

The inverse problem in the calculus of variations: new developments

Thoan Do, Geoff Prince

Abstract. We deal with the problem of determining the existence and uniqueness of Lagrangians for systems of n second order ordinary differential equations. A number of recent theorems are presented, using exterior differential systems theory (EDS). In particular, we indicate how to generalise Jesse Douglas’s famous solution for $n = 2$. We then examine a new class of solutions in arbitrary dimension n and give some non-trivial examples in dimension 3.

Olga Rossi and the Ostrava Seminar

It has been a great privilege to have worked with Olga Rossi over many years. In addition to being an outstanding mathematician and academic, Olga was a remarkably generous and warm individual. We have both enjoyed the hospitality of the Department at the University of Ostrava under her leadership and the seminars have always been a highlight of our visits. We thank Pasha Zusmanovich for his stewardship of the seminar series and for his invitation to make this contribution.

1 The inverse problem in the calculus of variations

The inverse problem in the calculus of variations involves deciding whether the solutions of a given system of second-order ordinary differential equations (SODEs)

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b), \quad a, b = 1, \dots, n$$

2020 MSC: Primary 37J06, 70H03, 58A15 Secondary 49N45

Key words: Inverse problem in the calculus of variations, Helmholtz conditions, Exterior differential systems, Lagrangian system.

Affiliation:

Thoan Do – Department of Mathematics and Statistics, La Trobe University,
 Victoria 3086, Australia and Department of Mathematics, Can Tho University,
 Vietnam

E-mail: dtkthoan@ctu.edu.vn

Geoff Prince – Department of Mathematics and Statistics, La Trobe University,
 Victoria 3086, Australia

E-mail: g.prince@latrobe.edu.au

are the solutions of a set of Euler-Lagrange equations

$$\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b} \ddot{x}^b + \frac{\partial^2 L}{\partial x^b \partial \dot{x}^a} \dot{x}^b + \frac{\partial^2 L}{\partial t \partial \dot{x}^a} = \frac{\partial L}{\partial x^a}$$

for some Lagrangian function $L(t, x^b, \dot{x}^b)$. Clearly the Hessian matrix $\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b}$ should be invertible on some domain. The problem dates to the end of the 19th century and it still has deep importance for mathematics and mathematical physics (see [17], [14]).

Because the Euler-Lagrange equations are not generally in normal form, the problem is to find a so-called multiplier matrix $g_{ab}(t, x^c, \dot{x}^c)$ which is invertible on some domain and such that

$$g_{ab}(\dot{x}^b - F^b) \equiv \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^a} \right) - \frac{\partial L}{\partial x^a}.$$

The most commonly used set of necessary and sufficient conditions for the existence of the g_{ab} are the so-called *Helmholtz conditions* due to Douglas [11] and put in the following form by Sarlet [20]:

$$g_{ab} = g_{ba}, \quad \Gamma(g_{ab}) = g_{ac}\Gamma_b^c + g_{bc}\Gamma_a^c, \quad g_{ac}\Phi_b^c = g_{bc}\Phi_a^c, \quad \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b},$$

where

$$\Gamma_b^a := -\frac{1}{2} \frac{\partial F^a}{\partial \dot{x}^b}, \quad \Phi_b^a := -\frac{\partial F^a}{\partial x^b} - \Gamma_b^c \Gamma_c^a - \Gamma(\Gamma_b^a),$$

and where

$$\Gamma := \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a}.$$

When a solution g_{ab} exists a corresponding Lagrangian is recovered from $\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b} = g_{ab}$.

A full review of our perspective on the inverse problem as at 2008 and the role of exterior differential system theory (EDS) can be found in the article by Krupková and Prince [17] which includes reference to other approaches. A full account of the latest developments by the current authors can be found in [9], [10].

1.1 Timeline

There have been too many books and papers written about this inverse problem for us to list. Instead, we offer a brief time-line of milestones in the development of our particular approach.

1886 Sonin solves the inverse problem for one equation ($n = 1$) [23]

1887 Helmholtz states the problem [12]

1898 Hirsch states the problem [15]

1941 Douglas solves the inverse problem for $n = 2$ [11]

- 1982 Henneaux & Shepley propose an algorithm for solving the general inverse problem, identify quantum mechanical difficulties [13], [14]
- 1982 Sarlet reformulates the Helmholtz conditions [20]
- 1984 Crampin, Prince, Thompson geometrize the problem [7]
- 1990 Morandi et al develop the geometric framework [19]
- 1992 Anderson & Thompson apply the EDS technique and solve the first arbitrary n subcase [3]
- 1994 Crampin et al reframe Douglas' $n = 2$ analysis in geometric terms [8]
- 1994 Massa and Pagani introduce their linear connection for SODEs [18]
- 1999 Crampin, Prince, Sarlet & Thompson solve more arbitrary n cases [21], [22]
- 2003 Aldridge applies EDS to Douglas $n = 2$ and some arbitrary n [1], [2]
- 2016 Do and Prince identify the classification structure for arbitrary n and apply it to $n = 3$, 75 years after Douglas [9], [10]

2 Geometric formulation and EDS

We will provide only enough of the geometric setting of the inverse problem to make the later discussion viable; more complete descriptions and further references can be found in [2], [16], [17].

2.1 2nd order o.d.e's

Suppose that M is some differentiable manifold with generic local co-ordinates (x^a) . The *evolution space* is defined as $E := \mathbb{R} \times TM$, with projection onto the first factor being denoted by $t : E \rightarrow \mathbb{R}$ and bundle projection $\pi : E \rightarrow \mathbb{R} \times M$. E has adapted co-ordinates (t, x^a, u^a) associated with t and (x^a) .

A system of second order differential equations with local expression

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b), \quad a, b, = 1, \dots, n$$

is associated with a smooth vector field Γ on E given in the same co-ordinates by

$$\Gamma := \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a}.$$

Γ is called a *second order differential equation field* or SODE. It can be thought of as the total derivative operator associated with the differential equations. The integral curves of Γ are just the parametrised and lifted solution curves of the differential equations. When the system admits a Lagrangian as described in section 1, Γ is called *the Euler-Lagrange field*.

The evolution space E is equipped with the *vertical endomorphism* S , defined locally by $S := V_a \otimes \theta^a$ (see [7] for an intrinsic characterisation). S combines the *contact structure* and *vertical sub-bundle*, $V(E)$, of E , θ^a being the local contact

forms $\theta^a := dx^a - u^a dt$ and $V_a := \frac{\partial}{\partial u^a}$ forming a basis for vector fields tangent to the fibres of $\pi : E \rightarrow \mathbb{R} \times M$ (the vertical sub-bundle).

It is natural to study the deformation of S produced by the flow of Γ , $\mathcal{L}_\Gamma S$. The eigenspaces of this $(1, 1)$ tensor field produce a direct sum decomposition of each tangent space of E . It is shown in [7] that $\mathcal{L}_\Gamma S$ (acting on vectors) has eigenvalues $0, +1$ and -1 . The eigenspace at a point of E corresponding to the eigenvalue 0 is spanned by Γ , while the eigenspace corresponding to $+1$ is the *vertical subspace* of the tangent space. The remaining eigenspace (of dimension n) is called the *horizontal subspace*. Unlike the vertical subspaces these eigenspaces are not integrable; their failure to be so is due to the curvature of this nonlinear connection (induced by Γ) which itself has components

$$\Gamma_b^a := -\frac{1}{2} \frac{\partial F^b}{\partial u^a}.$$

The most useful basis for the horizontal eigenspaces has elements with local expression

$$H_a := \frac{\partial}{\partial x^a} - \Gamma_a^b \frac{\partial}{\partial u^b}$$

so that a local basis of vector fields for the direct sum decomposition of the tangent spaces of E is $\{\Gamma, H_a, V_a\}$ with corresponding dual basis $\{dt, \theta^a, \psi^a\}$ where

$$\psi^a := du^a - F^a dt + \Gamma_b^a \theta^b.$$

The components of the curvature appear in the commutators of the horizontal fields:

$$[H_a, H_b] = R_{ab}^d V_d$$

where

$$R_{ab}^d := \frac{1}{2} \left(\frac{\partial^2 F^d}{\partial x^a \partial u^b} - \frac{\partial^2 F^d}{\partial x^b \partial u^a} + \frac{1}{2} \left(\frac{\partial F^c}{\partial u^a} \frac{\partial^2 F^d}{\partial u^c \partial u^b} - \frac{\partial F^c}{\partial u^b} \frac{\partial^2 F^d}{\partial u^c \partial u^a} \right) \right).$$

In our chosen basis the curvature tensor is

$$R = R_{ab}^d \theta^a \wedge \theta^b \otimes V_d$$

It will be useful to have some other commutators:

$$[H_a, V_b] = -\frac{1}{2} \left(\frac{\partial^2 F^c}{\partial u^a \partial u^b} \right) V_c = V_b(\Gamma_a^c) V_c = V_a(\Gamma_b^c) V_c = [H_b, V_a],$$

$$[\Gamma, H_a] = \Gamma_a^b H_b + \Phi_a^b V_b, \quad [\Gamma, V_a] = -H_a + \Gamma_a^b V_b,$$

and, of course, $[V_a, V_b] = 0$.

Denoting the projectors defined by the $\mathcal{L}_\Gamma S$ -induced direct sum decomposition as P_Γ, P_V and P_H , the *Jacobi endomorphism*, Φ , is

$$\Phi = P_V \circ \mathcal{L}_\Gamma P_H = \Phi_b^a V_a \otimes \theta^b.$$

The normal forms of the component matrix $\Phi = (\Phi_b^a)$, of Φ are fundamental to the analysis of the inverse problem. While the (1,1) tensor Φ itself clearly has no real eigenspaces, the closely related *Shape Map*, A_Γ [16], captures the real eigenspaces of Φ_b^a :

$$A_\Gamma = -\Phi - P_H \circ \mathcal{L}_\Gamma P_V = -\Phi_b^a V_a \otimes \theta^b + H_a \otimes \psi^a$$

and

$$A_\Gamma(X) = \mu X \iff \mu^2 \theta^a(X) = -\Phi_b^a \theta^b(X) \text{ and } \psi^a(X) = \mu \theta^a(X).$$

In what follows we will denote by $X^{V/H}$ the vertical, respectively horizontal, copies of eigenvectors X^a of Φ_b^a belonging to μ^2 :

$$X^V := X^a V_a \quad X^H := X^a H_a$$

so that $X^H + \mu X^V$ belongs to the corresponding eigenvalue μ of A_Γ . Similarly for the eigenforms $\phi^{V/H}$.

Note: In a more complete presentation mathematical framework for the inverse problem we would also introduce the Massa and Pagani connection [18], the shape map [16] and the (jet bundle) calculus along the projection [5]. For an extensive review see [17].

2.2 The Helmholtz conditions

The Helmholtz conditions given in section 1 are the necessary and sufficient conditions that a two form $g_{ab} \psi^a \wedge \theta^b$ be closed and of maximal rank on some domain. This can be given an even more geometric framing in the following theorem from [7]:

Theorem 1. *Given a SODE Γ , the necessary and sufficient conditions for there to be Lagrangian for which Γ is the Euler–Lagrange field is that there should exist a 2–form Ω such that*

$$\Omega(V_1, V_2) = 0, \quad \forall V_1, V_2 \in V(E)$$

$$\Gamma \lrcorner \Omega = 0$$

$$d\Omega = 0$$

Ω is of maximal rank.

The simplest way to see how the Helmholtz conditions arise from theorem 1 is to put $\Omega := g_{ab} \psi^a \wedge \theta^b$ and compute $d\Omega$:

$$\begin{aligned} d\Omega &= (\Gamma(g_{ab}) - g_{cb} \Gamma_a^c - g_{ac} \Gamma_b^c) dt \wedge \psi^a \wedge \theta^b \\ &+ (H_d(g_{ab}) - g_{cb} V_a(\Gamma_d^c)) \psi^a \wedge \theta^b \wedge \theta^d \\ &+ V_c(g_{ab}) \psi^c \wedge \psi^a \wedge \theta^b \\ &+ g_{ab} \psi^a \wedge \psi^b \wedge dt \\ &+ g_{ca} \Phi_b^c \theta^a \wedge \theta^b \wedge dt \\ &+ g_{ca} H_b(\Gamma_d^c) \theta^a \wedge \theta^b \wedge \theta^d. \end{aligned}$$

The four Helmholtz conditions are

$$\begin{aligned} d\Omega(\Gamma, V_a, V_b) &= 0, & d\Omega(\Gamma, V_a, H_b) &= 0, \\ d\Omega(\Gamma, H_a, H_b) &= 0, & d\Omega(H_a, V_b, V_c) &= 0. \end{aligned}$$

The remaining conditions arising from $d\Omega = 0$, namely

$$d\Omega(H_a, H_b, V_c) = 0 \quad \text{and} \quad d\Omega(H_a, H_b, H_c) = 0,$$

can be shown to be derivable from the first four (notice that this last condition is void in dimension 2).

2.3 The EDS approach

The 1991 book by Bryant, Chern et al [4] is a comprehensive reference for exterior differential systems; in the context of the inverse problem the landmark reference is the 1992 memoir by Anderson and Thompson [3].

In exterior differential systems terms, the inverse problem is

“Find all closed, maximal rank 2-forms in $\Sigma := Sp\{\psi^a \wedge \theta^b\} \subset \wedge^2(E)$ ”

There are three steps in the EDS process:

1. Find the largest differential ideal generated by the submodule Σ . An algebraic and iterative process.
2. Create a Pfaffian system from the closure condition on this ideal. A differential process.
3. Apply the Cartan-Kähler theorem to determine the generality of the solution of this Pfaffian system. A somewhat intuitive process!

So we must find all the closed, maximal rank 2-forms on E of the form

$$g_{ab}\psi^a \wedge \theta^b,$$

where we may as well assume that g_{ab} is symmetric. So let Σ be the submodule of two forms $Sp\{\psi^a \wedge \theta^b + \psi^b \wedge \theta^a\}$, and let $\{\Omega^k\}$ be a subset of two forms in Σ . Initially we take $\{\Omega^k : k = 1, \dots, n(n+1)/2\}$ to be some basis for Σ . Then the inverse problem becomes that of finding the submodule of closed, maximal rank two forms in Σ , i.e. finding functions r_k such that $d(r_k\Omega^k) = 0$. *Note that $\{\Omega^k\}$ is a working subset of Σ which will shrink as we progress.*

The first EDS step is to find the maximal submodule, Σ' , of Σ that generates a differential ideal (that is, an ideal closed under exterior differentiation). We will find (or not) our closed two forms in this ideal.

We use the following recursive process: starting with the submodule $\Sigma^0 := \Sigma$ and a basis $\{\Omega^k\}$, find the submodule $\Sigma^1 \subseteq \Sigma^0$ such that $d\Omega \in \langle \Sigma^0 \rangle$ for all non-zero $\Omega \in \Sigma^1$. That is, find the functions r_k on E such that $d(r_k\Omega^k) \in \langle \Sigma^0 \rangle$ and hence $r_k d\Omega^k \in \langle \Sigma^0 \rangle$. This is an algebraic problem.

Having found these r_k and hence Σ^1 , we check if $\Sigma^1 = \Sigma^0$ and so is already a differential ideal. If not, we iterate the process, finding the submodule $\Sigma^2 \subset \Sigma^1 \subset \Sigma^0$ and so on until at some step, a differential ideal is found or the empty set is reached. If, at any point during this process, it is not possible to create a maximal rank two form, then the inverse problem has no solution. That is, if $\{\Omega^1, \dots, \Omega^d\}$ is a basis for Σ^i , then $\wedge^n(\sum_{k=1}^d \Omega^k)$ must be non-zero at each step.

3 Significant results from the differential ideal step

As in [10] this paper again concentrates on the case where the matrix representation, $\Phi = (\Phi_b^a)$, of Φ is diagonalisable, which corresponds to Douglas cases I, IIa or III (see [8] and [22]). Our choice of the basis for $\mathfrak{X}(E)$ is $\{\Gamma, X_a^V, X_a^H\}$, where X_a^V and X_a^H are vertical and horizontal copies of eigenvectors X_a of diagonalisable Φ (belonging to eigenvalue λ_a , possibly repeated but with a distinct label a per repetition). The corresponding copied eigenforms ϕ^{aV} and ϕ^{aH} , together with dt , form the dual basis $\{dt, \phi^{aV}, \phi^{aH}\}$. While it's not strictly accurate we will call $X_a^{V/H}$ and $\phi^{aV/H}$ eigenvectors and eigenforms of Φ .

So we start the EDS process with the module $\Sigma^0 := Sp\{\omega^{ab}\}$, where

$$\omega^{ab} := \frac{1}{2}(\phi^{aV} \wedge \phi^{bH} + \phi^{bV} \wedge \phi^{aH}), \quad 1 \leq a \leq b \leq n,$$

and look for the (final) differential ideal generated by Σ^f .

Then, having found a non-degenerate, closed 2-form $\omega = \sum_{a \leq b} r_{ab} \omega^{ab} \in \Sigma^f$, the multiplier g_{ab} is given by

$$g_{cd} = r_{ab} \phi_c^a \phi_d^b,$$

where ϕ_c^a and ϕ_d^b are components of eigenforms ϕ^a and ϕ^b respectively. In this section we review significant results obtained by applying the first step of exterior differential systems, namely the differential ideal step. See the paper [10] for details.

The exterior derivatives of eigenforms ϕ^{aV} and ϕ^{aH} are:

$$\begin{aligned} d\phi^{aV} &= -\tau_b^{a\Gamma} dt \wedge \phi^{bV} - \lambda_a dt \wedge \phi^{aH} + \tau_{cb}^{aH} \phi^{bV} \wedge \phi^{cH} + \tau_{cb}^{aV} \phi^{bV} \wedge \phi^{cV} \\ &\quad - \frac{1}{2} \phi^{aV} (R(X_b^H, X_c^H)) \phi^{bH} \wedge \phi^{cH}, \\ d\phi^{aH} &= dt \wedge \phi^{aV} - \tau_b^{a\Gamma} dt \wedge \phi^{bH} + \tau_{cb}^{aH} \phi^{bH} \wedge \phi^{cH} - \tau_{bc}^{aV} \phi^{bV} \wedge \phi^{cH}, \end{aligned}$$

The structure functions $\tau_b^{a\Gamma}$, τ_{cb}^{aH} and τ_{cb}^{aV} are defined by these expressions and the curvature tensor is that given in section 1.

Proposition 1. *The differential ideal step finishes at Σ^0 if and only if Φ is a function multiple of the identity.*

In the remainder of this section we assume that Φ is diagonalisable with distinct eigenvalues.

Proposition 2. [10] *Suppose that Φ is diagonalisable with distinct eigenvalues and eigenforms ϕ^a . Take $\Sigma^0 = Sp\{\omega^{ab}\}$ and $\omega \in \Sigma^0$. Then $\omega \in \Sigma^1$ if and only if $\omega := \sum_{d=1}^n r_d \omega^{dd}$ and the curvature satisfies*

$$\sum_{\text{cyclic } abc} r_a \phi^{aV} (R(X_b^H, X_c^H)) = 0, \quad \text{for all distinct } a, b, c, \text{ (no sum on } a). \quad (1)$$

As discussed in [10], we introduce $\tilde{\Sigma}^1 := Sp\{\omega^a := \omega^{aa}, a = 1, \dots, n\}$, not necessarily satisfying (1), so that $\Sigma^1 \subseteq \tilde{\Sigma}^1 \subset \Sigma^0$. The results show that for the case where Φ is diagonalisable with distinct eigenvalues, $\tilde{\Sigma}^1$ is the more effective option to start the differential ideal step. As we will see, this will generate an intermediate sequence of submodules of significant value.

Proposition 3. *Let Φ be diagonalisable with distinct eigenvalues. Then the necessary and sufficient conditions for $\omega = \sum_a r_a \phi^{aV} \wedge \phi^{aH} \in \tilde{\Sigma}^1$ to have its exterior derivative in the ideal $\langle \tilde{\Sigma}^1 \rangle$ are that, for all distinct a, b and c (no sum),*

$$\begin{aligned}
 r_a \tau_b^{a\Gamma} + r_b \tau_a^{b\Gamma} &= 0, \\
 r_a(\tau_{bc}^{aV} - \tau_{cb}^{aV}) - r_b \tau_{ca}^{bV} + r_c \tau_{ba}^{cV} &= 0, \\
 r_a(\tau_{bc}^{aH} - \tau_{cb}^{aH}) - r_b \tau_{ca}^{bH} + r_c \tau_{ba}^{cH} &= 0, \\
 r_a \phi^{aV}(R(X_c^H, X_b^H)) + r_b \phi^{bV}(R(X_a^H, X_c^H)) + r_c \phi^{cV}(R(X_b^H, X_a^H)) &= 0.
 \end{aligned} \tag{2}$$

The last of these is just (1).

If these conditions are satisfied for all r_a we have:

Corollary 1. *For diagonalisable Φ with distinct eigenvalues, the necessary and sufficient conditions for $\tilde{\Sigma}^1$ to generate a differential ideal are that, for all distinct a, b and c ,*

$$\tau_b^{a\Gamma} = 0, \tau_{bc}^{aV} = 0$$

In the remaining differential ideal steps, we define $\tilde{\Sigma}^{i+1} := \{\omega \in \tilde{\Sigma}^i : d\omega \in \langle \tilde{\Sigma}^i \rangle\}$. Thus $\tilde{\Sigma}^2$ is the submodule of 2-forms in $\tilde{\Sigma}^1$ which further satisfy the conditions in (2) and so $\tilde{\Sigma}^2 \subseteq \Sigma^1 \subseteq \tilde{\Sigma}^1$. The relation between the sequences $\tilde{\Sigma}^1 \supset \tilde{\Sigma}^2 \supset \dots \supset \tilde{\Sigma}^p \supset \dots$ and $\Sigma^1 \supset \Sigma^2 \supset \dots \supset \Sigma^p \supset \dots$ is as follows.

Lemma 1. $\tilde{\Sigma}^1 \supseteq \Sigma^1 \supseteq \tilde{\Sigma}^2 \supseteq \Sigma^2 \supseteq \dots \supseteq \tilde{\Sigma}^p \supseteq \Sigma^p \supseteq \dots$

This lemma makes clear the computational value of the $\tilde{\Sigma}^i$.

The following proposition indicates the sufficient condition for degenerate solutions in the distinct eigenvalue case. This will be used to exclude the cases where there are no regular solutions.

Proposition 4. *Suppose a submodule Σ^f generates a differential ideal. If any ω^a is missing from Σ^f then there is no regular solution to the inverse problem.*

Now we identify one of the key factors in our classification for the inverse problem: *integrable eigen co-distributions*.

Definition 1. The eigen co-distribution $D_a^\perp = Sp\{\phi^{aV}, \phi^{aH}\}$ of (copied) eigenforms of Φ is said to be (Frobenius) integrable if

$$d\phi^{aV}, d\phi^{aH} \equiv 0 \pmod{\phi^{aV}, \phi^{aH}},$$

equivalently

$$d\omega^a = \xi_a^a \wedge \omega^a \text{ (no sum on } a), \text{ i.e. } d\omega^a \equiv 0 \pmod{\omega^a}. \tag{3}$$

Note that

$$d\xi_a^a \equiv 0 \pmod{\phi^{aV}, \phi^{aH}}. \tag{4}$$

Proposition 5. *The necessary and sufficient conditions for an eigen co-distribution $D_a^\perp = Sp\{\phi^{aV}, \phi^{aH}\}$ of Φ to be (Frobenius) integrable are:*

$$\tau_b^{a\Gamma} = 0, \tau_{bc}^{aV} = 0, \tau_{bc}^{aH} = 0, \phi^{aV}(R(X_b^H, X_c^H)) = 0$$

for all $b, c \neq a$.

The following important result resolves the major problem of dealing with an arbitrary number of non-integrable eigendistributions of Φ .

Theorem 2. *Let Φ be diagonalisable with distinct eigenvalues. Suppose there are q non-integrable eigen co-distributions. If the sequence $\langle \tilde{\Sigma}^1 \rangle, \dots, \langle \tilde{\Sigma}^q \rangle$ does not contain a differential ideal then there is no non-degenerate solution.*

Proof. Suppose that the eigen co-distributions are ordered so that the first q are non-integrable. Firstly, if $\langle \tilde{\Sigma}^q \rangle$ is not a differential ideal, then no earlier $\langle \tilde{\Sigma}^p \rangle$ can be a differential ideal. Now each of the $n - q$ integrable $\omega^b := \phi^{bV} \wedge \phi^{bH}$ has remained in $\tilde{\Sigma}^q$ since $d\omega^b = \xi_b^b \wedge \omega^b$. However $\langle \tilde{\Sigma}^q \rangle$ is not a differential ideal so that $\dim(\tilde{\Sigma}^q) > n - q$. Now $\dim(\tilde{\Sigma}^{p+1}) < \dim(\tilde{\Sigma}^p)$ for $p < q + 1$ and so $\dim(\tilde{\Sigma}^q) \leq n - (q - 1)$. Thus $\dim(\tilde{\Sigma}^q) = n - q + 1$. But $\langle \tilde{\Sigma}^q \rangle$ is not a differential ideal by assumption and hence $\dim(\Sigma^{q+1}) = n - q$ and so $\omega^1, \dots, \omega^q$ are missing and no solution exists. \square

4 A new Classification Scheme

By observation from the results of the differential ideal step, in particular from proposition 1 and theorem 2, we suggest a more practical classification compared with that of Douglas, especially for higher dimensional problems. Our classification is based on the diagonalisability of Φ firstly, then the number of distinct eigenvalues and integrability of eigen co-distributions of Φ and lastly the step at which a differential ideal is obtained.

Case A $\Phi = \lambda I_n$. This is equivalent to $\langle \Sigma^0 \rangle$ being a differential ideal (see proposition 1).

Case B Φ is diagonalisable with distinct real-valued eigenvalues. Further subcases will be divided according to the integrability of the lifted two-dimensional eigen co-distributions of Φ i.e. q co-distributions are non-integrable and $n - q$ are integrable. According to our theorem 2, if up to and including $\langle \tilde{\Sigma}^q \rangle$ there is no differential ideal, then there is no non-degenerate multiplier. Hence, for each q , the subcases to be considered are that a differential ideal is generated at step 1, step 2, ..., up to step q .

Case C Φ is diagonalisable with repeated eigenvalues. Further subdivision according to integrability will be similar to case B above.

Case D Φ is not diagonalisable. Further subdivision depends on the integrability of normal form distributions of Φ .

As an example, we will provide here our suggested classification for the inverse problem in dimension 2 compared with the classification of Douglas. Firstly, if Φ is diagonalisable with only one eigenvalue, then Φ is the multiple of the identity which is Douglas case I. Secondly, if Φ is diagonalisable with two distinct eigenvalues, we divide it into three subcases (recall that as it is shown in proposition 1 and corollary 1 that in this case $\langle \Sigma^0 \rangle$ is not a differential ideal and $\tilde{\Sigma}^1$ is a differential ideal if and only if $\tau_2^{1\Gamma} = 0$ and $\tau_1^{2\Gamma} = 0$): the first subcase is where Φ has both integrable eigen co-distributions, that is $\tau_b^{a\Gamma} = 0, \tau_{bb}^{aV} = 0$ for all $a \neq b$, then this corresponds to the “separated” case IIa1 of Douglas; the second subcase is where Φ has one integrable and one non-integrable eigen co-distributions and a differential ideal is found at step 1, that is $\tau_2^{1\Gamma} = 0, \tau_1^{2\Gamma} = 0$ and one of the τ_{22}^{1V} and τ_{11}^{2V} is non-zero, which corresponds to Douglas case IIa2 (“semi separated”); the third subcase is where Φ has both non-integrable eigen co-distributions which is the most difficult case. We divide this case into two further subcases depending on the step at which a differential ideal is found as follows.

1. A differential ideal is found at step 1, that is $\tau_2^{1\Gamma} = \tau_1^{2\Gamma} = 0$ and both τ_{22}^{1V} and τ_{11}^{2V} are non-zero. This corresponds to Douglas case IIa3 (“non-separated”),
2. A differential ideal is found at step 2. This may correspond to case III of Douglas because it is not the case that both $\tau_2^{1\Gamma}$ and $\tau_1^{2\Gamma}$ are zero which then implies $[\bar{\nabla}\Phi, \Phi] \neq 0$.

The remaining case is where Φ is not diagonalisable, and this corresponds to case IIb of Douglas.

For a full classification and solutions for the inverse problem in dimension 2 we refer to chapter 5 of [9].

5 Case BNII

Until recently, only the two easiest cases of Douglas, case I and case IIa1, had been solved in arbitrary dimension (see [21], [6] and [3]). In [10], we investigated in details an extension of Douglas case IIa2 in arbitrary dimension n , where the matrix Φ is diagonalisable with distinct eigenvalues with exactly $n - 1$ co-distributions being integrable. We also gave two examples of the case where Φ is diagonalisable with distinct eigenvalues with two non-integrable co-distributions in dimension 3 without giving any analysis. In this section we shall provide an analysis for this case.

Case BNII is where Φ is diagonalisable with distinct eigenvalues (label ‘B’) and has 2 non-integrable eigen co-distributions (label ‘II’), in dimension n (label ‘N’). As we will see there are 3 subcases. Without loss of generality, we assume that the eigen co-distributions are ordered with the 2 non-integrable eigen co-distributions are $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$, and the other $n - 2$ eigen co-distributions, $Sp\{\phi^{\alpha V}, \phi^{\alpha H} : \alpha = 3, \dots, n\}$, are integrable. According to proposition 1, $\langle \Sigma^0 \rangle$ is not a differential ideal and the differential ideal step of EDS starts with $\tilde{\Sigma}^1 := Sp\{\omega^a := \phi^{aV} \wedge \phi^{aH} : a = 1, \dots, n\}$. Furthermore, according to theorem 2 the problem has no solution if up to $\tilde{\Sigma}^2$, the differential ideal is not found. We will discuss this in a bit more detail.

Now starting with $\tilde{\Sigma}^1 := Sp\{\omega^a := \phi^{aV} \wedge \phi^{aH} : a = 1, \dots, n\}$ and computing $d\omega^a$ for each $a = 1, \dots, n$ we have (with no sum on a)

$$\begin{aligned} d\omega^a &= d(\phi^{aV} \wedge \phi^{aH}) = d\phi^{aV} \wedge \phi^{aH} - \phi^{aV} \wedge d\phi^{aH} \\ &= \xi_a^a \wedge \omega^a + \xi_b^a \wedge \omega^b \\ &\quad - \tau_b^{a\Gamma} dt \wedge (\phi^{bV} \wedge \phi^{aH} + \phi^{aV} \wedge \phi^{bH}) \\ &\quad - \tau_{cb}^{aH} \phi^{cH} \wedge (\phi^{bV} \wedge \phi^{aH} + \phi^{aV} \wedge \phi^{bH}) \\ &\quad - \tau_{cb}^{aV} \phi^{cV} \wedge (\phi^{bV} \wedge \phi^{aH} + \phi^{aV} \wedge \phi^{bH}) \\ &\quad + \frac{1}{2} \phi^{aV} (R(X_c^H, X_b^H)) \phi^{bH} \wedge \phi^{cH} \wedge \phi^{aH}, \end{aligned}$$

for distinct a, b, c and where

$$\begin{aligned} \xi_a^a &:= A_{ab}^{aV} \phi^{bV} + A_{ab}^{aH} \phi^{bH}, \\ \xi_b^a &:= \tau_{bb}^{aV} \phi^{aV} + \tau_{bb}^{aH} \phi^{aH} \end{aligned}$$

where for all a, b, c , $A_{bc}^{aV/H} = \tau_{bc}^{aV/H} - 2\tau_{cb}^{aV/H}$.

Let $\omega = r_a \omega^a \in \tilde{\Sigma}^1$, where $\omega^a := \phi^{aV} \wedge \phi^{aH}$, $a = 1, \dots, n$. According to proposition 3, $d\omega \in \langle \tilde{\Sigma}^1 \rangle$ if and only if the homogeneous system of equations (2) are satisfied by the r_a .

With the assumption that $Sp\{\phi^{\alpha V}, \phi^{\alpha H} : \alpha = 3, \dots, n\}$ are all integrable, we have

$$\tau_c^{\alpha\Gamma} = 0, \tau_{cb}^{\alpha V} = 0, \tau_{cb}^{\alpha H} = 0, \phi^{\alpha V} (R(X_c^H, X_b^H)) = 0,$$

for distinct α, c, b with $\alpha = 3, \dots, n$. It follows then the system (2) is equivalent to

$$\begin{aligned} r_1 \tau_2^{1\Gamma} + r_2 \tau_1^{2\Gamma} &= 0, \\ r_1 \tau_\alpha^{1\Gamma} &= 0, \\ r_2 \tau_\alpha^{2\Gamma} &= 0, \\ r_1 (\tau_{2\alpha}^{1V} - \tau_{\alpha 2}^{1V}) - r_2 \tau_{\alpha 1}^{2V} &= 0, \\ r_1 \tau_{2\alpha}^{1V} - r_2 \tau_{1\alpha}^{2V} &= 0, \\ r_1 (\tau_{2\alpha}^{1H} - \tau_{\alpha 2}^{1H}) - r_2 \tau_{\alpha 1}^{2H} &= 0, \\ r_1 \tau_{2\alpha}^{1H} - r_2 \tau_{1\alpha}^{2H} &= 0, \\ r_1 \phi^{1V} (R(X_2^H, X_\alpha^H)) - r_2 \phi^{2V} (R(X_1^H, X_\alpha^H)) &= 0, \end{aligned} \tag{5}$$

for all $\alpha = 3, \dots, n$.

Note that the r_1 and r_2 are unknowns in the system (5), and they must all be non-zero for non-degenerate solutions. Let \mathbf{A}_1 denote the matrix of coefficients of the system (5). Now the problem can be divided into three subcases depending on the rank of \mathbf{A}_1 , which is 0, 1 or 2. Subcase 1: if $rank(\mathbf{A}_1) = 0$, then $\tilde{\Sigma}^1$ generates a differential ideal. Subcase 2: if $rank(\mathbf{A}_1) = 1$, then the system (5) gives a relation between the r_1 and r_2 , $r_2 = h_2 r_1$ and so affects the dimensions of the submodule in the next step, $\tilde{\Sigma}^2$, that is $dim(\tilde{\Sigma}^2) = n - 1$ and $\tilde{\Sigma}^2 := Sp\{\tilde{\omega}^1, \omega^\alpha :$

$\alpha = 3, \dots, n\}$, where $\tilde{\omega}^1 := \omega^1 + h_2\omega^2$, and where h_2 is a known function. Subcase 3: if $\text{rank}(\mathbf{A}_1) = 2$, then the solutions of the system (5) are $r_1 = 0$ and $r_2 = 0$ which is a non-existence case.

5.1 Case BNIII

Now we analyse the subcase 2 mentioned above, where Φ is diagonalisable with distinct (real) eigenvalues with exactly two non-integrable eigen co-distributions and $\text{rank}(\mathbf{A}_1) = 1$. Thus a differential ideal is not obtained at the first step, that is, $\tilde{\Sigma}^1 := Sp\{\phi^{cV} \wedge \phi^{cH} : c = 1, \dots, n\}$ does not generate a differential ideal. The results are given in theorem 3 at the end of the section followed by illustrative examples.

Solving this system (5) with the condition that r_1 and r_2 are non-zero for a non-degenerate solution, with the assumption that $\text{rank}(\mathbf{A}_1) = 1$, we get an equation relating r_1 and r_2 , $r_2 = h_2r_1$, and the conditions on the τ 's are as follows,

$$\tau_\alpha^{1\Gamma} = 0, \quad \tau_\alpha^{2\Gamma} = 0, \quad \text{for all } \alpha = 3, \dots, n \quad (6)$$

and in each of equation in the system (5), the coefficients of r_1 and r_2 must be both non-zero or both zero. Besides, since $\text{rank}(\mathbf{A}_1) = 1$ by assumption, at least one of the ratios

$$-\frac{\tau_2^{1\Gamma}}{\tau_1^{2\Gamma}}, \frac{\tau_{2\alpha}^{1V} - \tau_{\alpha 2}^{1V}}{\tau_{\alpha 1}^{2V}}, \frac{\tau_{2\alpha}^{1V}}{\tau_{1\alpha}^{2V}}, \frac{\tau_{2\alpha}^{1H} - \tau_{\alpha 2}^{1H}}{\tau_{\alpha 1}^{2H}}, \frac{\tau_{2\alpha}^{1H}}{\tau_{1\alpha}^{2H}}, \frac{\phi^{1V}(R(X_2^H, X_\alpha^H))}{\phi^{2V}(R(X_1^H, X_\alpha^H))} \quad (7)$$

for all $\alpha = 3, \dots, n$, must be well-defined and non-zero and when they are non-zero, they must be equal for all those α . Therefore h_2 equals these non-zero expressions. So we assume that the conditions (6) and (7) hold, we have

$$\tilde{\Sigma}^2 := Sp\{\tilde{\omega}^1, \omega^\alpha : \alpha = 3, \dots, n\}, \quad \text{where } \tilde{\omega}^1 = \omega^1 + h_2\omega^2.$$

Now consider the conditions for $\langle \tilde{\Sigma}^2 \rangle$ to be a differential ideal. Let $\omega \in \tilde{\Sigma}^2$, then $\omega = \tilde{r}_1\tilde{\omega}^1 + r_\alpha\omega^\alpha$, α summed $3, \dots, n$. Calculating the exterior derivative of ω , we have

$$\begin{aligned} d\omega &= d\tilde{r}_1 \wedge \tilde{\omega}^1 + \tilde{r}_1 d\tilde{\omega}^1 + dr_\alpha \wedge \omega^\alpha + r_\alpha d\omega^\alpha \\ &= (d\tilde{r}_1 + \tilde{r}_1\tilde{\xi}_1^1) \wedge \tilde{\omega}^1 + (dr_\alpha + \tilde{r}_1\tilde{\xi}_\alpha^1 + r_\alpha\xi_\alpha^\alpha) \wedge \omega^\alpha + \tilde{r}_1(dh_2 + \tilde{\xi}_2^1 - h_2\tilde{\xi}_1^1) \wedge \omega^2, \end{aligned}$$

for $\alpha = 3, \dots, n$ and where

$$\tilde{\xi}_c^1 := \xi_c^1 + h_2\xi_c^2,$$

for each $c = 1, \dots, n$. Thus $d\omega \in \langle \tilde{\Sigma}^2 \rangle$ for all $\omega \in \tilde{\Sigma}^2$ (so $\langle \tilde{\Sigma}^2 \rangle$ is a differential ideal) if and only if,

$$d\tilde{\omega}^1 = \tilde{\xi}_1^1 \wedge \tilde{\omega}^1 + \tilde{\xi}_\alpha^1 \wedge \omega^\alpha, \quad \alpha = 3, \dots, n.$$

This condition is equivalent to

$$\begin{aligned} dh_2 + \xi_2^1 + h_2(\xi_2^2 - \xi_1^1 - h_2\xi_1^2) &\equiv 0 \pmod{\phi^{2V}, \phi^{2H}} \\ \Leftrightarrow dh_2 + \xi_2^1 + h_2(\xi_2^2 - \xi_1^1) &\equiv 0 \pmod{\phi^{2V}, \phi^{2H}}, \end{aligned} \quad (8)$$

as $\xi_1^2 = \tau_{11}^{2V} \phi^{2V} + \tau_{11}^{2H} \phi^{2H} \equiv 0 \pmod{\phi^{2V}, \phi^{2H}}$.

Now let us assume that the condition (8) holds, this means $\tilde{\Sigma}^2$ is the final submodule. The next step is to find the non-degenerate and closed forms in $\tilde{\Sigma}^2$ by solving the system of Pfaffian equations

$$d\tilde{r}_1 + \tilde{r}_1 \tilde{\xi}_1^1 = 0 \quad (9)$$

$$dr_\alpha + \tilde{r}_1 \tilde{\xi}_\alpha^1 + r_\alpha \xi_\alpha^\alpha = -P_\alpha \phi^{\alpha V} - Q_\alpha \phi^{\alpha H} \quad (\text{no sum on } \alpha) \quad (10)$$

where $\alpha = 3, \dots, n$ and P_α, Q_α are arbitrary functions.

Following the EDS procedure, we extend E to a new manifold N with coordinates $(t, x^c, u^c, r_c, P_c, Q_c)$ and now the problem is to find the integrable distributions on N with $\sigma_\alpha = 0, \alpha = 3, \dots, n$ and $\sigma_1 = 0$ where

$$\sigma_1 := d\tilde{r}_1 + \tilde{r}_1 \tilde{\xi}_1^1 \quad (11)$$

$$\sigma_\alpha := dr_\alpha + \tilde{r}_1 \tilde{\xi}_\alpha^1 + r_\alpha \xi_\alpha^\alpha + P_\alpha \phi^{\alpha V} + Q_\alpha \phi^{\alpha H} \quad (12)$$

Continuing the EDS process, set $\pi_\alpha^P := dP_\alpha$ and $\pi_\alpha^Q := dQ_\alpha, \alpha = 3, \dots, n$. Using this a co-frame on N is $(dt, \phi^{dV}, \phi^{dH}, \sigma_d, \pi_d^P, \pi_d^Q)$ for $d = 1, \dots, n$. So the next step is to calculate $d\sigma_1$ and $d\sigma_\alpha$ modulo $\{\sigma_1, \sigma_\alpha : \alpha = 3, \dots, n\}$, as follows:

Taking the exterior derivative of (11) and (12) gives:

$$d\sigma_1 \equiv \tilde{r}_1 d\tilde{\xi}_1^1 \pmod{\sigma_1}$$

and, for each $\alpha = 3, \dots, n$,

$$\begin{aligned} d\sigma_\alpha &= d\tilde{r}_1 \wedge \tilde{\xi}_\alpha^1 + \tilde{r}_1 d\tilde{\xi}_\alpha^1 + dr_\alpha \wedge \xi_\alpha^\alpha + r_\alpha d\xi_\alpha^\alpha \\ &+ dP_\alpha \wedge \phi^{\alpha V} + P_\alpha d\phi^{\alpha V} + dQ_\alpha \wedge \phi^{\alpha H} + Q_\alpha d\phi^{\alpha H} \quad (\text{no sum on } \alpha) \\ &\equiv -\tilde{r}_1 \tilde{\xi}_1^1 \wedge \tilde{\xi}_\alpha^1 + \tilde{r}_1 d\tilde{\xi}_\alpha^1 \\ &+ (-\tilde{r}_1 \tilde{\xi}_\alpha^1 - r_\alpha \xi_\alpha^\alpha - P_\alpha \phi^{\alpha V} - Q_\alpha \phi^{\alpha H}) \wedge \xi_\alpha^\alpha + r_\alpha d\xi_\alpha^\alpha \\ &+ \pi_\alpha^P \wedge \phi^{\alpha V} + P_\alpha d\phi^{\alpha V} + \pi_\alpha^Q \wedge \phi^{\alpha H} + Q_\alpha d\phi^{\alpha H} \pmod{\sigma_1, \sigma_\alpha} \quad (\text{no sum on } \alpha) \end{aligned}$$

Now we see which terms in $d\sigma_\alpha$ can be absorbed into π_α^P and π_α^Q . Note that in each $d\sigma_\alpha$, any term that can be written as $\beta \wedge \phi^{\alpha V}$ or $\beta \wedge \phi^{\alpha H}$ can be absorbed into terms $\pi_\alpha^P \wedge \phi^{\alpha V}$ and $\pi_\alpha^Q \wedge \phi^{\alpha H}$ respectively. After this absorption these terms are denoted as $\tilde{\pi}_\alpha^P \wedge \phi^{\alpha V}$ and $\tilde{\pi}_\alpha^Q \wedge \phi^{\alpha H}$ and the remainder that can't be absorbed represents the 'torsion' of the system.

Recall that each eigen co-distribution $Sp\{\phi^{\alpha V}, \phi^{\alpha H}\}, \alpha = 3, \dots, n$, is integrable, so as given at (3)-(4) we have that

$$d\phi^{\alpha V} \equiv 0, \quad d\phi^{\alpha H} \equiv 0, \quad d\xi_\alpha^\alpha \equiv 0 \pmod{\phi^{\alpha V}, \phi^{\alpha H}}.$$

So the terms $d\phi^{\alpha V}, d\phi^{\alpha H}$ and $d\xi_\alpha^\alpha$ can be absorbed.

Thus, we have

$$\begin{aligned} d\sigma_\alpha &\equiv \tilde{\pi}_\alpha^P \wedge \phi^{\alpha V} + \tilde{\pi}_\alpha^Q \wedge \phi^{\alpha H} \\ &+ \tilde{r}_1 [(\xi_\alpha^\alpha - \tilde{\xi}_1^1) \wedge \tilde{\xi}_\alpha^1 + d\tilde{\xi}_\alpha^1] \pmod{\sigma_1, \sigma_\alpha} \end{aligned}$$

The torsion must be zero for nontrivial solutions and enforcing non-degeneracy results in the following conditions

$$d\tilde{\xi}_1^1 = 0 \Leftrightarrow d(\xi_1^1 + h_2\xi_1^2) = 0, \quad (13)$$

and

$$\begin{aligned} & (\xi_\alpha^\alpha - \tilde{\xi}_1^1) \wedge \tilde{\xi}_\alpha^1 + d\tilde{\xi}_\alpha^1 \equiv 0 \pmod{\phi^{\alpha V}, \phi^{\alpha H}} \\ \Leftrightarrow & (\xi_\alpha^\alpha - \xi_1^1 - h_2\xi_1^2) \wedge (\xi_\alpha^1 + h_2\xi_\alpha^2) + d(\xi_\alpha^1 + h_2\xi_\alpha^2) \equiv 0 \pmod{\phi^{\alpha V}, \phi^{\alpha H}} \end{aligned} \quad (14)$$

for each $\alpha = 3, \dots, n$.

If we assume that all conditions (6), (7), (8), (13) and (14) are satisfied, we have

$$\begin{aligned} d\sigma_1 & \equiv 0 \pmod{\sigma} \\ d\sigma_\alpha & \equiv \tilde{\pi}_\alpha^P \wedge \phi^{\alpha V} + \tilde{\pi}_\alpha^Q \wedge \phi^{\alpha H} \pmod{\sigma} \end{aligned} \quad (15)$$

We change the basis $\{\phi^{cV}, \phi^{cH}\}$ to the basis $\{\gamma^{cV}, \gamma^{cH}\}$ using:

$$\begin{aligned} \gamma^{1V/H} & = \phi^{1V/H} + \phi^{2V/H} + \dots + \phi^{nV/H} \\ \gamma^{dV/H} & = \phi^{1V/H} - \phi^{dV/H}, \quad d = 2, \dots, n \end{aligned}$$

We then get the optimal tableau:

	γ^{1V}	γ^{1H}	...	γ^{pV}	γ^{pH}	$\tilde{\Pi} =$ $\gamma^{(p+1)V}$	$\gamma^{(p+1)H}$...	γ^{nV}	γ^{nH}
σ_1	0	0	...	0	0	0	0	...	0	0
σ_3	$\tilde{\pi}_3^P$	$\tilde{\pi}_3^Q$...	0	0	$-\tilde{\pi}_3^P$	$-\tilde{\pi}_3^Q$...	0	0
\vdots	\vdots	\vdots	...	\vdots	\vdots	\vdots	\vdots	...	\vdots	\vdots
σ_n	$\tilde{\pi}_n^P$	$\tilde{\pi}_n^Q$...	0	0	0	0	...	$-\tilde{\pi}_n^P$	$-\tilde{\pi}_n^Q$

This tableau gives Cartan characters: $s_1 = n - 2$, $s_2 = n - 2$, $s_i = 0$ for $i = 3, \dots, n$.

The final step is to check for involution. To do this, we let t be the number of ways that $\tilde{\pi}_\alpha^P$ and $\tilde{\pi}_\alpha^Q$ can be altered such that (15) are unchanged. It can be seen that if we write:

$$\begin{aligned} \tilde{\pi}_\alpha^P & = \tilde{\pi}_\alpha^P + f_\alpha^1 \phi^{\alpha V} + f_\alpha^2 \phi^{\alpha H}, \\ \tilde{\pi}_\alpha^Q & = \tilde{\pi}_\alpha^Q + f_\alpha^3 \phi^{\alpha H} + f_\alpha^4 \phi^{\alpha V}, \end{aligned}$$

then (15) would be unchanged if we replace $\tilde{\pi}_\alpha^{P/Q}$ by $\tilde{\pi}_\alpha^{P/Q}$. Thus for each $\alpha = 3, \dots, n$ we have three degrees of freedom in modifying $\tilde{\pi}_\alpha^P$ and $\tilde{\pi}_\alpha^Q$, giving $3(n - 2)$ degrees of freedom for all $\tilde{\pi}_\alpha^{P/Q}$. Therefore in this case, $t = 3(n - 2)$, which equal to $s_1 + 2s_2$ as required for involution. So the solution depends on $n - 2$ functions of two variables in this case.

In summary, the result of this case is given in the following theorem.

Theorem 3. Assume that Φ is diagonalisable with distinct (real) eigenvalues and with exactly 2 non-integrable eigen co-distributions. Suppose that eigen co-distributions are ordered such that $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable. Suppose further that $\langle \tilde{\Sigma}^1 \rangle$ where $\tilde{\Sigma}^1 := Sp\{\phi^{cV} \wedge \phi^{cH} : c = 1, \dots, n\}$ is not a differential ideal. Then the necessary and sufficient for the existence of a solution for the associated inverse problem are that the conditions (6), (7), (8), (13) and (14) hold. Furthermore, the solution (if it exists) depends on $n - 2$ arbitrary functions of 2 variables each.

Example 1. This is a non-existence example of the case B3III as the condition for $\langle \tilde{\Sigma}^2 \rangle$ to be a differential ideal (8) fails.

We consider the following system

$$\ddot{x} = x\dot{y}, \quad \ddot{y} = \dot{x}, \quad \ddot{z} = 0, \tag{16}$$

on an appropriate domain. Denoting $\dot{x}, \dot{y}, \dot{z}$ by u, v, w , we find that Φ is diagonalisable with distinct eigenvalues and corresponding re-scaled eigenvectors X_a as follows,

$$\begin{aligned} \lambda_1 &= -\frac{x}{4} & \text{and} & & X_1 &= \left(\frac{u}{\sqrt[4]{v^3}}, \sqrt[4]{v}, 0\right), \\ \lambda_2 &= -\frac{4v+x}{4} & \text{and} & & X_2 &= \left(\frac{1}{\sqrt[4]{v}}, 0, 0\right), \\ \lambda_3 &= 0 & \text{and} & & X_3 &= (0, 0, 1). \end{aligned}$$

The structure functions τ 's are zero except for

$$\begin{aligned} \tau_2^{1\Gamma} &= -\frac{\sqrt{v}}{4v}, \quad \tau_1^{2\Gamma} = -\frac{3u^2}{4v\sqrt{v}}, \quad \tau_{11}^{1H} = \frac{u}{8v\sqrt[4]{v^3}}, \quad \tau_{11}^{1V} = \frac{1}{2\sqrt[4]{v^3}}, \quad \tau_{21}^{1H} = \frac{1}{8v\sqrt[4]{v}} \\ \tau_{11}^{2H} &= \frac{2xv^2 - u^2}{2v^2\sqrt[4]{v}}, \quad \tau_{11}^{2V} = \frac{-u}{v\sqrt[4]{v}}, \quad \tau_{12}^{2H} = \frac{-u}{8v\sqrt[4]{v^3}}, \quad \tau_{12}^{2V} = -\frac{1}{2\sqrt[4]{v^3}} \\ \tau_{22}^{2V} &= -\frac{1}{8v\sqrt[4]{v}}, \quad \phi^{2V}(R(X_1^H, X_2^H)) = -\sqrt[4]{v} \end{aligned}$$

These results show that the eigen co-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$, $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable, and the third one is integrable by proposition 5 and $\langle \Sigma^1 \rangle$ is not a differential ideal by corollary 1 and that the conditions (6) and (7) hold with

$$h_2 = -\frac{\tau_2^{1\Gamma}}{\tau_1^{2\Gamma}} = -\frac{v}{3u^2}.$$

Further examination is whether or not the condition (8) holds. Calculations gives

$$\begin{aligned} dh_2 &= d\left(-\frac{v}{3u^2}\right) \equiv \frac{2xv^2 - u^2}{3u^3} dt + \frac{x(4v^2 - u^3)}{6u^5\sqrt[4]{v^3}} \phi^{1H} \pmod{\phi^{2V}, \phi^{2H}}, \\ \xi_2^2 &= A_{21}^{2V} \phi^{1V} + A_{21}^{2H} \phi^{1H} + A_{23}^{2V} \phi^{3V} + A_{23}^{2H} \phi^{3V} \\ &= \frac{1}{\sqrt[4]{v^3}} \phi^{1V} + \frac{u}{4v\sqrt[4]{v^3}} \phi^{1H}, \end{aligned}$$

$$\begin{aligned} \xi_1^1 &= A_{12}^{1V} \phi^{2V} + A_{12}^{1H} \phi^{2H} + A_{13}^{1V} \phi^{3V} + A_{13}^{1H} \phi^{3H} \\ &= -\frac{1}{\sqrt[4]{v}} \phi^{2H}, \\ \xi_2^1 &= 0. \end{aligned}$$

Thus, the condition (8) does not hold and so $\langle \tilde{\Sigma}^2 \rangle$ is not a differential ideal. Therefore, the corresponding inverse problem of this system of second-order ordinary differential equations (16) has no regular solutions.

Example 2. We consider another example of the subcase BNII1 analysed above. This B3II1 system was introduced by us in [10],

$$\ddot{x} = zt, \quad \ddot{y} = 0, \quad \ddot{z} = x,$$

on an appropriate domain. Denoting the derivatives by u, v, w , we find that Φ is diagonalisable with distinct eigenvalues and corresponding eigenvectors X_a as follows,

$$\begin{aligned} \lambda_1 = \sqrt{t} & \quad \text{and} \quad X_1 = (-\sqrt{t}, 0, 1), \\ \lambda_2 = -\sqrt{t} & \quad \text{and} \quad X_2 = (\sqrt{t}, 0, 1), \\ \lambda_3 = 0 & \quad \text{and} \quad X_3 = (0, 1, 0). \end{aligned}$$

The structure functions τ 's are zero except for

$$\tau_1^{1\Gamma} = \tau_2^{2\Gamma} = -\tau_2^{1\Gamma} = -\tau_1^{2\Gamma} = \frac{1}{4t}.$$

These results show that the eigen co-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$, $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable, and the third one is integrable by proposition 5. Also $\langle \tilde{\Sigma}^1 \rangle$ is not a differential ideal by corollary 1, and that the conditions (6) and (7) hold with $h_2 = -1$. Further examination gives

$$d\tilde{\omega}^1 = -\frac{1}{2t} dt \wedge \tilde{\omega}^1, \quad \tilde{\omega}^1 = \omega^1 - \omega^2,$$

that is, the condition (5.1) holds with $\tilde{\xi}_1^1 = -\frac{1}{2t} dt$ and $\tilde{\xi}_3^1 = 0$ and so $\tilde{\Sigma}^2 := Sp\{\tilde{\omega}^1, \omega^3\}$ generates a differential ideal. The remaining conditions (13) and (14) also hold for solution as $\tilde{\xi}_1^1 = -\frac{1}{2t} dt$ and $\tilde{\xi}_3^1 = 0$. Therefore this system is variational and the solution depends on one arbitrary function of two variables.

To determine the explicit expression of the Cartan two-form for this example, we examine the Pfaffian equations (9) and (10). Explicitly, in this example, they are

$$\begin{aligned} d\tilde{r}_1 + \tilde{r}_1 \tilde{\xi}_1^1 &= 0, \\ dr_3 + P_3 \phi^{3V} + Q_3 \phi^{3H} &= 0 \end{aligned}$$

We then find that $\tilde{r}_1 = G\sqrt{t}$ where G is a constant and $r_3 = r_3(u_3^1, u_3^2)$ is an arbitrary function of two variables $u_3^1 = y - vt$ and $u_3^2 = v$. Thus the Cartan 2-form finally is

$$\omega = G\sqrt{t}(\omega^1 - \omega^2) + r_3(u_3^1, u_3^2)\omega^3.$$

In the next section, we will present the results for subcase 1 of case BNII in which the rank of the system (5) is zero, that is, $\langle \tilde{\Sigma}^1 \rangle$ is differential ideal.

5.2 Case BNII0

This is the case where Φ is diagonalisable with distinct eigenvalues, two non-integrable eigen co-distributions, $n - 2$ integrable eigen co-distributions and the rank of A_1 is zero, that is, $\tilde{\Sigma}^1 := Sp\{\omega^a : a = 1, \dots, n\}$ generates a differential ideal. In the case $n = 2$, this corresponds to the most difficult case of Douglas, case IIa3, and not entirely complete in his paper. Again, we assume that $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable and $Sp\{\phi^{\alpha V}, \phi^{\alpha H} : \alpha = 3, \dots, n\}$ are integrable.

A full analysis can be found in [9], we restrict ourselves to stating an abbreviated version of the main result and an example in $n = 3$. The ‘certain conditions’ referred to in the theorem below correspond for this case to the conditions in theorem 3 for case BNII1.

Theorem 4. *Assume that Φ is diagonalisable with distinct eigenvalues having 2 non-integrable eigen co-distributions, $n - 2$ integrable co-distributions and the rank of A_1 is zero. The existence of solutions to the inverse problem depends on whether or not certain conditions (see [9]) are satisfied. The solution (if it exists) depends on $n - 2$ functions of two variables.*

Example 3. This example is a straightforward modification of a case B2II0 example where the added equation produces an integrable eigen co-distribution. Consider the system

$$\ddot{x} = x\dot{z}, \quad \dot{y} = x, \quad \dot{z} = x$$

on an appropriate domain. Again denoting the derivatives by u, v, w , we find

$$\Phi = \begin{pmatrix} -w & 0 & \frac{u}{2} \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

is diagonalisable with distinct eigenvalues and corresponding eigenvectors X_a chosen so that $\hat{\nabla}_\Gamma X_a^V = 0$:

$$\begin{aligned} \lambda_1 &= \sqrt{-2u + w^2} - w & \text{and} & & X_1 &= (-\sqrt{-2u + w^2} + w, 2, 2), \\ \lambda_2 &= -\sqrt{-2u + w^2} - w & \text{and} & & X_2 &= (\sqrt{-2u + w^2} + w, 2, 2), \\ \lambda_3 &= 0 & \text{and} & & X_3 &= (0, 1, 0). \end{aligned}$$

The non-zero functions τ_{bc}^{aV} and τ_{bc}^{aH} are

$$\begin{aligned} \tau_{11}^{1V} &= -\tau_{11}^{2V} = \frac{\sqrt{-2u + w^2} - w}{2(2u - w^2)}, & \tau_{11}^{1H} &= -\tau_{11}^{2H} = \frac{x}{2(2u - w^2)}, \\ \tau_{12}^{1V} &= -\tau_{12}^{2V} = \frac{3\sqrt{-2u + w^2} + w}{2(2u - w^2)}, & \tau_{12}^{1H} &= -\tau_{12}^{2H} = \frac{-x}{2(2u - w^2)}, \\ \tau_{21}^{1V} &= -\tau_{21}^{2V} = \frac{3\sqrt{-2u + w^2} - w}{2(2u - w^2)}, & \tau_{21}^{1H} &= -\tau_{21}^{2H} = \frac{x}{2(2u - w^2)}, \end{aligned}$$

$$\tau_{22}^{1V} = -\tau_{22}^{2V} = \frac{\sqrt{-2u + w^2} + w}{2(2u - w^2)}, \quad \tau_{22}^{1H} = -\tau_{22}^{2H} = \frac{-x}{2(2u - w^2)}.$$

These results show that the eigen co-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$, $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable, and the third one is integrable and $\langle \tilde{\Sigma}^1 \rangle$ is differential ideal. Furthermore, the existence conditions of theorem 4 are also satisfied. So, the solution depends on one arbitrary function of two variables.

Acknowledgements

We thank Willy Sarlet for useful discussions and his continuing interest.

References

- [1] J.E. Aldridge: Aspects of the Inverse Problem in the Calculus of Variations. (2003)
- [2] J.E. Aldridge, G. E. Prince, W. Sarlet, G. Thompson: An EDS approach to the inverse problem in the calculus of variations. *J. Math. Phys.* 47 (2006).
- [3] I. Anderson, G. Thompson: The inverse problem of the calculus of variations for ordinary differential equations. *Memoirs Amer. Math. Soc.* 98 (473) (1992).
- [4] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, P.A. Griffiths: *Exterior Differential Systems*. Springer (1991).
- [5] M. Crampin, E. Martínez, W. Sarlet: Linear connections for systems of second-order ordinary differential equations. *Ann. Inst. H. Poincaré Phys. Théor.* 65 (1996) 223–249.
- [6] M. Crampin, G.E. Prince, W. Sarlet, G. Thompson: The inverse problem of the calculus of variations: separable systems. *Acta Appl. Math.* 57 (1999) 239–254.
- [7] M. Crampin, G.E. Prince, G. Thompson: A geometric version of the Helmholtz conditions in time-dependent Lagrangian dynamics. *J. Phys. A: Math. Gen.* 17 (1984) 1437–1447.
- [8] M. Crampin, W. Sarlet, E. Martínez, G.B. Byrnes, G.E. Prince: Toward a geometrical understanding of Douglas’s solution of the inverse problem in the calculus of variations. *Inverse Problems* 10 (1994) 245–260.
- [9] T. Do.: The Inverse Problem in the Calculus of Variations via Exterior Differential Systems. (2016)
- [10] T. Do, G.E. Prince: New progress in the inverse problem in the calculus of variations. *Diff. Geom. Appl.* 45 (2016) 148–179.
- [11] J. Douglas: Solution of the inverse problem of the calculus of variations. *Trans. Am. Math. Soc.* 50 (1941) 71–128.
- [12] H. Helmholtz: Über der physikalische Bedeutung des Princips der kleinsten Wirkung. *J. Reine Angew. Math.* 100 (1887) 137–166.
- [13] M. Henneaux: On the inverse problem of the calculus of variations. *J. Phys. A: Math. Gen.* 15 (1982) L93–L96.
- [14] M. Henneaux, L. C. Shepley: Lagrangians for spherically symmetric potentials. *J. Math. Phys.* 23 (1988) 2101–2107.
- [15] A. Hirsch: Die Existenzbedingungen des verallgemeinerten kinetischen Potentialen. *Math. Ann.* 50 (1898) 429–441.
- [16] M. Jerie, G.E. Prince: Jacobi fields, linear connections for arbitrary second order ODE’s. *J. Geom. Phys.* 43 (2002) 351–370.

- [17] O. Krupková, G.E. Prince: *Second order ordinary differential equation in jet bundles, the inverse problem of the calculus of variation*. in: *HandBook of Global Analysis*, Elsevier (2008).
- [18] E. Massa, E. Pagani: Jet bundle geometry, dynamical connections,, the inverse problem of Lagrangian mechanics. *Ann. Inst. Henri Poincaré, Phys. Theor.* 61 (1994) 17–62.
- [19] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo, C. Rubano: The inverse problem in the calculus of variations, the geometry of the tangent bundle. *Phys. Rep.* 188 (1990) 147–284.
- [20] W. Sarlet: The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics. *J. Phys. A: Math. Gen.* 15 (1982) 1503–1517.
- [21] W. Sarlet, M. Crampin, E. Martínez: The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations. *Acta Appl. Math.* 54 (1998) 233–273.
- [22] W. Sarlet, G. Thompson, G.E. Prince: The inverse problem of the calculus of variations: the use of geometrical calculus in Douglas’s analysis. *Trans. Amer. Math. Soc.* 354 (2002) 2897–2919.
- [23] N. Ya. Sonin: On the definition of maximal, minimal properties. *Warsaw Univ. Izvestiya* 1–2 (1886) 1–68.

Received: 16 November 2019

Accepted for publication: 17 January 2020

Communicated by: Pasha Zusmanovich