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Metric groups, unitary representations and continuous logic

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Abstract. We describe how properties of metric groups and of unitary representations of metric groups can be presented in continuous logic. In particular we find $L_{\omega_1\omega}$-axiomatization of amenability. We also show that in the case of locally compact groups some uniform version of the negation of Kazhdan's property (T) can be viewed as a union of first-order axiomatizable classes. We will see when these properties are preserved under taking elementary substructures.

1 Introduction

In this paper we study the behavior of amenability and Kazhdan's property **(T)** under logical constructions. We view these tasks as a part of investigations of properties of basic classes of topological groups appeared in measurable and geometric group theory, see [9], [10], [13]. The fact that some logical constructions, for example ultraproducts, have become common in group theory, gives additional flavour for this topic. We concentrate on properties of metric groups which can be expressed in continuous logic [2].

Since we want to make the paper available for non-logicians, in Section 2 we briefly remind the reader some preliminaries of continuous logic. In Section 3 we apply it to amenability. In Section 4 we consider unitary representations of locally compact groups.

2 Basic continuous logic

2.1 Continuous structures

We fix a countable continuous signature

$$L = \{d, R_1, \dots, R_k, \dots, F_1, \dots, F_l, \dots\}.$$

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Let us recall that a *metric L-structure* is a complete metric space (M, d) with d bounded by 1, along with a family of uniformly continuous operations F_i on M and a family of predicates R_i, i.e. uniformly continuous maps from appropriate M^{k_i} to $[0, 1]$. It is usually assumed that to a predicate symbol R_i a continuity modulus γ_i is assigned so that when $d(x_j, x'_j) < \gamma_i(\varepsilon)$ with $1 \leq j \leq k_i$ the corresponding predicate of M satisfies

$$|R_i(x_1, \ldots, x_j, \ldots, x_{k_i}) - R_i(x_1, \ldots, x'_j, \ldots, x_{k_i})| < \varepsilon.$$

It happens very often that γ_i coincides with id. In this case we do not mention the appropriate modulus. We also fix continuity moduli for functional symbols. Each classical first-order structure can be considered as a complete metric structure with the discrete $\{0, 1\}$-metric.

By completeness, continuous substructures of a continuous structure are always closed subsets.

Atomic formulas are the expressions of the form $R_i(t_1, \ldots, t_r)$, $d(t_1, t_2)$, where t_i are terms (built from functional L-symbols). In metric structures they can take any value from $[0, 1]$. *Statements* concerning metric structures are usually formulated in the form

$$\phi = 0$$

(called an *L-condition*), where ϕ is a *formula*, i.e. an expression built from 0,1 and atomic formulas by applications of the following functions:

$$x/2 \ , \ x \dot{-} y = \max(x - y, 0) \ , \ \min(x, y) \ , \ \max(x, y) \ , \ |x - y| \ ,$$

$$\neg(x) = 1 - x \ , \ x \dot{+} y = \min(x + y, 1) \ , \ \sup_x \text{ and } \inf_x .$$

A *theory* is a set of L-conditions without free variables (here $\sup_x$ and $\inf_x$ play the role of quantifiers).

It is worth noting that any formula is a γ-uniformly continuous function from the appropriate power of M to $[0, 1]$, where γ is the minimum of continuity moduli of L-symbols appearing in the formula.

The condition that the metric is bounded by 1 is not necessary. It is often assumed that d is bounded by some rational number d_0. In this case the (truncated) functions above are appropriately modified.

We sometimes replace conditions of the form $\phi \dot{-} \varepsilon = 0$ where $\varepsilon \in [0, d_0]$ by more convenient expressions $\phi \leq \varepsilon$.

In several places of the paper we use continuous $L_{\omega_1\omega}$-logic. It extends the first-order logic by new connectives applied to countable families of formulas : $\bigvee$ is the infinitary min and $\bigwedge$ corresponds to the infinitary max. When we apply these connectives we only demand that the formulas of the family all obey the same continuity modulus, see [3].

2.2 Metric groups

Below we always assume that our metric groups are continuous structures with respect to bi-invariant metrics (see [2]). This exactly means that (G, d) is a complete metric space and d is bi-invariant. Note that the continuous logic approach takes

weaker assumptions on d. Along with completeness it is only assumed that the operations of a structure are uniformly continuous with respect to d. Thus it is worth noting here that

- any group which is a continuous structure has an equivalent bi-invariant metric.

Indeed assuming that (G, d) is a continuous metric group which is not discrete one can apply the following function:

$$d^*(x, y) = \sup_{u,v} d(u \cdot x \cdot v, u \cdot y \cdot v).$$

See Lemma 2 and Proposition 4 of [11] for further discussions concerning this observation.

2.3 The approach

In Section 3 we apply the recent paper [14] for $L_{\omega_1\omega}$-axiomatization of (non-) amenability of metric groups. The case of property **(T)** looks slightly more complicated, because unbounded metric spaces are involved in the definition.

Typically unbounded metric spaces are considered in continuous logic as many-sorted structures of n-balls of a fixed point of the space ($n \in \omega$). Section 15 of [2] contains nice examples of such structures. If the action of a bounded metric group G is isometric and preserves these balls we may consider the action as a sequence of binary operations where the first argument corresponds to G. In such a situation one just fixes a sequence of continuity moduli for G (for each n-ball). We will see in Section 4 that this approach works sufficiently well for the negation of property (**T**) (non-(**T**)) in the class of locally compact groups.

It is well-known that a locally compact group with property **(T)** is amenable if and only if it is compact. Thus it is natural to consider these properties together.

2.4 Uniform continuity

Actions of metric groups which can be analyzed by tools of continuous logic must be uniformly continuous for each sort appearing in the presentation of the space by metric balls. This slightly restricts the field of applications.

2.5 Hilbert spaces in continuous logic

We treat a Hilbert space over $\mathbb{R}$ exactly as in Section 15 of [2]. We identify it with a many-sorted metric structure

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r\in\mathbb{R}}, +, -, \langle\rangle),$$

where B_n is the ball of elements of norm $\leq n$, $I_{mn} : B_m \to B_n$ is the inclusion map, $\lambda_r : B_m \to B_{km}$ is scalar multiplication by r, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |r| < k$; furthermore, $+, - : B_n \times B_n \to B_{2n}$ are vector addition and subtraction and $\langle\rangle : B_n \to [-n^2, n^2]$ is the predicate of the inner product. The metric on each sort is given by $d(x, y) = \sqrt{\langle x - y, x - y\rangle}$. For every operation the continuity modulus is standard. For example in the case of λ_r this is $\frac{z}{|r|}$. Note that

in this version of continuous logic we do not assume that the diameter of a sort is bounded by 1. It can become any natural number.

Stating existence of infinite approximations of orthonormal bases (by a countable family of axioms, see Section 15 of [2]) we assume that our Hilbert spaces are infinite dimensional. By [2] they form the class of models of a complete theory which is κ-categorical for all infinite κ, and admits elimination of quantifiers.

This approach can be naturally extended to complex Hilbert spaces,

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{C}}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}).$$

We only extend the family $\lambda_r : B_m \to B_{km}$, $r \in \mathbb{R}$, to a family $\lambda_c : B_m \to B_{km}$, $c \in \mathbb{C}$, of scalar products by $c \in \mathbb{C}$, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |c| < k$. We also introduce Re- and Im-parts of the inner product.

3 Metric groups and amenability

Although closed subgroups of amenable locally compact groups are amenable, amenability is not preserved under elementary extensions. For example there are locally finite countable groups having elementary extensions containing free groups (i.e. in the discrete case amenability is not axiomatizable). In this section we apply the description of amenable topological groups found by F.M. Schneider and A. Thom in [14] in order to axiomatize in $L_{\omega_1\omega}$ amenability for metric groups which are continuous structures. In fact we will see that this property is $\sup\bigvee\inf$ in terms of [7] [1]. This kind of axiomatization is essential in model-theoretic forcing, see Proposition 2.6 in [7]. Our results imply that (typical) elementary substructures of (non-) amenable groups are (non-) amenable. In particular these properties are bountiful, see [12].

Let G be a topological group, $F_1, F_2 \subset G$ are finite and U be an identity neighbourhood. Let R_U be a binary relation defined as follows:

$$R_U = \{(x, y) \in F_1 \times F_2 : yx^{-1} \in U\}.$$

This relation defines a bipartite graph on (F_1, F_2). Let

$$\mu(F_1, F_2, U) = |F_1| - \sup\{|S| - |N_R(S)| : S \subseteq F_1\},$$

where $N_R(S) = \{y \in F_2 : (\exists x \in S)(x, y) \in R_U\}$. By Hall's matching theorem this value is the *matching number* of the graph (F_1, F_2, R_U). Theorem 4.5 of [14] gives the following description of amenable topological groups.

> *Let G be a Hausdorff topological group. The following are equivalent.*
> *(1) G is amenable.*
> *(2) For every $\theta \in (0, 1)$, every finite subset $E \subseteq G$, and every identity neighbourhood U, there is a finite non-empty subset $F \subseteq G$ such that*
>
> $$\forall g \in E(\mu(F, gF, U) \geq \theta|F|).$$

[1] in the discrete case this was observed in [8]

(3) There exists $\theta \in (0,1)$ such that for every finite subset $E \subseteq G$, and every identity neighbourhood U, there is a finite non-empty subset $F \subseteq G$ such that

$$\forall g \in E(\mu(F, gF, U) \geq \theta|F|).$$

It is worth noting here that when an open neighbourhood V contains U the number $\mu(F, gF, U)$ does not exceed $\mu(F, gF, V)$. In particular in the formulation above we may consider neighbourhoods U from a fixed base of identity neighbourhoods. For example in the case of a continuous structure (G, d) with an invariant d we may take all U in the form of metric balls $B_{<q} = \{x : d(1, x) < q\}$, $q \in \mathbb{Q} \cap (0, 1)$. It is also clear that we can restrict all θ by rational ones. From now on we work in this case.

In Lemma 1 we consider the Schneider-Thom theorem in the formulation where all U are closed balls $B_q = \{x : d(1, x) \leq q\}$, $q \in \mathbb{Q} \cap (0, 1)$. Notice that the corresponding versions of statement (2) above are equivalent for U of the form $B_{<q}$ and of the form B_q. Indeed this follows from the observation that $\mu(F, gF, B_{<q}) \leq \mu(F, gF, B_q)$ and $\mu(F, gF, B_q) \leq \mu(F, gF, B_{<r})$ for $q < r$.

Lemma 1. *Given $k \in \mathbb{N}$ and rational numbers $q, \theta \in (0, 1)$ there is a quantifier free formula $\phi_{k,q,\theta}(\bar{x}, y)$ depending on variables $x_1, \ldots, x_k$ and y such that in the structure (G, d) the 0-statement $\phi_{k,q,\theta}(\bar{x}, y) \leq 0$ is equivalent to the condition that $x_1, \ldots, x_k$ form a set F with $\mu(F, yF, B_q) \geq \theta|F|$.*

Moreover the identity function is a continuity modulus of y in $\phi_{k,q,\theta}(\bar{x}, y)$.

Proof. To guarantee the inequality $\mu(F, gF, B_q) \geq \theta|F|$ for an $F = \{f_1, \ldots, f_k\}$ we only need to demand that for every $S \subseteq F$ the following inequality holds:

$$|S| - k + \theta \cdot k \leq |N_R(S)|,$$

where $N_R(S)$ is defined with respect to (F, gF) and $U = B_q$.

To satisfy this inequality we will use the observation that when $S' \subseteq gF$ and ρ is a function $S' \to S$ such that $\max\{d(gf, \rho(gf)) : gf \in S'\} \leq q$ then $|S'| \leq |N_R(S))|$. Let us assume that S corresponds to some tuple $x_{i_1}, \ldots, x_{i_l}$ of elements of $\{x_1, \ldots, x_k\}$, the subset S' corresponds to some tuple of terms from $\{yx_1, \ldots, yx_k\}$ (recovered by ρ^{-1}) and let

$$\mathrm{dist}_{S,S',\rho}(x_1, \ldots x_k, y) = \max\{d(yx_i, \rho(yx_i)) : yx_i \in S'\}$$

(a max-formula of continuous logic). Then the statement formalizing the inequality $|S'| \leq |N_R(S))|$ can be expressed that the formula $\mathrm{dist}_{S,S',\rho}(x_1, \ldots x_k, y)$ takes value $\leq q$ with respect to the realization of $\bar{x}$ by the tuple $f_1, \ldots f_k$ and y by g.

Thus the following formula $\phi_{k,q,\theta}(\bar{x}, y)$ satisfies the statement of the lemma:

$$\max_{S \subseteq \{x_1, \ldots, x_k\}} \min\{\mathrm{dist}_{S,S',\rho}(x_1, \ldots x_k, y) : S' \subseteq \{yx_1, \ldots, yx_k\} \ , \ \rho : S' \to S \ ,$$

$$|S| - k + \theta \cdot k \leq |S'|\} \dot{-} q.$$

To see the last statement of the lemma it suffices to notice that the identity function is a continuity modulus of y in each $\mathrm{dist}_{S,S',\rho}(x_1, \dots x_k, y)$. The latter follows from the definition of $\mathrm{dist}_{S,S',\rho}(x_1, \dots x_k, y)$ and the fact that d is invariant and satisfies the triangle inequality. □

Theorem 1. *The class of all amenable groups which are continuous structures with invariant metrics, is axiomatizable by all $L_{\omega_1\omega}$-statements of the following form:*

$$\sup_{y_1 \dots y_l} \bigvee \{ \inf_{x_1 \dots x_k} \max\{\phi_{k,q,\theta}(\bar{x}, y_i) : 1 \le i \le l\} : k \in \omega\} \le 0,$$

where $\theta, q \in \mathbb{Q} \cap (0,1)$ and $l \in \omega$.

In particular every first order elementary substructure of a continuous structure which is an amenable group is also an amenable group.

Proof. The tuple $y_1, \dots, y_l$ consists of all free variables of the formula

$$\inf_{x_1 \dots x_k} \max\{\phi_{k,q,\theta}(\bar{x}, y_i) : 1 \le i \le l\}.$$

By the last statement of Lemma 1 the identity function is a continuity modulus of each y_i in this formula. This implies the same statement concerning the infinite disjunction in the formulation. Thus the formula in the formulation belongs to $L_{\omega_1\omega}$.

By Theorem 4.5 of [14] and the discussion after the formulation of that theorem above we see that all amenable groups satisfy the statements in the formulation.

Let us prove the contrary direction. Let (G, d) satisfy the axioms from the formulation. We want to apply condition (2) of Theorem 4.5 of [14] in the case of balls B_q. Fix θ, q and E as in the formulation so that $E = \{g_1, \dots, g_l\}$. Choose $q' \in \mathbb{Q} \cap (0,1)$ so that $q' < q$. Since (G, d) satisfies

$$\sup_{y_1 \dots y_l} \bigvee \{ \inf_{x_1 \dots x_k} \max\{\phi_{k,q',\theta}(\bar{x}, y_i) : 1 \le i \le l\} : k \in \omega\} \le 0,$$

we find $f_1, \dots, f_k \in G$ so that

$$\max\{\phi_{k,q',\theta}(\bar{f}, g_i) : 1 \le i \le l\} < q - q'.$$

By the definition of the formula $\phi_{k,q',\theta}(\bar{f}, g_i)$ we see that condition (2) of Theorem 4.5 of [14] holds for θ, $U = B_q$ and E. This proves that (G, d) is amenable.

To see the last statement of the theorem assume that (G, d) is amenable and $(G_1, d) \preceq (G, d)$. We repeat the argument of the previous paragraph for $E \subseteq G_1$. Then having found $f_1, \dots, f_k \in G$ as above we can apply the definition of an elementary substructure in order to obtain $f'_1, \dots, f'_k \in G_1$ so that

$$\max\{\phi_{k,q',\theta}(\bar{f}', g_i) : 1 \le i \le l\} < q - q'.$$

The rest is clear. □

To have a similar theorem for non-amenability we need the following lemma.

Lemma 2. *Given $k \in \mathbb{N}$ and rational numbers $q, \theta \in (0,1)$ there is a quantifier free formula $\phi^-_{k,q,\theta}(\bar{x}, y)$ depending on variables $x_1, \dots, x_k$ and y such that in the structure (G, d) the 0-statement $\phi_{k,q,\theta}(\bar{x}, y) \leq 0$ is equivalent to the condition that $x_1, \dots, x_k$ form a set F with $\mu(F, yF, B_{<q}) < \theta|F|$.*

Moreover the identity function is a continuity modulus of y in $\phi^-_{k,q,\theta}(\bar{x}, y)$.

Proof. To guarantee the inequality $\mu(F, gF, B_{<q}) < \theta|F|$ for an $F = \{f_1, \dots, f_k\}$ we only need to demand that there is $S \subseteq F$ such that

$$|N_R(S)| < |S| - k + \theta \cdot k,$$

where $N_R(S)$ is defined with respect to (F, gF) and $U = B_{<q}$.

To formalize this inequality we will use the observation that when $S' \subseteq gF$ satisfies the inequality

$$q \leq \inf\{d(gf, f')) : f' \in S \text{ and } gf \notin S'\},$$

then $|S'| \geq |N_R(S))|$. Thus if we associate to S and S' some tuple of elements of $\{x_1, \dots, x_k\}$ and some tuple of terms from $\{yx_1, \dots, yx_k\}$ respectively, then the statement $|S'| \geq |N_R(S))|$ can be expressed that the formula

$$\overline{\text{dist}}_{S,S'}(x_1, \dots x_k, y) = \inf\{d(yx_i, x_j)) : x_j \in S \text{ and } yx_i \notin S'\}$$

takes value $\geq q$ with respect to the realization of $\bar{x}$ by the tuple $f_1, \dots f_k$ and y by g.

Thus the following formula $\phi^-_{k,q,\theta}(\bar{x}, y)$ satisfies the statement of the lemma.

$$\min_{S \subseteq \{x_1, \dots, x_k\}} \min\{q \dot{-} \overline{\text{dist}}_{S,S'}(x_1, \dots x_k, y) : S' \subseteq \{yx_1, \dots, yx_k\} \text{ ,}$$
$$|S'| < |S| - k + \theta \cdot k\}.$$

The statement that the identity function is a continuity modulus of y in the formula $\phi^-_{k,q,\theta}(\bar{x}, y)$ follows from the definition of this formula and the assumption that d is an invariant metric. □

Theorem 2. *The class of all non-amenable groups which are continuous structures with invariant metrics, is axiomatizable by all $L_{\omega_1\omega}$-statements of the following form:*

$$\bigvee_q \bigvee_{l \in \omega} \{\inf_{y_1 \dots y_l} \bigwedge \{\sup_{x_1 \dots x_k} \min\{\phi^-_{k,q,\theta}(\bar{x}, y_i) : 1 \leq i \leq l\} : k \in \omega\} : q \in \mathbb{Q} \cap (0,1)\} \leq 0,$$
$$\text{where } \theta \in \mathbb{Q} \cap (0,1).$$

Moreover every subset A of a continuous structure (G, d) which is a non-amenable group is contained in a first order elementary substructure of (G, d) of density character $\leq |A| + \aleph_0$ which also a non-amenable group.

Proof. As in the proof of Theorem 1 one can show that the formula from the formulation of the theorem satisfies the requirements to be an $L_{\omega_1\omega}$-formula of continuous logic.

By Theorem 4.5 (3) of [14] and the discussion after the formulation of that theorem above we see that all non-amenable groups satisfy the statements in the formulation.

Let us prove the contrary direction. Let (G, d) satisfy the axioms from the formulation. To apply condition (3) of Theorem 4.5 of [14] fix $\theta \in \mathbb{Q} \cap (0, 1)$. For every $\varepsilon > 0$ one can choose q and $E = \{g_1, \dots, g_l\}$ so that (G, d) satisfies

$$\bigwedge\{\sup_{x_1 \dots x_k} \min\{\phi^-_{k,q,\theta}(\bar{x}, g_i) : 1 \le i \le l\} : k \in \omega\} \le \varepsilon.$$

This obviously means that

$$\bigwedge\{\sup_{x_1 \dots x_k} \min\{\phi^-_{k,q-\varepsilon,\theta}(\bar{x}, g_i) : 1 \le i \le l\} : k \in \omega\} \le 0.$$

By the definition of the formula $\phi^-_{k,q-\varepsilon,\theta}(\bar{x}, g_i)$ we see that condition (3) of Theorem 4.5 of [14] does not hold for θ, $U = B_{<q-\varepsilon}$ and E. This proves that (G, d) is not amenable.

To see the last statement of the theorem assume that (G, d) is not amenable. Then applying Theorem 4.5 (2) of [14] we find $\theta \in (0, 1)$, a finite subset $E = \{g_1, \dots, g_l\} \subseteq G$, and an identity neighbourhood $U = B_{<q}$, such that for every finite non-empty subset $F \subseteq G$

$$\exists g \in E(\mu(F, gF, U) < \theta|F|).$$

Let $(G_1, d) \preceq (G, d)$ and $E \subseteq G_1$. Then by Lemma 2

$$G_1 \models \bigwedge\{\sup_{x_1 \dots x_k} \min\{\phi^-_{k,q,\theta}(\bar{x}, g_i) : 1 \le i \le l\} : k \in \omega\} \le 0.$$

By Theorem 4.5 (2) of [14] the group (G_1, d) is not amenable. It remains to note that given A as in the formulation above by the Löwenheim-Skolem Theorem for continuous logic ([2], Proposition 7.3) the substructure G_1 can be chosen of density character $\le |A| + \aleph_0$ with $A \cup E \subset G_1$. □

4 Negating (T)

It is well-known that a locally compact group with property **(T)** of Kazhdan is amenable if and only if it is compact. Thus axiomatization of property **(T)** (non-**(T)**) is natural in the context of axiomatization of amenability (at least for locally compact groups). In this section we apply continuous logic to **(T)**/non-**(T)**. Our results are partial. On the one hand they are restricted to the class of locally compact groups and on the other one they mainly concern property non-**(T)**.

4.1 Introduction

Let a topological group G have a continuous unitary representation on a complex Hilbert space $\mathbf{H}$. A closed subset $Q \subset G$ has an **ε-invariant unit vector** in $\mathbf{H}$ if

$$\text{there exists } v \in \mathbf{H} \text{ such that } \sup_{x \in Q} \parallel x \circ v - v \parallel < \varepsilon \text{ and } \parallel v \parallel = 1.$$

A closed subset Q of the group G is called a **Kazhdan set** if there is $\varepsilon > 0$ with the following property: for every unitary representation of G on a Hilbert space where Q has an ε-invariant unit vector there is a non-zero G-invariant vector. The following statement is Proposition 1.1.4 from [1].

> Let G be a topological group. The pair $(G, \sqrt{2})$ is Kazhdan pair, i.e. if a unitary representation of G has a $\sqrt{2}$-invariant unit vector then G has a non-zero invariant vector.

If the group G has a compact Kazhdan subset then it is said that G **has property (T) of Kazhdan**.

Proposition 1.2.1 of [1] states that the group G has property (**T**) of Kazhdan if and only if any unitary representation of G which weakly contains the unit representation of G in $\mathbb{C}$ has a fixed unit vector.

By Corollary F.1.5 of [1] the property that the unit representation of G in $\mathbb{C}$ is **weakly contained** in a unitary representation π of G (this is denoted by $1_G \prec \pi$) is equivalent to the property that for every compact subset Q of G and every $\varepsilon > 0$ the set Q has an ε-invariant unit vector with respect to π.

The following example shows that in the first-order logic property (**T**) is not elementary: there are two groups G_1 and G_2 which satisfy the same sentences of the first-order logic but $G_1 \models (\mathbf{T})$ and $G_2 \not\models (\mathbf{T})$.

Example. Let $n > 2$. According Example 1.7.4 of [1] the group $SL_n(\mathbb{Z})$ has property (**T**). Let G be a countable elementary extension of $SL_n(\mathbb{Z})$ which is not finitely generated. Then by Theorem 1.3.1 of [1] the group G does not have (**T**).

The main result of this section, Theorem 3, shows that in the context of continuous logic the class of unitary representations of locally compact groups with property non-**(T)** can be viewed as the union of axiomatizable classes.

4.2 Unitary representations in continuous logic

We apply methods announced in the introduction. In order to treat axiomatizability question in the class of locally compact groups satisfying some uniform version of property non-(**T**) we need the preliminaries of continuous model theory of Hilbert spaces from Section 2.5. Moreover since we want to consider unitary representations of metric groups G in continuous logic we should fix continuity moduli for the corresponding binary functions $G \times B_n \to B_n$ induced by G-actions on metric balls of the corresponding Hilbert space.

Remark 1. Continuous unitary actions of G on B_1 obviously determine their extensions to $\bigcup\{B_i : i > 1\}$:

$$g(r \cdot x) = r \cdot g(x) \text{ where } x \in B_1 \text{ and } r \cdot x \in B_n.$$

Thus a continuity modulus, say F, for the corresponding function $G \times B_1 \to B_1$ can be considered as a family of continuity moduli for $G \times B_i \to B_i$ as follows:

$$F_i(\varepsilon) = F(\frac{\varepsilon}{i}).$$

Using this observation we simplify the approach by considering only continuity moduli for $G \times B_1 \to B_1$. When we fix such F we call the corresponding continuous unitary action of G an F-**continuous** action.

We now define a uniformly continuous versions of the notion of a Kazhdan set.

Definition 1. Let G be a metric group of diameter ≤ 1 which is a continuous structure in the language $(d, \cdot, ^{-1}, 1)$. Let F be a continuity modulus for the G-variable of continuous functions $G \times B_1 \to B_1$.

We call a closed subset Q of the group G an F-**Kazhdan set** if there is ε with the following property: every F-continuous unitary representation of G on a Hilbert space with (Q, ε)-invariant unit vectors also has a non-zero invariant vector.

It is clear that for any continuity modulus F a subset $Q \subset G$ is F-Kazhdan if it is Kazhdan. We will say that G has property F-**non-(T)** if G does not have a compact F-Kazhdan subset.

To study such actions in continuous logic let us consider a class of many-sorted continuous metric structures which consist of groups G together with metric structures of complex Hilbert spaces

$$(d, \cdot, ^{-1}, 1) \cup (\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{C}}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}).$$

Such a structure also contains a binary operation $\circ$ of an action which is defined by a family of appropriate maps $G \times B_m \to B_m$ (in fact $\circ$ is presented by a sequence of functions $\circ_m$ which agree with respect to all I_{mn}). When we add the obvious continuous sup-axioms that the action is linear and unitary, we obtain an axiomatizable class $\mathcal{K}_{GH}$. Given unitary action of G on $\mathbf{H}$ we denote by $A(G, \mathbf{H})$ the member of $\mathcal{K}_{GH}$ which is obtained from this action. When we fix a continuity modulus, say F, for the G-variables of the operation $G \times B_1 \to B_1$ we denote by $\mathcal{K}_{GH}(F)$ the corresponding subclass of $\mathcal{K}_{GH}$.

Definition 2. The class $\bigcup\{\mathcal{K}_\delta(F) : \delta \in (0,1) \cap \mathbb{Q}\}$. Let $\mathcal{K}_\delta(F)$ be the subclass of $\mathcal{K}_{GH}(F)$ axiomatizable by all axioms of the following form

$$\sup_{x_1,\dots,x_k\in G} \inf_{v\in B_m} \sup_{x\in\bigcup x_i K_\delta} \max(\parallel x \circ v - v \parallel \dot{-} \frac{1}{n}, |1 - \parallel v \parallel|) = 0,$$

$$\text{where } k, m, n \in \omega \setminus \{0\} \text{ and } K_\delta = \{g \in G : d(1, g) \leq \delta\}.$$

It is easy to see that the axiom of Definition 2 implies that each finite union $\bigcup_{i=1}^k g_i K_\delta$ has a $\frac{1}{n}$-invariant unit vector in $\mathbf{H}$. To see that it can be written by a formula of continuous logic note that $\sup_{x\in\bigcup x_i K_\delta}$ can be replaced by $\sup_x$ with simultaneous including of the quantifier-free part together with $\max(\delta \dot{-} d(x, x_i) : 1 \leq i \leq k)$ into the corresponding min-formula.

In fact the following theorem shows that in the class of locally compact metric groups condition F-non-**(T)** is a union of axiomatizable classes.

Theorem 3. *Let F be a continuity modulus for the G-variable of continuous functions $G \times B_1 \to B_1$.*

(a) In the class of all unitary F-representations of locally compact metric groups the condition of weak containment of the unit representation 1_G coincides with the condition that the corresponding structure $A(G, \mathbf{H})$ belongs to $\bigcup\{\mathcal{K}_\delta(F) : \delta \in (0,1) \cap \mathbb{Q}\}$.

*(b) In the class of all unitary F-representations of locally compact metric groups the condition of witnessing F-non-**(T)** corresponds to a union of axiomatizable classes of structures of the form $A(G, \mathbf{H})$.*

Proof. (a) Let G be a locally compact metric group and let the ball $K_\delta = \{g \in G : d(g,1) \leq \delta\} \subseteq G$ be compact. If a unitary F-representation of G weakly contains the unit representation 1_G, then considering it as a structure $A(G, \mathbf{H})$ we see that this structure belongs to $\mathcal{K}_\delta(F)$.

On the other hand if some structure of the form $A(G, \mathbf{H})$ belongs to $\mathcal{K}_\varepsilon(\mathcal{F})$, then assuming that $\varepsilon \leq \delta$ we easily see that the corresponding representation weakly contains 1_G. If $\delta < \varepsilon$, then K_ε may be non-compact. However since $K_\delta \subseteq K_\varepsilon$ any compact subset of G belongs to a finite union of sets of the form xK_ε. Thus the axioms of $\mathcal{K}_\varepsilon(F)$ state that the corresponding structure $A(G, \mathbf{H})$ defines a representation weakly containing 1_G.

(b) The condition

$$\sup_{v \in B_1} \inf_{x \in G} (1 \dot{-} (\| x \circ v - v \| + |1 - \| v \| |)) \leq 0$$

(we call it **NIV**) obviuosly implies that G does not have invariant unit vectors. For the contrary direction we use the fact mentioned in the introduction of Section 4 that the absence of G-invariant unit vectors implies that G does not have $\sqrt{2}$-invariant unit vectors. In particular for every $v \in B_1$ there is $x \in G$ such that

$$\sqrt{2} \| v \| \leq \| x \circ v - v \| .$$

This obviously implies **NIV**.

Adding **NIV** to each $\mathcal{K}_\delta(F)$ we obtain axiomatizable classes as in the statement of (b). Below we call it $\mathcal{K}_\delta^{\mathbf{NIV}}(F)$. □

5 Comments

(I) It is clear that the classes $\bigcup\{\mathcal{K}_\delta(F) : \delta \in (0,1) \cap \mathbb{Q}\}$ and $\bigcup\{\mathcal{K}_\delta^{\mathbf{NIV}}(F) : \delta \in (0,1) \cap \mathbb{Q}\}$ can be considered without the restriction of local compactness. However it does not look likely that then they axiomatize weak containment of the unit representation 1_G or witnessing non-**(T)**.

(II) In spite of axiomatizability issues in our paper it is still not clear how large the class of locally compact non-compact groups with bi-invariant merics and property **(T)**. This is especially interesting in the case of connected groups, since some standard Lie group examples do not admit compatible bi-invariant metrics.

(III) Using the method of Proposition 1.2.1 from [1] one can show that if for every compact subset Q of a locally compact metric group (G, d) and every $\varepsilon > 0$ there is an expansion of G to a structure from $\mathcal{K}_{GH}(F)$ with a (Q, ε)-invariant unit vector

but without non-zero invariant vectors, then G has a unitary F-representation which belongs to $\mathcal{K}^{\mathbf{NIV}}_\delta(F)$ for appropriate $\delta \in (0,1) \cap \mathbb{Q}$.
(IV) Since the class of locally compact metric groups is not axiomatizable [2], the subclasses of $\mathcal{K}_\delta(F)$ and $\mathcal{K}^{\mathbf{NIV}}_\delta(F)$ appearing in Theorem 3 are only *relatively axiomatizable*. On the other hand they have some nice properties of axiomatizable classes. For example the following statement holds.

Proposition 1. *Any elementary substructure of any* $A(G, \mathbf{H}) \in \bigcup\{\mathcal{K}^{\mathbf{NIV}}_\delta(F) : \delta \in (0,1) \cap \mathbb{Q}\}$ *with a locally compact* G *also belongs to* $\bigcup\{\mathcal{K}^{\mathbf{NIV}}_\delta(F) : \delta \in (0,1) \cap \mathbb{Q}\}$ *and is of the form* $A(G_0, \mathbf{H}_0)$*, where* $G_0 \preceq G$, $\mathbf{H}_0 \preceq \mathbf{H}$ *and* G_0 *is locally compact.*

In the proof we use an additional tool from model theory. Let M be a continuous metric structure. A tuple $\bar{a}$ from M^n is *algebraic* in M over A if there is a compact subset $C \subseteq M^n$ such that $\bar{a} \in C$ and the distance predicate $\mathrm{dist}(\bar{x}, C)$ is definable (in the sense of continuous logic, [2]) in M over A. Let $\mathsf{acl}(A)$ be the set of all elements algebraic over A. In continuous logic the concept of algebraicity is parallel to that in traditional model theory (see Section 10 of [2]).

Proof. Let $M \in \bigcup\{\mathcal{K}^{\mathbf{NIV}}_\delta(F) : \delta \in (0,1) \cap \mathbb{Q}\}$ and G be the group sort of M. Choose $\delta > 0$ so that the δ-ball $K = \{g \in G : d(g,1) \leq \delta\}$ in G is compact. Note that since the condition $d(g,1) \leq \delta$ defines a totally bounded complete subset in any elementary extension of G, the set K is a definable subset of $acl(\emptyset)$.

Let $M_0 \preceq M$ and G_0 be the sort of M_0 corresponding to G. It remains to verify that for any compact subset $D \subset G_0$ and any $\varepsilon > 0$ the representation M_0 always has a (D, ε)-invariant unit vector. To see this note that since $G_0 \prec G$ and K is compact and algebraic, the ball $\{g \in G_0 : d(g,1) \leq \delta\} \subset G_0$ is a compact neighborhood of 1 which coincides with K. In particular D is contained in a finite union of sets of the form gK. The rest follows from the conditions that $M_0 \in \mathcal{K}_\delta(F)$ and $G_0 \prec G$. □

We do not know if the statement of this proposition holds without the assumtion that G locally compact.
(V) In Section 8.4 of [13] it is proved that if Γ is a discrete group with property **(T)**, then the direct power Γ^ω also has property **(T)** as a topological group. On the other hand the topological group Γ^ω is a continuous metric group under the obvious metric. There is also a certain class of 'trivial examples' of non-locally compact groups with bi-invariant metrics that have property **(T)**. Namely, there are abelian metrizable groups that admit no non-trivial unitary representations, i.e. satisfying property **(T)**. Such an example can be found in [5]. Since it is extremely amenable it does not admit non-trivial unitary representations. The author does not know other non-locally compact groups which are continuous metric groups with property **(T)**. In particular are there non-trivial connected examples? This remark originally appeared in a discussion with Michal Doucha and then was extended by the referee.
(VI) The author thinks that the following question is basic in this topic.

[2] an ultraproduct of compact metric groups is not necessarily locally compact

Let (G, d) be a metric group which is a continuous structure. Assume that property **(T)** *holds in G. Does every elementary substructure of (G, d) satisfy* **(T)***?*

According the previous remark it looks reasonable to start with the discrete case:

Does an elementary substructure of a discrete group with property **(T)** *also have property* **(T)***?*

It is natural to consider this question in the case of linear groups, where property **(T)** and elementary equivalence are actively studied, see [4] and [6].
(VII) Property **FH** states that every action of G by affine isometries on a Hilbert space has a fixed point. It is equivalent to property **(T)** for σ-compact locally compact groups. Axiomatization of **FH** is studied in arXiv paper [12]. Since in this case unbounded actions appear, the approach is different there.
(VIII) One of the definitions of non-amenability says that a topological group is non-amenable if there is a locally convex topological vector space V and a continuous affine representation of G on V such that some non-empty invariant convex compact subset K of V does not contain a G-fixed point ([1], Theorem G.1.7). If we restrict ourselves just by linear representations on normed/metric vector spaces we obtain a property which is stronger than non-amenability. We call it **strong non-FP**. The paper [12] contains some results showing that the approach to non-**(T)** presented in Sections 4 and 5 can be applied to strong non-FP too. We would also mention that this arXiv paper also considers the class of groups which are not extremely amenable in some uniform way.

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