a *connection*, a section Γ of the affine bundle $\pi_{1,0} : J^1\pi \to E$. Regarding $\Gamma : E \to J^1\pi$ as a fibred map projecting to the identity on M, we obtain $J^1\Gamma : J^1\pi \to J^1\pi_1$, and we find that the image of the composite map $J^1\Gamma \circ \Gamma$ is just $J^1\pi_{\Gamma(E)}$, and that this is in fact a submanifold of the semiholonomic manifold $\hat{J}^2\pi \subset J^1\pi_1$. We may therefore define the *curvature* of the connection Γ to be the component of $J^1\Gamma \circ \Gamma$ in $V\pi \otimes \bigwedge^2 T^*M$ (see equation (2) above). If the curvature vanishes then, as $J^1\Gamma \circ \Gamma$ is injective, $J^1\pi_{\Gamma(E)} \cap J^2\pi$ projects surjectively to E, and we may deduce from Frobenius’ Theorem that there are no further integrability conditions. In coordinates, if $\Gamma^\alpha_i = u^\alpha_i \circ \Gamma$ then

$$(j^1\Gamma)^\alpha_i = \Gamma^\alpha_i\,, \qquad (j^1\Gamma)^\alpha_{\cdot;j} = u^\alpha_j\,, \qquad (j^1\Gamma)^\alpha_{ij} = \frac{\partial \Gamma^\alpha_i}{\partial x^j} + (j^1\Gamma)^\beta_{\cdot;j}\frac{\partial \Gamma^\alpha_i}{\partial u^\beta}$$

so that $(j^1\Gamma\circ\Gamma)^\alpha_{\cdot;j} = (j^1\Gamma\circ\Gamma)^\alpha_j = \Gamma^\alpha_j$, and then

$$(j^1\Gamma\circ\Gamma)^\alpha_{ij} - (j^1\Gamma\circ\Gamma)^\alpha_{ji} = \frac{\partial \Gamma^\alpha_i}{\partial x^j} - \frac{\partial \Gamma^\alpha_j}{\partial x^i} + \Gamma^\beta_j\frac{\partial \Gamma^\alpha_i}{\partial u^\beta} - \Gamma^\beta_i\frac{\partial \Gamma^\alpha_j}{\partial u^\beta}$$

is the classic expression for curvature.

2.4 Infinite jets

We sometimes need to use 'infinite jets': these are equivalence classes of maps where the value and all the derivatives (rather than those up to some given maximal order) are equal at a specified point.

In the case of a fibred manifold $\pi : E \to M$, the set of infinite jets $j^\infty_p\phi$ of local sections ϕ is denoted $J^\infty\pi$, with projections to the various finite-dimensional manifolds being denoted $\pi_{\infty,k} : J^\infty\pi \to J^k\pi$ and $\pi_\infty : J^\infty\pi \to M$. It may be shown that $J^\infty\pi$ is an infinite-dimensional Fréchet manifold, modelled on the topological vector space R^∞ of infinite sequences, the inverse limit of the sequence

$$\cdots \to \mathbb{R}^l \to \mathbb{R}^{l-1} \to \cdots \to \mathbb{R}^2 \to \mathbb{R} \to 0$$

in the category of topological vector spaces and continuous linear maps (see [30]). In order to confirm this, we need to be sure that every infinite sequence describes the Taylor polynomial of some smooth function (which need not be convergent anywhere away from the origin); but this is Borel's Theorem (see [4] for the original result, applicable to functions of a single variable, and [20] for a general proof).

The topological dual of $\mathbb{R}^\infty$ is $\mathbb{R}^{(\infty)}$, the space of infinite sequences with finitely many terms nonzero. Thus a tangent vector on $J^\infty\pi$ may have infinitely many nonzero components, whereas a contangent vector can have only finitely many.

3 Vector fields and differential forms on jet manifolds

3.1 Contact forms

We continue with a fibred manifold $\pi : E \to M$.

On any such manifold E there is a distinguished class of vector fields, the vertical (over M) vector fields, which are tangent to the fibres of π. There is correspondingly a distinguished class of differential 1-forms, the horizontal 1-forms, which are annihilated by the vertical vector fields. In fact these are pointwise concepts: at any point $q \in E$ the tangent space T_qE has a distinguished subspace of vertical vectors, and the cotangent space T^*_qE has a distinguished subspace of horizontal covectors.

In general there are no complementary concepts of horizontal vector fields and vertical 1-forms, and indeed the existence of these complementary structures arises when there is a connection (in the classical sense) defined on π. In the previous section we defined a connection on π in a different way, as a section

$$\Gamma : E \to J^1\pi$$

of $\pi_{1,0}$, and we can see how this is equivalent to the classical formulation by introducing *contact forms*.

A contact form on the jet manifold $J^1\pi$ is a differential r-form θ satisfying the condition that if ϕ is any local section of π then the pullback $(j^1\phi)^*\theta$ vanishes on M. (Of course this definition has content only when $r \leq m$.) As with horizontal forms, this is a pointwise concept; we can, for example, describe a cotangent vector $\eta \in T^*_a J^1\pi$ as contact if, for any prolongation $j^1\phi$ such that $j^1_p\phi = a$ (where $p = \pi_1(a)$), we have $T^*(j^1\phi)(\eta) = 0 \in T^*_p M$. In coordinates $(x^i, u^\alpha, u^\alpha_i)$ on $J^1\pi$, a local basis for contact 1-forms is given by $\theta^\alpha = du^\alpha - u^\alpha_i dx^i$.

If Γ is a connection and θ is a contact 1-form on $J^1\pi$ then $\Gamma^*\theta$ is a 1-form on E. In terms of the local basis contact forms we see that $\Gamma^*\theta^\alpha = du^\alpha - \Gamma^\alpha_i dx^i$ where $\Gamma^\alpha_i = u^\alpha_i \circ \Gamma$ is the coordinate expression of Γ, showing that the forms $\Gamma^*\theta^\alpha$ are the vertical forms on E defining a connection in the classical sense. Indeed, given a general 1-form ω on E, its pullback $\pi^*_{1,0}\omega$ on $J^1\pi$ may be expressed as the sum of two components $\pi^*_{1,0}\omega = \omega_h + \omega_c$ where ω_h is horizontal over M and ω_c is a contact form: in coordinates, if $\omega = \omega_i dx^i + \omega_\alpha du^\alpha$ then

$$\pi^*_{1,0}\omega = (\omega_i + \omega_\alpha u^\alpha_i)dx^i + \omega_\alpha\theta^\alpha\,.$$

More generally, an r-form on $J^1\pi$ is said to be *at least s-contact* for $2 \leq s \leq r$ if its contraction with any vector field on $J^1\pi$ is at least $(s-1)$-contact, where a contact form as defined above is said to be at least 1-contact.

We can also say when an r-form on $J^1\pi$ horizontal over E is 'exactly' s-contact, as well as 'at least' s-contact. We say that it is *exactly 1-contact* if it is at least 1-contact, and its contraction with any vector field on $J^1\pi$ vertical over M is horizontal over M. We say that it is *exactly s-contact* for $2 \leq s \leq r$ if it is at least s-contact, and its contraction with any vector field on $J^1\pi$ vertical over M is exactly $(s-1)$-contact. For example, the 3-forms $\theta^\alpha \wedge \theta^\beta \wedge dx^i$ and $\theta^\alpha \wedge \theta^\beta \wedge du^\gamma$ are both at least 2-contact; they are both horizontal over E; but the first is exactly 2-contact whereas the second is not.

Where there is no possibility of confusion, we often say simply that the form is *s-contact* to mean that it is exactly s-contact.

We have described contact forms on $J^1\pi$, but a similar definition applies to forms on $J^k\pi$: we say that a differential r-form θ on $J^k\pi$ is a contact form if, whenever ϕ is a local section of π, then the pullback $(j^k\phi)^*\theta$ vanishes on M. We use coordinates (x^i, u^α_I) on $J^k\pi$, where the length of the multi-index I ranges from zero (giving the coordiates u^α) to k, and then a local basis for the contact 1-forms is given by $\theta^\alpha_J = du^\alpha_J - u^\alpha_{J+1_i}dx^i$ for $0 \leq |J| \leq k-1$. If ω is a general 1-form on $J^{k-1}\pi$ then now $\pi^*_{k,k-1}\omega = \omega_h + \omega_c$, where $\pi_{k,k-1} : J^k\pi \to J^{k-1}\pi$ is the natural map sending $j^k_p\phi$ to $j^{k-1}_p\phi$, and where ω_h is horizontal over M and ω_c is a contact form. If $\omega = \omega_i dx^i + \sum_{|J|=0}^{k-1}\omega^J_\alpha du^\alpha_J$ then

$$\pi^*_{k,k-1}\omega = \left(\omega_i + \sum_{|J|=0}^{k-1}\omega^J_\alpha u^\alpha_{J+1_i}\right)dx^i + \sum_{|J|=0}^{k-1}\omega^J_\alpha\theta^\alpha_J\,.$$

More generally, if ω is an r-form on $J^{k-1}\pi$ with $r \leq m$ then $\pi^*_{k,k-1}\omega$ may be expressed uniquely as a sum

$$\pi_{k,k-1}^{*}\omega = \omega_0 + \omega_1 + \cdots + \omega_r$$

of r-forms on $J^k\pi$, where ω_0 is horizontal over M, ω_1 is exactly 1-contact, and so on, so that ω is a contact form when $\omega_0 = 0$. If $r > m$ then we may similarly write

$$\pi_{k,k-1}^{*}\omega = \omega_{r-m} + \omega_{r-m+1} + \cdots + \omega_r ;$$

in this case we say that ω is *strongly contact* if $\omega_{r-m} = 0$.

The ideal of differential forms generated by the contact 1-forms on $J^k\pi$ is not differentially closed. The local 1-form

$$\theta_I^\alpha = du_I^\alpha - u_{I+1}^\alpha dx^i , \qquad |I| = k-1$$

is a contact form, but $d\theta_I^\alpha = -du_{I+1_i}^\alpha \wedge dx^i$ is a local contact 2-form which is not a linear combination of the contact 1-forms. We may see a similar phenomenon for global contact forms by using bump functions. Nevertheless it can be shown that if θ is an arbitrary contact r-form on $J^k\pi$ then θ may be expressed as a linear combination of contact 1-forms and their exterior derivatives; locally we may write θ as

$$\sum_{|J|=0}^{k-1} \xi_\alpha^J \wedge \theta_J^\alpha + \sum_{|J|=0}^{k-1} \zeta_\alpha^J \wedge d\theta_J^\alpha$$

where ξ_α^J are local $(r-1)$-forms and ζ_α^J are local $(r-2)$-forms [24].

The situation is slightly simpler in the infinite-order case. If ω is an r-form on $J^\infty\pi$ with $r \le m$ then ω itself (without any pullback) may be expressed uniquely as a sum

$$\omega = \omega_0 + \omega_1 + \cdots + \omega_r \tag{3}$$

of r-forms on $J^\infty\pi$, where as before ω_0 is horizontal over M, ω_1 is exactly 1-contact, and so on. It is also the case that, on $J^\infty\pi$, any contact form can be expressed as a linear combination of contact 1-forms, without the need to use their exterior derivatives; in this case the ideal generated by the contact 1-forms is differentially closed.

Although we have concentrated on jets of fibred manifolds, there are also contact forms on manifolds of contact elements, and on velocity manifolds. On $T_m^k E$ an r-form θ is said to be a contact form if $(\bar{\jmath}^k\gamma)^*\theta = 0$, just as in the case of fibred manifolds. As an example, on $\mathring{T}_m^1 E$ (it is convenient for technical reasons to restrict attention to regular velocities) we find that the contact 1-forms are spanned locally by forms given by determinant expressions

$$\begin{vmatrix} u_1^{a_1} & u_1^{a_2} & \cdots & u_1^{a_{m+1}} \\ u_2^{a_1} & u_2^{a_2} & \cdots & u_2^{a_{m+1}} \\ \vdots & \vdots & & \vdots \\ u_m^{a_1} & u_m^{a_2} & \cdots & u_m^{a_{m+1}} \\ du^{a_1} & du^{a_2} & \cdots & du^{a_{m+1}} \end{vmatrix}$$

so that the coordinate expressions for contact forms on velocity manifolds look very different from the coordinate expressions for such forms on manifolds of jets of sections.

As we do not assume that E is fibred over some other manifold, there is no base manifold over which a form on a velocity manifold can be 'horizontal', and there is therefore no decomposition of lifted forms into horizontal and contact components. Nor is there a concept of a form being exactly s-contact, as there is similarly no base manifold over which vector fields can be vertical. We can, though, still say that a form is at least s-contact.

Surprisingly, perhaps, it is not true that contact r-forms on $T^1_m E$ are linear combinations of contact 1-forms and their exterior derivatives. To see an example of this, take $E = \mathbb{R}^3$ with coordinates (u, v, w) and consider the manifold of regular first-order 2-velocities on E with coordinates

$$(u, v, w; u_1, v_1, w_1; u_2, v_2, w_2) .$$

On this manifold the contact 1-forms are multiples of

$$\theta = (u_1 v_2 - u_2 v_1) dw + (v_1 w_2 - v_2 w_1) du + (w_1 u_2 - w_2 u_1) dv ;$$

but

$$\omega = (u_1 dv - v_1 du) \wedge dw_2 - (u_2 dv - v_2 du) \wedge dw_1$$

is a contact 2-form because $(\bar{j}^1\gamma)^*\omega = 0$ for every map $\gamma : B \to E$, but ω is clearly not a linear combination of θ and $d\theta$. It is, however, the case that if we restrict θ and ω to the affine submanifold $A \subset \mathring{T}^1_m E$ given by $u_1 = v_2 = 1$, $u_2 = v_1 = 0$ (so that we are pretending that E is a fibred manifold over $\mathbb{R}^2$ with A is its first jet manifold) then

$$\theta|_A = dw - w_1 du - w_2 dv , \qquad \omega|_A = d\theta|_A .$$

For essentially this reason the contact forms on manifolds of contact elements have the same coordinate expressions, in split charts, as they do on manifolds of jets of sections.

3.2 Vector fields

We return to the case of a fibred manifold $\pi : E \to M$ and its first jet manifold. In this case, dual to the contact 1-forms are the *total derivatives*. These are vector fields along a map rather than on a manifold, and, as with contact forms, may be defined pointwise. If $v \in T_p M$ is a tangent vector and ϕ is a local section of π with p in its domain then $T\phi(v)$ is a tangent vector at $\phi(p) \in E$. If $\tilde{\phi}$ is another such local section then $T\tilde{\phi}(v) = T\phi(v)$ if $j^1_p\tilde{\phi} = j^1_p\phi$. Thus, given a vector field X on M, we obtain a map X_1 from $J^1\pi$ to TE satisfying $\tau_E \circ X_1 = \pi_{1,0}$, so that X_1 is a vector field along the source projection $\pi_{1,0}$. The local coordinate vector field $\partial/\partial x^i$ gives, in this way, the local total derivative

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} .$$

Given a connection $\Gamma : E \to J^1\pi$, a total derivative X_1 becomes a horizontal vector field $X_1 \circ \Gamma$ on E; locally

$$\frac{d}{dx^i} \circ \Gamma = \frac{\partial}{\partial x^i} + \Gamma_i^\alpha \frac{\partial}{\partial u^\alpha} .$$

It is evident that the contraction of a contact 1-form (necessarily horizontal over E, and so taking its values in T^*E) and a total derivative (taking its values in TE) must vanish.

In the same way, a vector field X on M determines a vector field X_k along $\pi_{k,k-1} : J^k\pi \to J^{k-1}\pi$; locally now $\partial/\partial x^i$ gives rise to

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{|J|=0}^{k-1} u^\alpha_{J+1_i} \frac{\partial}{\partial u^\alpha_J}$$

and again the contraction of such a vector field X_k with a contact 1-form on $J^k\pi$ (necessarily horizontal over $J^{k-1}\pi$) must vanish. It is not the case, though, that every vector field on $J^k\pi$ having a vanishing contraction with contact forms must be a linear combination of total derivatives, because vector fields vertical over $J^{k-1}\pi$ also satisfy this condition. The distribution on $J^k\pi$ annihilated by the contact forms and spanned by both classes of vector fields together is called the *contact distribution* (or also the *Cartan distribution*); it is not an integrable distribution, for the same reason that the ideal generated by the contact 1-forms is not differentially closed.

Once again, the situation is slightly simpler in the infinite-order case. The vector field X_∞ is a genuine vector field on the manifold $J^\infty\pi$, rather than a vector field along a map (although it need not have a flow: there is no guarantee that vector fields on Fréchet manifolds possess flows). In this case the rank of the contact distribution (the distribution annihilated by the contact forms) is $\dim M$, and the distribution is spanned by the total derivatives alone.

We return to the finite-dimensional case. A fibred map $g : J^k\pi \to J^k\pi$ will be called a *contact transformation* if it maps the contact distribution to itself, or equivalently if the pullback $g^*\theta$ of any contact form θ on $J^k\pi$ is again a contact form. Similarly a vector field Z on $J^k\pi$ will be called an *infinitesimal contact transformation* if its Lie bracket with any vector field taking values in the contact distribution (a total derivative, or a vector field vertical over $J^{k-1}\pi$) again takes values in the contact distribution; in terms of contact forms we require that the Lie derivative $d_Z\theta$ is again a contact form..

We may obtain contact transformations, or infinitesimal contact transformations, by prolongation. We mentioned earlier that the operation of taking jets of local sections is functorial; in particular, if $\pi : E \to M$ is a fibred manifold and $f : E \to E$ is a fibred map whose projection $\bar{f} : M \to M$ is a diffeomorphism then $J^k f : J^k\pi \to J^k\pi$ is another fibred map, its *prolongation*. Each such prolongation is a contact transformation.

If Y is a projectable vector field on E (that is to say projectable to a vector field $\bar{Y}$ on M) then its flow ψ_t is, locally, a family of fibred maps on E projecting to the local flow $\bar{\psi}_t$ of $\bar{Y}$. We may use the definition

$$J^k\psi_t(j^k_p\phi) = j^k_{\bar{\psi}_t(p)}(\psi_t \circ \phi \circ \bar{\psi}_{-t})$$

to specify a family of local fibred maps giving the *flow prolongation* Y^k on $J^k\pi$. This is again a vector field projectable to $\bar{Y}$, and in coordinates if

$$Y = Y^i \frac{\partial}{\partial x^i} + Y^\alpha \frac{\partial}{\partial u^\alpha}$$

then

$$Y^k = Y^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} \left(\frac{d^{|I|}Y^\alpha}{dx^I} - \sum_{\substack{J+K=I \\ J\neq 0}} \frac{I!}{J!K!} \frac{\partial^{|J|}Y^j}{\partial x^J} u^\alpha_{K+1_j} \right) \frac{\partial}{\partial u^\alpha_I} . \tag{4}$$

In fact Y need not be projectable in order for a prolongation Y^k to exist. We construct a prolongation for a general vector field on E by first noticing that there is an exchange map e_k between $V\pi_k$, the vectors on $J^k\pi$ vertical over M, and the k-jets of local sections of the composite fibred manifold $\nu_\pi : V\pi \to E \to M$. If Y is a vertical vector field on E, so that it is a fibred map $E \to V\pi$ projecting to the identity on M, then its prolongation $J^kY : J^k\pi \to J^k\nu_\pi$ is such that $e_k \circ J^kY : J^k\pi \to V\pi_k$ is just Y^k, the prolonged vector field. We then note that a similar construction may be used for a vertical vector field along the map $\pi_{1,0}$ to give a prolonged vector field along $\pi_{k+1,k}$. Finally, we observe that for an arbitrary vector field on Y we may obtain a related vertical vector field Y_v along $\pi_{1,0}$ by subtracting a suitable total derivative X_1, and if we then prolong to obtain Y^k_v as a vector field along $\pi_{k+1,k}$, we may obtain a prolongation Y^v as a vector field on $J^k\pi$ by adding the total derivative X_{k+1} (see [30] for details of this procedure). The coordinate formula for such a general prolongation of Y differs from that given in equation (4) only in that the partial derivatives $\partial^{|J|}Y^j/\partial x^J$ are replaced by total derivatives $d^{|J|}Y^j/dx^J$, because the functions Y^j need no longer be projectable to M.

Any such prolongation Y^k is an infinitesimal contact transformation. The Lie–Backlund Theorem asserts that if the fibre dimension of the fibred manifold $\pi : E \to M$ is greater than one then prolongations (of fibred maps $E \to E$, or of vector fields on E) are the only contact transformations (or infinitesimal contact transformations).

In the case where the fibre dimension equals one then there are contact transformations which are not projectable to E; for example if $M = \mathbb{R}^m$ and $E = \mathbb{R}^{m+1}$ then the *hodograph transformation*, given by $(x^i, u, u_i) \mapsto (u_i, x^iu_i - u, x^i)$, is a contact transformation on $J^1\pi$ which does not project to E. Nevertheless, even in the case where the fibre dimension is one, a contact transformation (or infinitesimal contact transformation) on $J^k\pi$ will be projectable to $J^1\pi$.

Similar constructions may be performed on velocity manifolds and on manifolds of contact elements, but the technicalities are perhaps a little easier. For instance, although the coordinate description of contact forms on velocity manifolds involves multilinear operations, the coordinate description of total derivatives is essentially the same as on manifolds of jets of sections:

$$\frac{d}{dt^i} = \sum_{|J|=0}^{k-1} u^a_{J+1_i} \frac{\partial}{\partial u^a_J} \quad \text{compared to} \quad \frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{|J|=0}^{k-1} u^\alpha_{J+1_i} \frac{\partial}{\partial u^\alpha_J} .$$

In addition the prolongation operation, for both maps and vector fields, is more straightforward, because there is no need to worry about projectability. The prolongation of a map $f : E \to E$ is simply the map $T^k_m f : T^k_m E \to T^k_m E$, and the prolongation of an arbitrary vector field on E is just its flow prolongation. The same approach may be used for prolongations to manifolds of contact elements.

3.3 Vertical and horizontal differentials

On the infinite jet manifold $J^\infty\pi$ we noted above that any 1-form ω can be written as the sum of horizontal and contact components $\omega = \omega_h + \omega_c$. The map $\omega \mapsto \omega_h$ is a pointwise linear operation, and may be considered as a vector-valued 1-form h on $J^\infty\pi$. The commutator of this operator with the exterior derivative d gives a derivation d_h of the exterior algebra $\Omega^* = \Omega^* J^\infty\pi$ (see [14]) called the *horizontal differential*; we write d_v for the complement $d - d_h$ and call it the *vertical differential*. In particular, for a function f on $J^\infty\pi$ we have $d_h f = (df)_h$ and $d_v f = (df)_c$.

More generally we may extend the decomposition given in equation (3) to write the exterior algebra Ω^* as a direct sum of components $\Omega^{r,s}$, where if $\omega \in \Omega^{r,s}$ then ω is an $(r+s)$-form on $J^\infty\pi$ which is exactly s-contact. Thus $\Omega^{r,s}$ is generated by forms with local coordinate expressions

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r} \wedge \theta^{\alpha_1}_{I_1} \wedge \theta^{\alpha_2}_{I_2} \wedge \cdots \wedge \theta^{\alpha_s}_{I_s} .$$

We see that $d_h : \Omega^{r,s} \to \Omega^{r+1,s}$ and $d_v : \Omega^{r,s} \to \Omega^{r,s+1}$, and these maps may be combined to give a construction known as the *variational bicomplex*. (See Figure 1 below; the horizontal differentials are shown vertically, and the vertical differentials horizontally, purely for typographical convenience.) In this bicomplex Φ^s denotes the quotient module $\Omega^{m,s}/d_h\Omega^{m-1,s}$, and the maps

$$\delta : \Phi^s \to \Phi^{s+1}$$

are induced by the maps $d_v : \Omega^{m,s} \to \Omega^{m,s+1}$. Note that d_h and d_v anticommute, so that most of the small squares in the diagram are anticommutative rather than commutative. An obvious question concerns the exactness of this bicomplex. It is fairly straightforward to see that d_v is locally exact by using a version of the standard Poincaré lemma with parameters, but proving local exactness of d_h is significantly more complicated. A comprehensive study of the variational bicomplex may be found in [1] and a useful summary with further references is given in [38]; these works also contain information about the global cohomology of the bicomplex.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & \\
 & \mathbb{R} \arrow[d] & \mathbb{R} \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^0 M \arrow[r, "\pi_\infty^*"] \arrow[d, "d"'] & \Omega^{0,0} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \Omega^{0,1} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \cdots \arrow[r, "d_v"] & \Omega^{0,s} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \cdots \\
0 \arrow[r] & \Omega^1 M \arrow[r, "\pi_\infty^*"] \arrow[d, "d"'] & \Omega^{1,0} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \Omega^{1,1} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \cdots \arrow[r, "d_v"] & \Omega^{1,s} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \cdots \\
 & \vdots \arrow[d, "d"'] & \vdots \arrow[d, "d_h"'] & \vdots \arrow[d, "d_h"'] & & \vdots \arrow[d, "d_h"'] & \\
0 \arrow[r] & \Omega^{m-1} M \arrow[r, "\pi_\infty^*"] \arrow[d, "d"'] & \Omega^{m-1,0} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \Omega^{m-1,1} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \cdots \arrow[r, "d_v"] & \Omega^{m-1,s} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \cdots \\
0 \arrow[r] & \Omega^m M \arrow[r, "\pi_\infty^*"] \arrow[d, "d"'] & \Omega^{m,0} \arrow[r, "d_v"] \arrow[d, "p"'] & \Omega^{m,1} \arrow[r, "d_v"] \arrow[d, "p"'] & \cdots \arrow[r, "d_v"] & \Omega^{m,s} \arrow[r, "d_v"] \arrow[d, "p"'] & \cdots \\
 & 0 \arrow[r] & \Phi^0 \arrow[r, "\delta"] \arrow[d] & \Phi^1 \arrow[r, "\delta"] \arrow[d] & \cdots \arrow[r, "\delta"] & \Phi^s \arrow[r, "\delta"] \arrow[d] & \cdots \\
 & & 0 & 0 & & 0 &
\end{tikzcd}
$$

Figure 1: The variational bicomplex on $J^\infty\pi$

It is important to realise that the variational bicomplex cannot work in the same way on finite-order jet manifolds, because if ω is a form on $J^k\pi$ then in general $d_h\omega$ and $d_v\omega$ are forms on $J^{k+1}\pi$; and sequences such as

$$\Omega^{1,0}E \xrightarrow{d_h} \Omega^{2,0}J^1\pi \xrightarrow{d_h} \Omega^{3,0}J^2\pi$$

are not even locally exact. To see an example of this, take $M = \mathbb{R}^2$ and $E = \mathbb{R}^4$, and put $\omega = (u^1_1 u^2_2 - u^1_2 u^2_1)dx^1 \wedge dx^2 \in \Omega^{2,0}J^1\pi$; then $d_h\omega = 0$, but there is no 1-form on E mapping to ω under d_h, because the image of any such form must have coefficients affine in the first derivative coordinates.

It is therefore of interest to consider forms which do not increase in order under d_h. It may be shown that, in coordinates, their coefficients must be polynomial in the highest order derivatives, and that such polynomials must involve determinants. These are called *Jacobian forms* in [1], and an elementary proof that if $\omega \in \Omega^{r,0}J^k\pi$ (where $0 \le r < m$) satisfies $d_h\omega \in \Omega^{r+1,0}J^k\pi$ then the coefficients of ω must be polynomial of degree not exceeding s in the k-th order derivative coordinates may be found in [34].

A different approach which avoids this problem is to consider, instead of horizontal forms, equivalence classes of arbitrary forms modulo contact forms or strongly contact forms. Consider a fixed order k, and now write Ω^r for $\Omega^r J^k\pi$. For $1 \le r \le m$ let Θ^r be the submodule of Ω^r containing the contact forms, so that if

$r > 1$ and $\theta \in \Theta^{r-1}$ then $d\theta \in \Theta^r$; if $r > m$ let Θ^r be generated by the strongly contact forms and $d\theta$ where $\theta \in \Theta^{r-1}$. We therefore have a subsequence

$$0 \to \Theta^1 \to \Theta^2 \to \cdots \to \Theta^m \to \cdots \to \Theta^M \to 0$$

of the de Rham sequence. By considering restrictions to open subsets of $J^k\pi$ of the form $\pi_{k,0}^{-1}(U)$ where $U \subset E$, the corresponding sheaf sequence is found to be exact; in addition each sheaf Θ^r is soft. We may therefore construct the following sheaf diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & & 0 \arrow[r] \arrow[d] & \Theta^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Theta^M \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Omega^0 \arrow[r, "d"] \arrow[dr] & \Omega^1 \arrow[r, "d"] \arrow[d, "p"] & \cdots \arrow[r, "d"] & \Omega^M \arrow[r, "d"] \arrow[d, "p"'] & \Omega^{M+1} \arrow[r, "d"] & \cdots \\
 & & & \Omega^1/\Theta^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^M/\Theta^M \arrow[ur] \arrow[d] & & \\
 & & & 0 & & 0 & &
\end{tikzcd}$$

where the sheaf sequence of quotients (with maps induced by d)

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0 \longrightarrow \Omega^1/\Theta^1 \longrightarrow \cdots \longrightarrow \Omega^M/\Theta^M \longrightarrow \Omega^{M+1} \longrightarrow \cdots$$

is exact. It is known as the *finite-order variational sequence*, and further details may be found in [23].

The construction of the finite-order variational sequence involves only contact forms, and so it also makes sense on manifolds of contact elements. In contrast the variational bicomplex uses horizontal forms, and so cannot be defined directly on manifolds of contact elements, although it is again possible to mimic horizontal forms by using quotient modules [28]. On velocity manifolds, though, a different type of bicomplex may be constructed using vector-valued forms: r-forms taking their values in $\bigwedge^s \mathbb{R}^{m*}$, with d taking the place of the vertical differential and a tensorial map constructed from total derivatives taking the place of the horizontal differential; further details may be found in [33].

4 The calculus of variations

4.1 Parametric and non-parametric variational problems

The machinery we have described above allows us to study variational problems of many kinds. There is, though, one important distinction which may be exemplified by considering two simple problems, one from mechanics and one from geometry. The problem in mechanics is to find the trajectory of a free particle moving in space (say, $\mathbb{R}^3$) from one point to another; the problem in geometry is to find the curve giving the shortest distance between two points in space.

The solution to both problems is, of course, the straight line between the two points; but the term 'straight line' has different meanings for the two problems. For the mechanics problem, the straight line is a map from a closed interval to $\mathbb{R}^3$, whereas for the geometric problem the straight line is a geometric line segment, perhaps with a given direction. In mechanics, traversing the path of the particle at a varying speed would represent a different trajectory, whereas in geometry there is no concept of 'speed' and any description of the line segment in terms of a map from a closed interval to $\mathbb{R}^3$ would involve an external parameter, not part of the original problem. We say that the geometric problem is *parametric*, and the mechanics problem is *non-parametric*.

The distinction may be seen when the two problems are expressed as integrals in classical notation. The mechanics problem would be expressed as

$$\int \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt$$

whereas the geometric problem would be expressed as

$$\int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt\,;$$

writing the integrand of the latter as $F(\dot{x}, \dot{y}, \dot{z})$, we see that it is 'positively homogeneous in the velocity variables', so that

$$F\bigl(\mu(\dot{x}, \dot{y}, \dot{z})\bigr) = \mu F(\dot{x}, \dot{y}, \dot{z})$$

for $\mu > 0$.

This geometric problem is an instance of a *Finsler geometry*. In a general Finsler geometry, one is given a manifold E and a positively homogeneous function $F : \mathring{T}E \to \mathbb{R}$, the *Finsler function*, satisfying a certain regularity condition; one seeks maps $\gamma : [a, b] \to E$ giving stationary values of the integral

$$\int_a^b (\bar{\jmath}^1\gamma)^* F\, dt\,.$$

It follows from the homogeneity property of F that if γ is such an extremal then so is any reparametrization $\gamma \circ \kappa$ where $\kappa : [a, b] \to [a, b]$ and $\kappa'(t) > 0$. In general, parametric variational problems may involve more independent variables (such as minimal surface problems) or higher order derivatives; they would be defined by a function on a bundle of regular velocities $\mathring{T}^k_m E$ satisfying a family of homogeneity conditions known as *Zermelo conditions* [37]. Such a problem may also be defined on the corresponding manifold of contact elements.

On the other hand, time is intrinsic to the description of the mechanics problem, so we may regard such problems as being formulated on $J^1\pi$, where $\pi : E \to \mathbb{R}$ is a fibred manifold. In general one is given a function $L : J^1\pi \to \mathbb{R}$, the *Lagrangian function* and seeks local sections $\phi : [a, b] \to E$ giving extreme values of the integral

$$\int_a^b (j^1\phi)^* L\, dt\,.$$

Now, though, t (the coordinate on the base manifold $\mathbb{R}$) is defined by pullback as a global coordinate on $J^1\pi$, so we may consider $\lambda = L\,dt$ as a 1-form on $J^1\pi$, the *Lagrangian 1-form*, horizontal over $\mathbb{R}$. Note that L, and hence λ, might have an explicit time dependence; if it does not, and E has the form of a Cartesian product $E = \mathbb{R} \times E_0$, then there is a canonical isomorphism $J^1\pi \cong \mathbb{R} \times TE_0$, and indeed many mechanics texts consider time-independent Lagrangian functions to be defined on TE_0 so that the techniques of symplectic geometry can be used on that even-dimensional manifold.

In general, a non-parametric variational problem may involve more independent variables (such as problems in field theory) or higher order derivatives; it would defined by a Lagrangian m-form on $J^k\pi$, where $\pi : E \to M$ is a fibred manifold, with M orientable and $m = \dim M$.

4.2 Lagrangians and Euler–Lagrange equations

We consider a non-parametric variational problem, given by the Lagrangian m-form λ on $J^k\pi$, horizontal over M (see [32]). Let $C \subset M$ be a compact connected m-dimensional submanifold, so that the variational problem defined by λ and C is to find *extremals*, local sections ϕ of $\pi : E \to M$ whose domains contain C, giving stationary values of the integral

$$\int_C (j^k\phi)^*\lambda$$

under small 'variations' of ϕ: that is to say, if ϕ_s is a one-parameter family of local sections, all of whose domains contain C, such that $\phi_0 = \phi$ and $\phi_s|_{\partial C} = \phi|_{\partial C}$, then

$$\frac{d}{ds}\int_C (j^k\phi_s)^*\lambda = 0\,.$$

Such a family of local sections may be constructed from the flow of a vertical vector field Y on E vanishing on $\pi^{-1}(\partial C)$, a *variation field*, and then we require

$$\int_C (j^k\phi)^* d_{Y^k}\lambda = 0 \tag{5}$$

where Y^k is the prolongation of Y to $J^k\pi$.

Now define an m-form ϑ_λ on $J^{2k-1}\pi$, horizontal over $J^{k-1}\pi$, to be a *Lepage equivalent* of λ if $\pi^*_{2k-1,k}\lambda - \vartheta_\lambda$ is a contact form, and also $i_X d\vartheta_\lambda$ is a contact form for every vector field X defined on $J^{2k-1}\pi$ and vertical over E. It may be shown that every Lagrangian m-form λ has a Lepage equivalent, although (unless $m = 1$) this will not be unique. Using a Lepage equivalent, equation (5) for extremals may be written as

$$\int_C (j^{2k-1}\phi)^* d_Z\vartheta_\lambda = 0 \tag{6}$$

whenever Z is a vector field on $J^{2k-1}\pi$ vanishing on $\pi^{-1}_{2k-1}(\partial C)$, with no requirement that Z be a prolongation.

Noting that $d_Z\vartheta_\lambda = di_Z\vartheta_\lambda + i_Z d\vartheta_\lambda$ and that

$$\int_C (j^{2k-1}\phi)^* di_Z\vartheta_\lambda = \int_C d\big((j^{2k-1}\phi)^* i_Z\vartheta_\lambda\big) = \int_{\partial C} (j^{2k-1}\phi)^* i_Z\vartheta_\lambda = 0$$

because Z vanishes on $\pi_{2k-1}^{-1}(\partial C)$, we find that only the 1-contact part of $\pi^*_{2k,2k-1} d\vartheta_\lambda$ makes any contribution to the integral (6), and we write ε_λ for this 1-contact part; it is an $(m+1)$-form on $J^{2k}\pi$ called the *Euler–Lagrange form.* In coordinates, if $\lambda = L\, dx^1 \wedge \cdots \wedge dx^m$ then

$$\varepsilon_\lambda = \left(\frac{\partial L}{\partial u^\alpha} - \sum_{|I|=1}^{2k-1} (-1)^{|I|-1} \frac{d^{|I|}}{dx^I} \frac{\partial L}{\partial u^\alpha_I} \right) du^\alpha \wedge dx^1 \wedge \cdots \wedge dx^m \,.$$

The zero set of this form is a differential equation of order (at most) $2k$, the *Euler–Lagrange equation* for λ.

We now consider this in the context of the constructions described earlier, the variational bicomplex and the finite-order variational sequence. We shall start with the bicomplex (recall that this is defined for infinite jets, so that everything must be pulled back to $J^\infty\pi$); the relevant part concerns m-forms and $(m+1)$-forms, and their predecessors and successors.

$$\begin{tikzcd}
 & \Omega^{m-1,0} \arrow[d, "d_h"] & & \\
 & \Omega^{m,0} \arrow[r, "d_v"] \arrow[d, "p"] & \Omega^{m,1} \arrow[r, "d_v"] \arrow[d, shift left, "p"] & \Omega^{m,2} \arrow[d, shift left, "p"] \\
0 \arrow[r] & \Phi^0 \arrow[r, "\delta"] \arrow[d] & \Phi^1 \arrow[r, "\delta"] \arrow[u, dashed, shift left, "j"] & \Phi^2 \arrow[u, dashed, shift left, "j"] \\
 & 0 & &
\end{tikzcd}$$

If $s \geq 1$ then each injection j selects a representative form from the equivalence class in Φ^s, and the composite maps $\mathcal{I} = j \circ p : \Omega^{m,s} \to \Omega^{m,s}$, with local coordinate expressions

$$\mathcal{I}(\omega) = \frac{1}{s} \sum_{|I|=0}^{\infty} (-1)^{|I|} \theta^\alpha \wedge \left(\frac{d^{|I|}}{dx^I} i_{\partial/\partial u^\alpha_I} \omega \right) \tag{7}$$

are globally defined projection maps and are called *interior Euler operators* [1]. In the particular case $s = 1$ any $(m+1)$-form in the image $\mathcal{I}(\Omega^{m,1})$ is called a *source form* and by construction is horizontal over E. The significance of the maps $\mathcal{I}$ is that, given a Lagrangian m-form $\lambda \in \Omega^{m,0}$, the Euler-Lagrange $(m+1)$-form ε_λ is the source form $\mathcal{I} d_v \lambda \in \Omega^{m,1}$.

Two questions arising in this context can be answered using local exactness of the variational bicomplex. First, if λ is a *null Lagrangian*, so that $\varepsilon_\lambda = \mathcal{I} d_v \lambda = 0$, we see that $\delta p \lambda = p d_v \lambda = 0$ as j is an injection, so that $p\lambda = 0$ as δ is an injection; thus $\lambda = d_h \mu$ for some $\mu \in \Omega^{m-1,0}$.

Secondly, if $\varepsilon \in \Omega^{m,1}$ is a source form satisfying the condition $\delta p\varepsilon = 0$ then $p\varepsilon = \delta p\lambda$ for some $\lambda \in \Omega^{m,0}$ by exactness at Φ^1 and surjectivity of p, so that $p\varepsilon = pd_v\lambda$ and therefore $\varepsilon = \mathcal{I}d_v\lambda = \varepsilon_\lambda$. This therefore gives a solution to the *inverse problem of the calculus of variations*, and the representative $\mathcal{I}d_v\varepsilon$ is called the *Helmholtz–Sonin form* of ε (see [38], where it is called the Helmholtz form). Note that this is a comparatively easy version of the inverse problem, because it requires the given differential equations to have the same format as the Euler–Lagrange equations of the constructed Lagrangian. The problem of determining the existence of a Lagrangian whose Euler–Lagrange equations are a matrix multiple of the given equations, and hence equivalent to them, is much harder and has only partial solutions.

Note that, in the absence of information about the cohomology of E, both these argument are purely local. In addition, as all the forms constructed in this way are defined on $J^\infty\pi$, information about projectability to any particular finite-order jet manifold must be obtained by other means.

Slightly different considerations are involved when using the finite-order variational sequence, as all the relevant terms are quotients, and selecting a canonical representative of an equivalence class might result in a form of a higher order than the order of the original sequence. We shall therefore indicate the order of each term by a subscript, so that the relevant part of the diagram, together with additional terms containing representatives, would be

$$\begin{tikzcd}
\Omega_k^{m-1} \arrow[r, "d"] \arrow[d, "p"] & \Omega_k^{m} \arrow[r, "d"] \arrow[d, "p"] & \Omega_k^{m+1} \arrow[r, "d"] \arrow[d, "p"] & \Omega_k^{m+2} \arrow[d, "p"] \\
\Omega_k^{m-1}/\Theta_k^{m-1} \arrow[r] & \Omega_k^{m}/\Theta_k^{m} \arrow[r] \arrow[d, dashed, "i"] & \Omega_k^{m+1}/\Theta_k^{m+1} \arrow[r] \arrow[d, dashed, "j"] & \Omega_k^{m+2}/\Theta_k^{m+2} \arrow[d, dashed, "j"] \\
 & \Omega_{k+1}^{m} \arrow[r, "\delta"] & \Omega_{2k+1}^{m+1} \arrow[r, "\delta"] & \Omega_{2k+1}^{m+2}
\end{tikzcd}$$

The map i gives the horizontal component ω_h of $\pi^*_{k+1,k}\omega$ corresponding to the equivalence class $[\omega]$, and the maps j gives rise to the finite-order versions of the interior Euler operator [21]

$$.\mathcal{I}(\omega) = \frac{1}{s}\sum_{|I|=0}^{k}(-1)^{|I|}\theta^\alpha \wedge \left(\frac{d^{|I|}}{dx^I} i_{\partial/\partial u_I^\alpha}\omega_s\right).$$

Note that the order of $\mathcal{I}(\omega)$ is in general $2k+1$ rather than $2k$ because $\omega_s \in \Omega_{k+1}^{m+s}$, whereas a Lagrangian form λ is a horizontal m-form so that $(d\lambda)_1 \in \Omega_k^{m+1}$ and therefore $\varepsilon_\lambda = \mathcal{I}(d\lambda) \in \Omega_{2k}^{m+1}$. (It is not necessary to take the s-contact component ω_s in the corresponding formula (7) for the interior Euler operator on the variational bicomplex, because elements of $\Omega^{m,s}$ are automatically s-contact.)

As before, we may consider null Lagrangians $\lambda \in \Omega_k^m$, and source forms $\varepsilon \in \mathcal{I}(\Omega_{2k}^{m+1})$ satisfying $\delta\varepsilon = 0$, and use exactness [25]. Indeed, for the second problem an explicit construction of a suitable form $\lambda = L dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$ is given by the *Vainberg–Tonti Lagrangian*

$$L = u^\alpha \int_0^1 \varepsilon_\alpha(x^i, su_I^\beta) ds .$$

Note that if λ is a Lagrangian form of order k with Euler–Lagrange form ε_λ of order $2k$, then the Vainberg–Tonti Lagrangian of ε_λ will generally have order $2k$ rather than k, and so will differ from λ by a null Lagrangian. There are, though, circumstances where ε_λ has order less than $2k$ but there is no lower-order form $\tilde{\lambda}$ such that $\lambda - \tilde{\lambda}$ is null; a necessary condition for this is that the coefficient of λ should be a polynomial in the k-th order derivatives, of degree not more than the number of distinct multi-indices I of length k (see [36], where the proof makes explicit use of the geometric properties of vector multi-indices). This condition is not, however, sufficient; a further condition of skew-symmetry in those derivatives is also needed.

5 Outlook

It is clear that the use of jet concepts has helped to clarify some of the underlying geometry of variational problems. There are still, though, many outstanding questions, particularly for multiple-integral problems ($m > 1$) on $J^k\pi$. For example, we mentioned above that although Lepage equivalents of multiple-integral problems are known to exist, they are not unique. This raises the question of whether it is possible to make a canonical choice of a global equivalent. It is known that this cannot be done in general when $k > 2$ [17], and that if $k = 1$ then there are certainly three canonical choices: the classical Poincaré–Cartan form [15], differing from the Lagrangian form by a term which is exactly 1-contact; the Carathéodory form [5], which is a decomposable form defined for non-vanishing Lagrangians, and the ‘Fundamental Lepage equivalent’ [3], [22] which is closed precisely when the Lagrangian is null. (The second and third of these may also be defined for variational problems on the manifold of first-order contact elements, whereas the condition of being exactly 1-contact makes no sense there.) There are also versions of the Poincaré–Cartan form and the Carathéodory form for second-order Lagrangians [6], [29], [31], defined on $J^2\pi$, but it is not known whether there is a canonical choice of Lepage equivalent of a second-order Lagrangian which is closed precisely when the Lagrangian is null.

Perhaps, though, one of the most interesting and important problems is the *inverse multiplier problem for first-order multiple-integral problems*: given a family of second-order partial differential equations, when do they represent (after multiplying by an as yet undetermined matrix) the Euler–Lagrange equations of a Lagrangian? Although there has been some work on this problem in the case of a single dependent variable (see [2] for some early results) very little is known about the case of a family of equations. Nevertheless, in view of recent developments in the inverse multiplier problem for ordinary differential equations [7], [8], one might hope to see some advances in due course.

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