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A look on some results about Camassa–Holm type equations

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Abstract. We present an overview of some contributions of the author regarding Camassa–Holm type equations. We show that an equation unifying both Camassa–Holm and Novikov equations can be derived using the invariance under certain suitable scaling, conservation of the Sobolev norm and existence of peakon solutions. Qualitative analysis of the two-peakon dynamics is given.

1 Introduction

From September to December of 2018 I had the opportunity to stay a while in the Mathematical Institute of the Silesian University in Opava, Czech Republic, as a visiting professor. It was a very nice and rich experience and, in particular, I had the opportunity to visit the University of Ostrava and deliver a talk in the Ostrava Seminar on Mathematical Physics. I want to express my gratitude to Professor Pasha Zusmanovich (Ostrava) for his invitation to give a talk in that seminar and to Professor Artur Sergyeyev (Opava) for his firm encouragement and help during my visit to Opava.

My talk was concerned with the Camassa–Holm (CH) equation

$$m_t + 2\,u_x\,m + u\,m_x = 0, \quad m := u - u_{xx}, \tag{1}$$

and other similar equations sharing common properties with it, and recently I received a very kind invitation to write a survey about my talk. This is the genesis of the present work which, like my seminar, is a review of some of my papers

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regarding the CH equation and partners. More precisely, in this review I will revisit the works [3], [6], [14], [15], [16], [17] which lead to the discovery of a one-parameter family of equations having the CH equation as a very special member. In view of the purposes of the present work, some parts of the presentation will closely follow the original references.

The CH equation (1) was named after the seminal work by Camassa and Holm [8], although, as far as I know, it was discovered a decade earlier in [24]. However, in [8] the equation was derived from a physical framework and some fascinating facts about it were reported there. Some of its interesting properties are:

P1. **Bi-Hamiltonian structure**, meaning that the equation has the representations

$$m_t = -\partial_x(1-\partial_x^2)\frac{\delta\mathcal{H}'}{\delta m} = -(m\partial_x + \partial_x m)\frac{\delta\mathcal{H}}{\delta m},$$

where

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}}(u^2+u_x^2)\,\mathrm{d}x, \tag{2}$$

$$\mathcal{H}' = \frac{1}{2}\int_{\mathbb{R}}(u^3+uu_x^2)\,\mathrm{d}x,$$

$$\frac{\delta\mathcal{H}}{\delta m} = (1-\partial_x^2)^{-1}E_u\left(\frac{u^2+u_x^2}{2}\right),\quad \frac{\delta\mathcal{H}'}{\delta m} = (1-\partial_x^2)^{-1}E_u\left(\frac{u^3+uu_x^2}{2}\right),$$

and

$$E_u := \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_t\frac{\partial}{\partial u_t} + D_x^2\frac{\partial}{\partial u_{xx}} + \cdots \tag{3}$$

is the Euler–Lagrange operator.

These two representations satisfy certain properties that we do not study further here. These properties, however, make the equation bi-Hamiltonian. For further details about bi-Hamiltonian equations, see Olver [41].

P2. **Infinite hierarchy of symmetries.** As a consequence of the bi-Hamiltonian structure, it has a recursion operator $\mathcal{R} = (m\partial_x + \partial_x m)(1-\partial_x^2)^{-1}\partial_x^{-1}$. The existence of the recursion operator implies on the existence of an infinite hierarchy of symmetries, although the presence of the operator $(1-\partial_x^2)^{-1}$ brings non-local terms into them. Again, we refer to Olver's book [41] for further details about this subject, as well as [40].

P3. **Infinite hierarchy of conservation laws.** One more consequence of the bi-Hamiltonian structure is the existence of infinitely many conservation laws.

P4 **Distributional wave solutions.** The equation has the solitary waves $u(t,x) = ce^{-|x-ct|}$, called *peakons*, as solutions.

P5 **Soliton-like solutions.** A sort of non-linear superposition of the peakon solutions, called *multipeakons*.

These properties were reported in [8] and since then they have been extensively investigated in the literature. Moreover, the items above do not give us an exhaustive list of the remarkable properties of the CH equation. From the perspective of Analysis, we can also mention the following salient features of the equation: Its peakon solutions are stable [11]; the solutions with sufficient regularity can be globally defined and depending on the value of the derivative of the initial data they can also describe wave breaking [10], [12], [13]. The reader is guided to [23], [35], [36] and references thereof for further readings about qualitative properties of the solutions of the CH equation. It is also worth observing that the properties **P1**, **P2**, and **P3** above are related to *integrability*. For a review on integrability and differential equations, see [34], [38].

For a while the CH equation was the only known equation having the above properties, until the appearance of the Degasperis–Procesi (DP) equation [20]

$$m_t + 3u_x m + u m_x = 0, \quad m = u - u_{xx}, \tag{4}$$

which differs from the CH equation by the coefficient in the term $u_x m$. Both CH and DP are members of the one-parameter family of equations

$$m_t + b u_x m + u m_x = 0, \quad m = u - u_{xx}, \tag{5}$$

commonly called *b-equation* [21]. Provided that b is an integer, equation (5) has the properties **P4** and **P5** above, see [21], [28], but not necessarily the others for arbitrary values of b. As far as I know, (5) was the first family of equations having such properties and containing two integrable members [37].

An influential work in the land of the CH-type equations was made by the end of the first decade of this century [39]. There, Novikov presented a list of quadratic and cubic nonlinear equations generalising the CH equation. Among the equations discovered by Novikov there was

$$m_t + 3u u_x m + u^2 m_x = 0, \quad m = u - u_{xx}, \tag{6}$$

which is today known as Novikov equation and, like the CH and DP equations, has the properties[1] **P1–P5** above.

The CH and Novikov equations still share other properties. Below we list some of those that will be of vital importance for our purposes:

P6 **Scaling symmetry.** They are invariant under the transformation $(t, x, u) \mapsto (\lambda^b t, x, \lambda^{-1} u)$, $\lambda > 0$, with $b = 1$ for the CH equation [9] and $b = 2$ for the Novikov equation [6], [15]. This transformation has the following generator:

$$X = bt\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{7}$$

P7 **Conservation of the $H^1(\mathbb{R})$-norm of its solutions.** Both equations have (2) as a Hamiltonian and this fact implies on the conservation of the $H^1(\mathbb{R})$-norm of their solutions decaying fast enough at infinity [8], [29].

[1]Note that they share the same properties, but not necessarily the same Hamiltonians or the same solutions.

P8 **Peakons.** Both equations have peakon solutions of the type

$$u(t,x)=c^{1/b}e^{-|x-ct|}\,,$$

$c>0$, [8], [29].

As already mentioned, this paper is mostly concerned with certain common properties of the CH and Novikov equations and how these properties, more specifically, **P6–P8**, enable us to obtain a family of equations unifying (1) and (6).

The paper is organised in the following way: in the next section we present an overview on symmetries, conservation laws and elements of functional analysis which will be relevant to us. Next, in Section 3 we consider an inverse problem and solve it. Its solution, actually, is the equation unifying the CH and Novikov equations. Then, in Section 4 we study the two-peakon dynamics of the equation obtained in the previous section. Final comments are presented in Section 5.

2 Preliminaries

In this section we present the basic tools of this work. We opt to make a *to the point presentation*, but we also suggest to the interested reader enough references with further and deeper discussions/presentations about the topics discussed here. Through this paper we assume that $x\in\mathbb{R}$.

2.1 Lie point symmetries

Here we present some facts about Lie symmetries of differential equations with two independent variables and one dependent variable $u=u(t,x)$ of the type (1). Further details and more general treatment can be found in [4], [5], [32], [33], [41].

We recall that a smooth function depending on (t,x,u) and derivatives of u up to a finite, but not necessarily fixed, order is called a differential function and the set of all of these functions is denoted by $\mathcal{A}$.

The total derivative operators with respect to t and x are given by

$$\begin{aligned}
D_t&=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+\cdots\,,\\
D_x&=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xt}\frac{\partial}{\partial u_t}+u_{xx}\frac{\partial}{\partial u_x}+\cdots\,.
\end{aligned}\tag{8}$$

Let

$$\begin{aligned}
\bar{t}(t,x,u,\epsilon)&=t+\epsilon\tau(t,x,u)+\mathcal{O}(\epsilon^2),\\
\overline{x}(t,x,u,\epsilon)&=x+\epsilon\xi(t,x,u)+\mathcal{O}(\epsilon^2),\\
\overline{u}(t,x,u,\epsilon)&=u+\epsilon\eta(t,x,u)+\mathcal{O}(\epsilon^2),
\end{aligned}\tag{9}$$

be a formal one-parameter group of transformations. The coefficients τ, ξ, η, which depend only on (t,x,u), define the infinitesimal generator

$$X=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}\tag{10}$$

of the group of transformations. The transformations (9) are said to be a Lie point symmetry of an equation

$$u_t - u_{txx} = F(u, u_x, u_{xx}, u_{xxx}) \tag{11}$$

if and only if

$$X^{(3)}(u_t - u_{txx} - F) = \lambda(u_t - u_{txx} - F), \tag{12}$$

for some function $\lambda \in \mathcal{A}$, where

$$X^{(3)} = X + \zeta^t \frac{\partial}{\partial u_t} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^{xx} \frac{\partial}{\partial u_{xx}} + \zeta^{xxx} \frac{\partial}{\partial u_{xxx}} + \zeta^{txx} \frac{\partial}{\partial u_{txx}} \tag{13}$$

is the third-order prolongation of the generator (10), and

$$\begin{aligned} \zeta^t &= D_t(\eta) - (D_t\tau)u_t - (D_t\xi)u_x, \\ \zeta^x &= D_x(\eta) - (D_x\tau)u_t - (D_x\xi)u_x, \\ \zeta^{xx} &= D_x(\zeta^x) - (D_x\tau)u_{xt} - (D_x\xi)u_{xx}, \\ \zeta^{xxx} &= D_x(\zeta^{xx}) - (D_x\tau)u_{xxt} - (D_x\xi)u_{xxx}, \\ \zeta^{txx} &= D_t(\zeta^{xx}) - (D_t\tau)u_{ttx} - (D_t\xi)u_{txx}. \end{aligned} \tag{14}$$

Remark 1. In principle the third-order prolongation of the generator (10) would have more components which we neglected because the equation (11) depends only on $t, x, u, u_t, u_x, u_{xx}, u_{txx}$ and u_{xxx} and therefore we only present the components of the prolongation that really interest us.

Example 1. Let us consider the generator (7) and the equation

$$u_t + \epsilon u_{txx} + f(u)u_x + g(u)u_x u_{xx} + h(u)u_{xxx} = 0, \tag{15}$$

where f, g and h are smooth functions and $\epsilon < 0$ is an arbitrary constant. Suppose that (7) is the generator of a one-parameter group of symmetries of (15). Substituting the components $\tau = bt$, $\xi = 0$ and $\eta = -u$ into (14) and next into (13), we have

$$X^{(3)} = u\frac{\partial}{\partial u} - bt\frac{\partial}{\partial t} + (b+1)u_t\frac{\partial}{\partial u_t} + u_x\frac{\partial}{\partial u_x} + u_{xx}\frac{\partial}{\partial u_{xx}} + u_{xxx}\frac{\partial}{\partial u_{xxx}} + (b+1)u_{txx}\frac{\partial}{\partial u_{txx}}. \tag{16}$$

A simple calculation yields

$$\begin{aligned} X^{(3)}(u_t + \epsilon u_{txx} \quad &+ \quad f(u)u_x + g(u)u_x u_{xx} + h(u)u_{xxx}) = \\ &(b+1)(u_t + \epsilon u_{txx}) + (uf)'u_x + [(ug)' + g]u_x u_{xx} + (uh)'u_{xxx}, \end{aligned}$$

where the prime $'$ means derivative with respect to u. Comparison of the latter expression with (12) gives

$$\begin{aligned} (b+1)(u_t + \varepsilon u_{txx}) \quad &+ \quad (uf)'u_x + [(ug)' + g]u_x u_{xx} + (uh)'u_{xxx} = \\ &\lambda(u_t + \varepsilon u_{txx} + f(u)u_x + g(u)u_x u_{xx} + h(u)u_{xxx}). \end{aligned}$$

From the coefficients of u_t, u_x, u_xu_{xx} we are forced to conclude that

$$\lambda = b+1,\ (uf)' = \lambda f,\ (ug)' + g = \lambda g,\ (uh)' = \lambda h,$$

which reads $f(u) = \gamma u^b$, $g(u) = \sigma u^{b-1}$ and $h(u) = \theta u^b$, where γ, σ and θ are arbitrary constants. As a consequence of these calculations, the generator (7) is a Lie point symmetry generator of (15) if and only if the equation takes the form

$$u_t + \varepsilon u_{txx} + \gamma u^b u_x + \sigma u^{b-1} u_x u_{xx} + \theta u^b u_{xxx} = 0, \tag{17}$$

where γ, σ and θ are arbitrary constants.

2.2 Conservation laws

Our purpose in this subsection is to present basic facts about conservation laws. For a more rigorous and detailed discussion about this point, see the papers [1], [2], [42], [43] and the books [4], [32], [33], [41].

A conserved current of the equation (11) is a pair $C = (C^0, C^1)$, where $C^0, C^1 \in \mathcal{A}$, such that $\mathrm{Div}(C) := D_tC^0 + D_xC^1$ vanishes identically on the solutions of (11). It is possible to prove that

$$D_tC^0 + D_xC^1 = Q(u_t - u_{txx} - F), \tag{18}$$

where $Q \in \mathcal{A}$ is called characteristic of the conservation law, while the expression in (18) is referred as the characteristic form of the conservation law corresponding to the conserved current C, *e.g*, see [43].

Let us explore (18) a bit more. On the solutions of (11) the relation (18) becomes $D_tC^0 + D_xC^1 \equiv 0$. The component C^0 is called *conserved density*, while the component C^1 is known as *conserved flux*. These terms are natural in view of the following observation: assume that u is a solution of (18) decaying to 0 at infinity, with the same property holding to its derivatives and C^1, and let us define

$$\mathcal{H}[u] := \int_{\mathbb{R}} C^0\,\mathrm{d}x. \tag{19}$$

We note that:

- $u \mapsto \mathcal{H}[u]$ is a functional, not necessarily linear;
- $\mathcal{H}[u]$ depends only on t.

It is easy to check that

$$\frac{d}{dt}\mathcal{H}[u] = \int_{\mathbb{R}} D_tC^0 = -\int_{\mathbb{R}} D_xC^1\,\mathrm{d}x = C^1\big|_{-\infty}^{+\infty} = 0.$$

If we denote $u_0(x) := u(0,x)$, the last relation yields

$$\mathcal{H}[u] = \mathcal{H}[u_0]. \tag{20}$$

In particular, this implies that $\mathcal{H}[u]$ is invariant in time and is completely determined if we know the function $u(t,x)$ for some value of t, which very often is at $t = 0$.

From Theorem 4.7 of [41], we know that $E_u(L) = 0$, where E_u is the Euler–Lagrange operator given in (3), if and only if there exist functions P^0 and P^1 depending on t, x, u and its derivatives such that $L = D_t P^0 + D_x P^1$. Then, taking (3) and (18) into account, we have

$$E_u(Q(u_t - u_{txx} - F)) = 0. \tag{21}$$

Equation (21) provides a necessary and sufficient condition for finding a characteristic of a conservation law for (11).

Example 2. Let us consider the equation (17) with $b > 0$ and $Q = u$. A simple computation yields

$$\begin{aligned} &E_u\left(u(u_t + \varepsilon u_{txx} + \gamma u^b u_x + \sigma u^{b-1} u_x u_{xx} + \theta u^b u_{xxx})\right) \\ &\qquad = [\sigma - (b+1)\theta] b \left[(b-1) u^{b-2} u_x^3 + 3u^{b-1} u_x u_{xx}\right]. \end{aligned} \tag{22}$$

Therefore, the right hand side of (22) vanishes if and only if $\sigma = (b+1)\theta$. If this is the case, the equation (17) becomes

$$u_t + \varepsilon u_{txx} + \gamma u^b u_x + (b+1)\theta u^{b-1} u_x u_{xx} + \theta u^b u_{xxx} = 0 \tag{23}$$

and, in particular, we have the identity

$$\begin{aligned} &u[u_t + \varepsilon u_{txx} + \gamma u^b u_x + (b+1)\theta u^{b-1} u_x u_{xx} + \theta u^b u_{xxx}] \\ &\qquad = D_t\left(u^2 - \epsilon^2 u_x^2\right) + D_x\left(\frac{2\gamma}{2+b} u^{b+2} - 2\theta u^{b+1} u_{xx} + 2\epsilon u u_{tx}\right). \end{aligned} \tag{24}$$

2.3 Review on functional analysis

We recall that $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space, while $\mathcal{S}'(\mathbb{R})$ denotes its (topological) dual space. The elements of $\mathcal{S}(\mathbb{R})$ are called test functions, while members of $\mathcal{S}'(R)$ are referred as tempered distributions.

The Sobolev space $H^1(\mathbb{R})$ is the set of functions $f \in L^2(\mathbb{R})$ such that its (weak) derivative $f' \in L^2(\mathbb{R})$. It is a Hilbert space when endowed with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}} (uv + u_x v_x)\, \mathrm{d}x.$$

Of great importance to us is the induced norm

$$\|u\|^2_{H^1(\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R})} + \|u_x\|^2_{L^2(\mathbb{R})}.$$

In particular, from the last paragraph and equation (2) we conclude that $\mathcal{H}[u] = \|u\|^2_{H^1(\mathbb{R})}/2$.

For further details, see [7], [23], [25], [31], [44].

2.4 Peakon solutions

Here we consider peakon solutions, which will be of capital importance in this review. They can be described as follows: a peakon is a continuous wave solution with a pointed crest, and as a consequence it is not a smooth function, but it has lateral derivatives, both finite but not equal.

A more formal characterisation is, see [36] for further details: Let $I \subseteq \mathbb{R}$ be an interval and suppose that a function ϕ is continuous on it. One says that ϕ has a *peak* at a point $x \in I$ if ϕ is smooth on both $I \cap \{z \in \mathbb{R} :\ z < x\}$ and $I \cap \{z \in \mathbb{R} :\ z > x\}$ and

$$0 \neq \lim_{\epsilon\to 0^+} \phi'(x+\epsilon) = -\lim_{\epsilon\to 0^+} \phi'(x-\epsilon) \neq \pm\infty.$$

Given an ordinary differential equation, we say that a distribution ϕ is its solution if it satisfies the equation in the distributional sense. If the function ϕ has a peak, we say that the solution ϕ is a peakon solution of the equation.

Example 3. Consider the ODE

$$-c(\phi' - \phi''') + \gamma\phi^b\phi' + (b+1)\theta\phi^{b-1}\phi'\phi'' + \theta\phi^b\phi''' = 0, \tag{25}$$

where b, c, γ, and θ are constants, $c > 0$ (for convenience) and $\phi = \phi(z)$.

The weak formulation of the equation (25) is given by (see [3, Section III-A])

$$\int_{-\infty}^{+\infty} [c(\psi''-\psi)\phi' + (\gamma\psi + \theta\psi'')\phi^b\phi' + \frac{2b-1}{2}\theta\psi'\phi^{b-1}(\phi')^2 - \frac{b-1}{2}\theta\psi\phi^{b-2}(\phi')^3]dz = 0,$$

where $\psi = \psi(z)$ is an arbitrary test function.

Substituting $\phi(z) = A\,e^{-|z|}$, with $A \neq 0$, and integrating by parts, we conclude that the integral equation above is identically satisfied if and only if

$$2A(c+\theta A^b)\psi'(0) - A^{b+1}((b+2)\theta + \gamma)\int_{-\infty}^{+\infty} \psi(z)\,\mathrm{sgn}(z)e^{-(b+1)|z|}dz = 0,$$

for any test function ψ, which forces us to conclude that $c + \theta A^b = 0$ and $\gamma = -(b+2)\theta$.

In conclusion, equation (25) has the peakon solution $\phi(z) = Ae^{-|z|}$ if and only if $\theta A^b = -c$ and $\gamma = -\theta(b+2)$. Therefore, it becomes

$$-c(\phi' - \phi''') - (b+2)\theta\phi^b\phi' + (b+1)\theta\phi^{b-1}\phi'\phi'' + \theta\phi^b\phi''' = 0. \tag{26}$$

We can extend the definition of peakon solutions for equations with more than one independent variable as follows: A continuous function $u(t,x)$ is said to have a peakon at a point (t_0, x_0) if at least one of the functions $x \mapsto u(t_0, x)$ or $t \mapsto u(t, x_0)$ has a peak at $x = x_0$ or $t = t_0$, respectively, see [18].

3 An inverse problem relating symmetries, conservation laws and weak solitary waves

We now arrive at the core of the present review. We firstly note that the CH and Novikov equations are members of the class (15) satisfying the properties **P6–P8** of the Introduction. A natural question is:

Question 1. What would be the most general member of the class (15), with $h(u) \neq 0$, having the properties **P6–P8** of the Introduction?

Note, in particular, that **P7** jointly with the Sobolev Embedding Theorem implies that the solutions of the equation are continuous (with respect to the variable x), but we cannot assure smoothness in the usual sense. Then, this means that the solutions of an equation satisfying **P6–P8** may lose usual differentiability and perhaps some of them can only be considered truly solutions for the equation in the sense of distributions.

A follow-up question is:

Question 2. From the members of the equation (15) having the properties in the Question 1, what would be the most general one having multi-peakon solutions, that is, solutions behaving like a superposition of the peakon solutions?

The answer to the first question is given right now.

Theorem 1. *Up to time and space scalings, the most general equation of the type* (15) *with $h(u) \neq 0$ having the properties* **P6–P8** *is the equation*

$$u_t - u_{txx} - (b+2)u^b u_x + (b+1)u^{b-1}u_x u_{xx} + u^b u_{xxx} = 0 \tag{27}$$

or its equivalent form

$$m_t + (b+1)u^{b-1}u_x m + u^b m_x = 0, \quad m = u - u_{xx}. \tag{28}$$

Proof. We firstly note that we can assume $\epsilon = -1$ in (15) since we can make a scaling in x which is equivalent to choosing this value of ϵ. By Example 1, (15) satisfies **P6** if and only if it belongs to the class (17). In addition, the $H^1(\mathbb{R})$-norm (Sobolev norm) of the solutions of the equation (15) is conserved if and only if (17) is reduced to (23), see Example 2. Finally, assume that it has a travelling wave solution $u(t,x) = \phi(z)$, $z = x - ct$. Then the equation (23) implies the ODE (25), which admits a peakon solution if and only if it takes the form (26), that is,

$$u_t - u_{txx} - (b+2)\theta u^b u_x + (b+1)\theta u^{b-1}u_x u_{xx} + \theta u^b u_{xxx} = 0,$$

where $\theta \neq 0$. This constant, on the other hand, can be eliminated under a suitable scale in the variable t and, therefore, we obtain (27). □

Let $N \geq 1$ be an integer and assume that

$$u(t,x) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{29}$$

where $p_1, \dots, p_N$ and $q_1, \dots, q_N$ are certain (piecewise differentiable) functions.

We note that the (weak) derivatives of u with respect to t and x are, respectively,

$$u_t = \partial_t \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|} = \sum_{j=1}^{N} \left(p_j'(t) - \operatorname{sgn}(x-q_j(t))q_j'(t)p_j(t)\right) e^{-|x-q_j(t)|},$$
$$u_x = \partial_x \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|} = -\sum_{j=1}^{N} p_j(t)\operatorname{sgn}(x-q_j(t))^{-|x-q_j(t)|}.$$

Therefore, we have

$$\begin{aligned} m &= u - u_{xx} = 2\sum_{i=1}^{N} p_i(t)\delta(x-q_i(t)),\\ m_t &= 2\sum_{i=1}^{N} p_i'(t)\delta(x-q_i(t)) - 2\sum_{i=1}^{N} p_i(t)q_i'(t)\delta'(x-q_i(t)),\\ m_x &= 2\sum_{i=1}^{N} p_i(t)\delta'(x-q_i(t)). \end{aligned} \tag{30}$$

We now observe that

$$u(t,x)^b\delta'(x-q_i(t)) = u(t,q_i(t))^b\delta'(x-q_i(t)) - bu(t,q_i(t))^{b-1}u_x(t,q_i(t))\delta(x-q_i(t))$$

and

$$u(t,x)^{b-1}u_x\delta(x-q_i(t)) = u(t,q_i(t))^{b-1}u_x(t,q_i(t))\delta(x-q_i(t)).$$

Then

$$\begin{aligned} u^b(t,x)m_x &= 2\sum_{i=1}^{N} p_i(t)\Big(u\big(t,q_i(t)\big)^b\delta'(x-q_i(t))\\ &\qquad - bu\big(t,q_i(t)\big)^{b-1}u_x\big(t,q_i(t)\big)\Big)\delta(x-q_i(t)),\\ u^{b-1}u_x m &= \sum_{i=1}^{N} u(t,q_i(t))^{b-1}u_x(t,q_i(t))\delta(x-q_i(t)). \end{aligned} \tag{31}$$

Substituting (29) and (30) into (28) and taking (31) into account we obtain

$$\sum_{i=1}^{N} \left[p_i'(t) + p_i(t)u^{b-1}(t,q_i)u_x(t,q_i)\right]\delta(x-q_i(t)) \\ - \sum_{i=1}^{N} p_i(t)\left[q_i'(t) - u(t,q_i(t))^b\right]\delta'(x-q_i(t)) = 0. \tag{32}$$

Our aim now is to obtain a set of equations describing the dynamics of $p_1,\dots,p_N$ and $q_1,\dots,q_N$. A simple argument is that both δ and δ' are linearly independent distributions and, therefore, (32) holds if and only if each term multiplying these distributions vanishes. A more formal and rigorous proof can be done using the ideas presented in [19, subsection 6.2] to prove that

$$\begin{aligned} p_j'(t) &= -p_i(t)u(t,q_i(t))^{b-1}u_x(t,q_i(t)),\\ q_i'(t) &= u(t,q_i(t))^b, 1 \le i \le N, \end{aligned} \tag{33}$$

see also [3].

For $b=1$ and $b=2$ system (33) is an integrable dynamical system, see for instance [22], [30] and references therein. This system is Hamiltonian with respect to the canonical Hamiltonian structure for $b=1$, see [22], [30], and a certain non-canonical Hamiltonian structure for $b=2$, see [29], [30], with the same Hamiltonian for both $b=1$ and $b=2$, see [22], [29], [30].

We conclude this section noting that the lines above answer Question 2: Equation (28), up to re-scaling, is the largest member of (15) having multi-peakon solutions.

Remark 2. Above we proceeded in a very naive way to obtain the dynamical system (33). In [3, Section III-B] the same set of equations is derived in a more rigorous way by taking the weak formulation of the equation (28).

4 Two-peakon dynamics

The analysis of the system (33) for arbitrary N is rather difficult, but the case $N=2$ is somewhat tractable and so we focus on it from now on. Let us assume a 2-peakon solution given by

$$u(x,t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}. \tag{34}$$

Henceforth we omit the dependence on t in the functions involved and $'$ means derivatives with respect to t. Equation (33) becomes

$$\begin{cases} p_1' &= \operatorname{sgn}(q_1-q_2)\, p_1\, p_2\, e^{-|q_1-q_2|}\left(p_1 + p_2\, e^{-|q_1-q_2|}\right)^{b-1},\\ p_2' &= -\operatorname{sgn}(q_1-q_2)\, p_1\, p_2\, e^{-|q_1-q_2|}\left(p_1\, e^{-|q_1-q_2|} + p_2\right)^{b-1},\\ q_1' &= \left(p_1 + p_2\, e^{-|q_1-q_2|}\right)^{b},\\ q_2' &= \left(p_1\, e^{-|q_1-q_2|} + p_2\right)^{b}. \end{cases} \tag{35}$$

We shall also impose another condition on the solution (34): that it satisfies (20), where $\mathcal{H}[u]$ is given by (2).

In order to find $\mathcal{H}[u]$ for (34), let us note that $um = u - u_{xx} = u + u_x^2 - D(uu_x)$. Then, we have

$$\frac{1}{2}\int_{\mathbb{R}} u\, m\, \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} (u^2 + u_x^2)\, \mathrm{d}x - \frac{1}{2}uu_x\Big|_{-\infty}^{\infty} = \mathcal{H}[u] - \frac{1}{2}uu_x\Big|_{-\infty}^{\infty}.$$

In case $uu_x|_{-\infty}^{\infty} = 0$ (and the function (34) satisfies this condition), we obtain

$$\mathcal{H}[u] = \frac{1}{2}\int_{\mathbb{R}} u\, m\, \mathrm{d}x.$$

Using m given by (30) with $N=2$, the last equation yields

$$\mathcal{H}[u]=p_1^2+2p_1p_2e^{-|q_1-q_2|}+p_2^2. \tag{36}$$

Let $p_{i0}:=p_i(0)$, $q_0:=|q_1(0)-q_2(0)|$ (this constant measures the separation of the pulses at $t=0$), and

$$\mathcal{H}_0:=p_{10}^2+2p_{10}^2p_{20}^2e^{-q_0}+p_{20}^2.$$

Then, equation (36) reads

$$p_1^2+2p_1p_2e^{-|q_1-q_2|}+p_2^2=\mathcal{H}_0. \tag{37}$$

As a consequence of (37), we have the inequality

$$0\leq e^{-|q_1-q_2|}=(\mathcal{H}_0-p_1^2-p_2^2)/(2p_1p_2)\leq 1. \tag{38}$$

Noting that

$$\begin{aligned} p_1+p_2e^{-|q_1-q_2|}&=p_1+\frac{\mathcal{H}_0-p_1^2-p_2^2}{2p_1}=\frac{\mathcal{H}_0+p_1^2-p_2^2}{2p_1}=:A_1,\\ p_1+p_2e^{-|q_1-q_2|}&=\frac{\mathcal{H}_0-p_1^2-p_2^2}{2p_2}+p_2=\frac{\mathcal{H}_0-p_1^2+p_2^2}{2p_2}=:A_2, \end{aligned} \tag{39}$$

and substituting the equations (39) and (37) into the equation (35), we obtain the following system:

$$\begin{aligned} q_1'&=A_1^b, \qquad q_2'=A_2^b,\\ p_1'&=\frac{1}{2}\,\mathrm{sgn}\,(q_1-q_2)A_1^{b-1}(\mathcal{H}_0-p_1^2-p_2^2),\\ p_2'&=-\frac{1}{2}\,\mathrm{sgn}\,(q_1-q_2)A_2^{b-1}(\mathcal{H}_0-p_1^2-p_2^2). \end{aligned}$$

Assume that $q_1(t)$ and $q_2(t)$ are very far one from the other as $t\gg 1$, so that we can assume $e^{-|q_1-q_2|}\approx 0$. From (35) we then conclude that $\mathcal{H}(t)=p_1^2+p_2^2$, that is, the coefficient functions p_1 and p_2 in (34) describe a circle of radius

$$\sqrt{\mathcal{H}_0}=\|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}\,,$$

where $u_0=u(0,x)$.

Let us now assume that $q_1(t)=q_2(t)$, that is, $e^{-|q_1-q_2|}=1$. Then $\mathcal{H}_0=(p_1+p_2)^2$ and from (34) we conclude that these "two-peakons" degenerates into a single peakon solution, since their positions coincide and the sum of their amplitudes is constant. In this case, the solution can be rewritten as

$$u(t,x)=\sqrt{\mathcal{H}_0}e^{-|x-\sqrt{\mathcal{H}_0^b}t|},$$

and we then obtain nothing but the well-known solution given by the Example 3. However, we can also obtain another solution, given by

$$u(t,x)=-\sqrt{\mathcal{H}_0}e^{-|x+\sqrt{\mathcal{H}_0^b}t|}.$$

The solutions above give us the relation between the conserved quantity, the wave speed and the amplitude of the wave: let $c := \sqrt{\mathcal{H}_0^b}$. Then

$$u(t,x) = \mp c^{1/b} e^{-|x \pm ct|}$$

and $\mathcal{H}[u] = c^b$. For further details, see [3, Section IV].

5 Final comments

In this paper we reviewed some contributions of the author regarding some CH-type equations. Our main focus was to derive the equation (27) using symmetries, conservation laws and assuming that it has peakon solutions.

The first two works reporting this equation were [27] and [14], see also [16]. In [27] the equation was considered from the point of view of analysis, more precisely, from the point of view of well-posedness and related topics. By the time P.L. da Silva and me reported our discovery in [14], [16] we did not know Himonas and Holliman's work. We only discovered this work by the time we were working on [3] in collaboration with S. Anco.

In [14], [16] we considered **Question 1** without imposing **P8**. As a consequence, we derived (23). However, we noted that taking $\gamma = -\theta(b+2)$ and next choosing $\theta = 1$ we would then obtain a one-parameter family of equations unifying both CH and Novikov equations. Then we obtained (27) in an *ad hoc* way.

In [3] we worked with a particular class of equations of the type (15) and we again derived (27), but differently from the previous reference, the derivation in [3] was very natural, once we were also investigating peakon solutions. The request that the equation has peakon solutions was the restriction missing in [14], [16] to obtain (27) without any *ad hoc* procedure.

We conclude this work by observing that the equation (27) has two known integrable cases: $b = 1$ and $b = 2$, corresponding to the Camassa-Holm and Novikov equations, respectively. We do not expect other integrable members belonging to this equation and this view is supported by recent results established in [26].

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