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On Balancing and Lucas-balancing Quaternions

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Abstract. The aim of this article is to investigate two new classes of quaternions, namely, balancing and Lucas-balancing quaternions that are based on balancing and Lucas-balancing numbers, respectively. Further, some identities including Binet's formulas, summation formulas, Catalan's identity, etc. concerning these quaternions are also established.

1 Introduction

Quaternions were introduced by W. R. Hamilton in the middle of 19th century; they are an extension of complex numbers. A quaternion q is a hyper-complex number defined by the equation

$$q = ae_0 + be_1 + ce_2 + de_3 = (a, b, c, d)$$

where $a, b, c,\ d$ are members of the set of real numbers $\mathbb{R}$ and e_0, e_1, e_2, e_3 with $e_0 = 1$ form a standard orthonormal basis in $\mathbb{R}^4$. The set of quaternions is usually denoted by $\mathbb{H}$ and constitutes a non-commutative field known as skew field that extends the complex field $\mathbb{C}$. The standard basis vectors e_0, e_1, e_2, e_3 satisfy the quaternion multiplication as per the following multiplication table (Table 1).

If p and q are any two quaternions in $\mathbb{H}$, say,

$$p = (p_0, p_1, p_2, p_3) \quad \text{and} \quad q = (q_0, q_1, q_2, q_3)\,,$$

then their addition and substraction are defined as

$$p \pm q = (p_0 \pm q_0)e_0 + (p_1 \pm q_1)e_1 + (p_2 \pm q_2)e_2 + (p_3 \pm q_3)e_3\,.$$

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$*$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

Table 1: The multiplication table for the basis of $\mathbb{H}$

Further, if we rewrite $p = p_0 + P$ and $q = q_0 + Q$ where $P = p_1e_1 + p_2e_2 + p_3e_3$ and $Q = q_1e_1 + q_2e_2 + q_3e_3$, then their multiplication is defined as

$$pq = p_0q_0 - P \cdot Q + p_0Q + q_0P + P \times Q\,.$$

Here " $\cdot$ " and " $\times$ " are respectively the scalar and vector products of the vectors. The complex conjugate of $q = q_0 + Q$, denoted by $\bar{q}$ is defined as $\bar{q} = q_0 - Q$, while the norm of q, denoted by $|q|$, is given as $|q| = \sqrt{q\bar{q}}$.

Fibonacci and Lucas quaternions were introduced by Horadam [8], and are defined by the equations

$$QF_n = F_ne_0 + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$QL_n = L_ne_0 + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3\,.$$

Here F_n and L_n denote the n^{th} Fibonacci and Lucas number, respectively. Some more properties including recurrence relation were studied in [9]. Iyer [10] derived some relations between the Fibonacci and Lucas quaternions. Halici [7] investigated the Fibonacci and Lucas quaternions and derived some identities of them which includes Binet's formulas and generating functions. Subsequently, Akyigit et al. [1] generalized the Fibonacci quaternions and studied many of their properties. Recently, Çimen and Ipek [5] defined the Pell and Pell-Lucas quaternions as follows:

$$QP_n = P_ne_0 + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3$$

and

$$QPL_n = Q_ne_0 + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3\,,$$

where P_n and Q_n are the n^{th} Pell and Pell-Lucas numbers respectively. As usual, Pell and Pell-Lucas numbers are defined recursively by

$$P_n = 2P_{n-1} + P_{n-2}$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}$$

for $n \geq 2$ with their respective initials

$$(P_0, P_1) = (0, 1) \quad \text{and} \quad (Q_0, Q_1) = (1, 1)\,.$$

Consequently, Szynal-Liana and Włoch [14] obtained several identities concerning QP_n and QPL_n using matrix methods. Motivated by the work of Szynal-Liana and Włoch, Catarino [4] introduced the Modified Pell and the Modified k-Pell quaternions and established some of their properties. Motivated by these works, in this paper we introduce the balancing and Lucas-balancing quaternions and establish some identities.

It is worth defining balancing and Lucas-balancing numbers. A balancing number B is a solution of the Diophantine equation

$$1 + 2 + 3 + \cdots + (B - 1) = (B + 1) + (B + 2) + \cdots + (B + R)$$

with R as a balancer corresponding to B [2]. For each balancing number B, the square root of $8B^2 + 1$ is called a Lucas-balancing number [11]. The n^{th} balancing number B_n and the n^{th} Lucas-balancing number C_n are defined recursively by

$$B_n = 6B_{n-1} - B_{n-2}$$

with $(B_0, B_1) = (0, 1)$ and

$$C_n = 6C_{n-1} - C_{n-2}$$

with $(C_0, C_1) = (1, 3)$ respectively for $n \geq 2$. The Binet formulas for B_n and C_n are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{and} \quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = \lambda_1^{-1}$.

In this article we introduce two new classes of quaternions, namely, balancing and Lucas-balancing quaternions and then derive some of their properties. Further, we also study various results of these classes of quaternions including recurrence relations, Binet's formulas, summation formulas, Catalan's identity etc.

2 Balancing and Lucas-balancing quaternions

In this section we define balancing and Lucas-balancing quaternions and derive some properties of these quaternions.

Definition 1. Let B_n and C_n denote the n^{th} balancing and the n^{th} Lucas-balancing numbers respectively. Then balancing and Lucas-balancing quaternions are respectively defined as

$$QB_n = B_n e_0 + B_{n+1} e_1 + B_{n+2} e_2 + B_{n+3} e_3 = \sum_{r=0}^{3} B_{n+r} e_r,$$

and

$$QC_n = C_n e_0 + C_{n+1} e_1 + C_{n+2} e_2 + C_{n+3} e_3 = \sum_{r=0}^{3} C_{n+r} e_r,$$

where e_0, e_1, e_2 and e_3 are the standard orthonormal basis vectors in $\mathbb{R}^4$.

We can observe from the above definition that addition and substraction of these quaternions can be obtained as follows:

$$QB_n \pm QC_n = \sum_{r=0}^{3} (B_r \pm C_r) e_r \,.$$

Balancing and Lucas-balancing quaternions satisfy similar recurrence relations as those of balancing and Lucas-balancing numbers. The following propositions demonstrate this fact.

Proposition 1. *The recurrence relations for balancing and Lucas-balancing quaternions are respectively*

$$QB_n = 6QB_{n-1} - QB_{n-2} \quad \text{and} \quad QC_n = 6QC_{n-1} - QC_{n-2}$$

for $n \geq 2$.

Proof. Using the recurrence relation of $\{B_n\}_{n\geq 2}$, we have

$$\begin{aligned} QB_n &= \sum_{r=0}^{3} B_{n+r} e_r \\ &= \sum_{r=0}^{3} (6B_{n-1+r} - B_{n-2+r}) e_r \\ &= 6QB_{n-1} - QB_{n-2} \,, \end{aligned}$$

which completes the proof. The proof is similar for Lucas-balancing quaternions. □

The following lemma is useful while deriving the Binet formulas for both QB_n and QC_n.

Lemma 1. *For any natural number* n,

$$QC_n + \sqrt{8} QB_n = A\lambda_1^n \quad \text{and} \quad QC_n - \sqrt{8} QB_n = B\lambda_2^n \,,$$

where

$$A = \sum_{r=0}^{3} \lambda_1^r e_r \quad \text{and} \quad B = \sum_{r=0}^{3} \lambda_2^r e_r \,.$$

Proof. Using the identity $C_n + \sqrt{8} B_n = \lambda_1^n$, we have

$$\begin{aligned} QC_n + \sqrt{8} QB_n &= \sum_{r=0}^{3} C_{n+r} e_r + \sqrt{8} \sum_{r=0}^{3} B_{n+r} e_r \\ &= \sum_{r=0}^{3} (C_{n+r} + \sqrt{8} B_{n+r}) e_r \\ &= \sum_{r=0}^{3} \lambda_1^{r+n} e_r \\ &= A\lambda_1^n \,, \end{aligned}$$

where $A = \sum_{r=0}^{3} \lambda_1^r e_r$. Similarly, using the identity $C_n - \sqrt{8}B_n = \lambda_2^n$, the second result can be obtained. $\square$

Theorem 1. *The Binet formulas for QB_n and QC_n are respectively given by*

$$QB_n = \frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{and} \quad QC_n = \frac{A\lambda_1^n + B\lambda_2^n}{2},$$

where $A = \sum_{r=0}^{3} \lambda_1^r e_r$ and $B = \sum_{r=0}^{3} \lambda_2^r e_r$ with $n \geq 0$.

Proof. By virtue of Lemma 1,

$$A\lambda_1^n - B\lambda_2^n = (\lambda_1 - \lambda_2)QB_n \quad \text{and} \quad A\lambda_1^n + B\lambda_2^n = 2QC_n,$$

and the results follow. $\square$

By using the Binet form of balancing and Lucas-balancing quaternions, we derive some identities concerning QB_n and QC_n. Before that we first define conjugates and norms of these quaternions.

Definition 2. The conjugates of QB_n and QC_n are respectively defined as

$$\overline{QB_n} = B_n e_0 - B_{n+1}e_1 - B_{n+2}e_2 - B_{n+3}e_3 = B_n - \sum_{r=1}^{3} B_{n+r}e_r,$$

$$\overline{QC_n} = C_n e_0 - C_{n+1}e_1 - C_{n+2}e_2 - C_{n+3}e_3 = C_n - \sum_{r=1}^{3} C_{n+r}e_r,$$

and the norms of QB_n and QC_n are respectively defined as

$$N_{QB_n} = \overline{QB_n}QB_n = B_n^2 + B_{n+1}^2 + B_{n+2}^2 + B_{n+3}^2 = \sum_{r=0}^{3} B_{n+r}^2,$$

$$N_{QC_n} = \overline{QC_n}QC_n = C_n^2 + C_{n+1}^2 + C_{n+2}^2 + C_{n+3}^2 = \sum_{r=0}^{3} C_{n+r}^2.$$

Proposition 2. *If $n \geq 2$, then*

(i) $QB_n + \overline{QB_n} = 2B_n$,

(ii) $QB_n^2 + QB_n\overline{QB_n} = 2B_nQB_n$,

(iii) $QB_n\overline{QB_n} = \frac{1}{32}(B_{2n+7} - B_{2n-1} - 8)$.

Proof. Using Definition 2, we have

$$QB_n + \overline{QB_n} = \sum_{r=0}^{3} B_{n+r}e_r + B_n - \sum_{r=1}^{3} B_{n+r}e_r = 2B_n,$$

which ends the proof of (i). Since

$$QB_n^2 = QB_nQB_n = QB_n(2B_n - \overline{QB_n}) = 2B_nQB_n - QB_n\overline{QB_n}$$

and so

$$QB_n^2 + QB_n\overline{QB_n} = 2B_nQB_n\,.$$

In order to prove (iii) we use the following identity for all positive integers n and m,

$$\sum_{r=0}^{m} B_{n+r}^2 = \frac{1}{32}(B_{2m+2n+1} - B_{2n-1} - 2(m+1)) \qquad \text{([6, Theorem 2.2]).}$$

Since $QB_n\overline{QB_n} = \sum_{r=0}^{3} B_{n+r}^2$, the identity follows by letting $m = 3$. □

Proposition 3. *If m and n are positive integers, then*

$$QB_{m+n} = B_mQC_n + C_mQB_n$$

and

$$QC_{m+n} = C_mQC_n + 8B_mQB_n\,.$$

Proof. Using the identity $B_{m+n} = B_mC_n + C_mB_n$, we have

$$\begin{aligned} QB_{m+n} &= \sum_{r=0}^{3} B_{m+n+r}e_r \\ &= B_m\sum_{r=0}^{3} C_{n+r}e_r + C_m\sum_{r=0}^{3} B_{n+r}e_r \\ &= B_mQC_n + C_mQB_n\,. \end{aligned}$$

Similarly,

$$QC_{m+n} = \sum_{r=0}^{3} C_{m+n+r}e_r\,.$$

Further simplification leads the right side expression to

$$\sum_{r=0}^{3}(C_mC_{n+r} + 8B_mB_{n+r})e_r\,.$$

It follows that

$$QC_{m+n} = C_m\sum_{r=0}^{3} C_{n+r}e_r + 8B_m\sum_{r=0}^{3} B_{n+r}e_r\,,$$

and the result follows. □

The following result can also be shown analogously.

Proposition 4. *If m and n are positive integers, then*

$$QB_{m-n} = C_n QB_m - B_n QC_m \quad \text{and} \quad QC_{m-n} = C_n QC_m - 8B_n QB_m\,.$$

Replacing n by $n+r$ in the identities $B_n = 3B_{n-1}+C_{n-1}$ and $C_{n+1} = 8B_n+3C_n$ [11], we have the following formulas that are useful while proving the subsequent results.

For any natural numbers n and r,

$$B_{n+r} = 3B_{n-1+r} + C_{n-1+r}\,, \tag{1}$$

$$C_{n+r} = 8B_{n-1+r} + 3C_{n-1+r}\,. \tag{2}$$

Using (2) and the recurrence relation for Lucas-balancing numbers, we have

$$B_{n+1+r} - B_{n-1+r} = 2C_{n+r}\,. \tag{3}$$

The following result demonstrates some relations between the balancing and Lucas-balancing quaternions.

Proposition 5. *For $n \geq 2$ we have the following identities,*

(i) $QB_n = 3QB_{n-1} + QC_{n-1}$,

(ii) $QC_n = 8QB_{n-1} + 3QC_{n-1}$,

(iii) $2QC_n = QB_{n+1} - QB_{n-1}$,

(iv) $QC_n - QC_{n-1} = 2(QB_{n-1} + QB_n)$.

Proof. From (1), we have

$$\begin{aligned} QB_n &= \sum_{r=0}^{3} B_{n+r} e_r \\ &= \sum_{r=0}^{3} (3B_{n-1+r} + C_{n-1+r}) e_r \\ &= 3QB_{n-1} + QC_{n-1}\,, \end{aligned}$$

which implies the first identity. Similarly, applying (2) and (3), (ii) and (iii) can be derived. (iv) immediately follows from (ii) and (iii). This completes the proof. □

Theorem 2 (Catalan's identity). *If $n, r \in \mathbb{N}$, then*

$$\begin{aligned} QB_n^2 - QB_{n+r}QB_{n-r} = {}& \frac{-1}{8}[(C_{2r} - C_0) + (C_{2r-1} - C_1)e_1 + (C_{2r+2} - C_2)e_2] \\ & + \frac{1}{16}[(C_{2r+3} + C_{2r-3} + C_{2r-1} - C_{2r+1} - 2C_3)]e_3 \end{aligned}$$

and

$$\begin{aligned} QC_n^2 - QC_{n+r}QC_{n-r} = {}& (C_{2r} - C_0) - (C_{2r-1} - C_1)e_1 - (C_{2r+2} - C_2)e_2 \\ & + (2C_3 + C_{2r+1} - C_{2r-3} - C_{2r-1} - C_{2r+3}/2)e_3\,. \end{aligned}$$

Proof. Using the Binet formula for balancing quaternions and the fact $\lambda_1\lambda_2 = 1$, we have

$$\begin{aligned}
QB_n^2 - QB_{n+r}QB_{n-r} \\
&= \left(\frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1-\lambda_2}\right)^2 - \left(\frac{A\lambda_1^{n+r} - B\lambda_2^{n+r}}{\lambda_1-\lambda_2}\right)\left(\frac{A\lambda_1^{n-r} - B\lambda_2^{n-r}}{\lambda_1-\lambda_2}\right)\\
&= \frac{AB\left(\left(\frac{\lambda_1}{\lambda_2}\right)^r - 1\right) + BA\left(\left(\frac{\lambda_2}{\lambda_1}\right)^r - 1\right)}{(\lambda_1-\lambda_2)^2}\\
&= \frac{AB(\lambda_1^{2r}-1) + BA(\lambda_2^{2r}-1)}{(\lambda_1-\lambda_2)^2}\\
&= \frac{\{-2 + 2\lambda_2 e_1 + 2\lambda_1^2 e_2 + (\lambda_1^3 + \lambda_2^3 + \lambda_2 - \lambda_1)e_3\}(\lambda_1^{2r}-1)}{(\lambda_1-\lambda_2)^2}\\
&\quad + \frac{\{-2 + 2\lambda_1 e_1 + 2\lambda_2^2 e_2 + (\lambda_1^3 + \lambda_2^3 + \lambda_1 - \lambda_2)e_3\}(\lambda_2^{2r}-1)}{(\lambda_1-\lambda_2)^2}\\
&= \left(\frac{1}{8} - \frac{\lambda_1^{2r} + \lambda_2^{2r}}{16}\right) + \left(\frac{\lambda_1^{2r-1} + \lambda_2^{2r-1}}{16} - \frac{\lambda_1 + \lambda_2}{16}\right)e_1\\
&\quad + \left(\frac{\lambda_1^{2r+2} + \lambda_2^{2r+2}}{16} - \frac{\lambda_1^2 + \lambda_2^2}{16}\right)e_2\\
&\quad + \left(\frac{\lambda_1^{2r+3} + \lambda_2^{2r+3}}{32} + \frac{\lambda_1^{2r-3} + \lambda_2^{2r-3}}{32}\right.\\
&\quad \left. + \frac{\lambda_1^{2r-1} + \lambda_2^{2r-1}}{32} - \frac{\lambda_1^{2r+1} + \lambda_2^{2r+1}}{32} - \frac{\lambda_1^3 + \lambda_2^3}{16}\right)e_3
\end{aligned}$$

which completes the proof of first part. The second part follows analogously. □

Since Cassini's identity is a special case of Catalan's identity where $r = 1$, the following result immediately follows from Theorem 2.

Corollary 1. *For any positive integer n, the Cassini identity for balancing quaternions is*

$$QB_{n+1}QB_{n-1} - QB_n^2 = \frac{AB(\lambda_1^2 - 1) + BA(\lambda_2^2 - 1)}{(\lambda_1-\lambda_2)^2} = -2 + 70e_2 + 192e_3\,,$$

whereas that for Lucas-balancing quaternions is

$$QC_{n+1}QC_{n-1} - QC_n^2 = \frac{AB(1-\lambda_1^2) + BA(1-\lambda_2^2)}{4} = 16 - 560e_2 - 1536e_3\,.$$

Theorem 3 (d'Ocagne's identity). *If $m, n \in \mathbb{N}$ with $n \geq m$, then*

$$\begin{aligned}
QB_{m+1}QB_n - QB_mQB_{n+1} &= 2(-B_{n-m}e_0 + B_{n-m+1}e_1 + B_{n-m-2}e_2)\\
&\quad + (B_{n-m+3} + B_{n-m-3} + B_{n-m+1} - B_{n-m-1})e_3
\end{aligned}$$

and

$$QC_{m+1}QC_n - QC_mQC_{n+1} = 16\,[B_{n-m} - B_{n-m+1}e_1 - B_{n-m-2}e_2] \\ - 8(B_{n-m+3} + B_{n-m+1} - B_{n-m-1} + B_{n-m+1})e_3\,.$$

Proof. Using the Binet formula for balancing quaternions and since $\lambda_1\lambda_2 = 1$, we have

$$\begin{aligned} QB_{m+1}QB_n - QB_mQB_{n+1} &= \left(\frac{A\lambda_1^{m+1} - B\lambda_2^{m+1}}{\lambda_1 - \lambda_2}\right)\left(\frac{A\lambda_1^{n} - B\lambda_2^{n}}{\lambda_1 - \lambda_2}\right) \\ &\quad - \left(\frac{A\lambda_1^{m} - B\lambda_2^{m}}{\lambda_1 - \lambda_2}\right)\left(\frac{A\lambda_1^{n+1} - B\lambda_2^{n+1}}{\lambda_1 - \lambda_2}\right) \\ &= \frac{BA\lambda_1^{n-m} - AB\lambda_2^{n-m}}{\lambda_1 - \lambda_2} \\ &= -2B_{n-m}e_0 + 2B_{n-m+1}e_1 + 2B_{n-m-2}e_2 \\ &\quad + (B_{n-m+3} + B_{n-m-3} + B_{n-m+1} - B_{n-m-1})e_3\,, \end{aligned}$$

which completes the proof of the first part. Similarly using the Binet formula for QC_n, the second result can be shown. □

An interesting observation from the above results is that the Catalan identities for balancing and Lucas-balancing quaternions are expressed in terms of Lucas--balancing numbers whereas the d'Ocagne identities for both these quaternions are in terms of balancing numbers.

Theorem 4. *The identity* $QC_n^2 - 8QB_n^2 = \frac{AB+BA}{2}$ *holds for* $n \geq 1$.

Proof. Applying the Binet formulas for balancing and Lucas-balancing quaternions, we get

$$\begin{aligned} QC_n^2 - 8QB_n^2 &= \left(\frac{A\lambda_1^n + B\lambda_2^n}{2}\right)^2 - 8\left(\frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1 - \lambda 2}\right)^2 \\ &= \frac{(A^2\lambda_1^{2n} + AB + BA + B^2\lambda_2^{2n}) - (A^2\lambda_1^{2n} - AB - BA + B^2\lambda_2^{2n})}{4} \\ &= \frac{AB + BA}{2}\,, \end{aligned}$$

which completes the proof. □

3 Sum formulas of balancing and Lucas-balancing quaternions

In this section, we derive some sum formulas involving QB_n and QC_n.

The following identity is available in [6].

Lemma 2. *For all positive integers* k *and* i,

$$\sum_{i=0}^{n} B_{k+i} = \frac{1}{4}[B_{(n+1)+k} - B_{n+k} - B_k + B_{k-1}]\,. \tag{4}$$

Theorem 5. *If P_r is the r^{th} Pell number, then*

$$\sum_{r=1}^{n} QB_r = \frac{1}{4}\left(QB_{n+2} - QB_{n+1} - \sum_{r=0}^{3} P_{2r+1}e_r\right).$$

Proof. Using (4), we have

$$\begin{aligned}
\sum_{r=0}^{n} QB_r &= \left(\sum_{r=0}^{n} B_r\right)e_0 + \left(\sum_{r=0}^{n} B_{r+1}\right)e_1 \\
&\quad + \left(\sum_{r=0}^{n} B_{r+2}\right)e_2 + \left(\sum_{r=0}^{n} B_{r+3}\right)e_3 \\
&= \left[\frac{1}{4}(B_{n+2} - B_{n+1} - 1)\right]e_0 + \left[\frac{1}{4}(B_{n+3} - B_{n+2} - 5)\right]e_1 \\
&\quad + \left[\frac{1}{4}(B_{n+4} - B_{n+3} - 29)\right]e_2 + \left[\frac{1}{4}(B_{n+5} - B_{n+4} - 169)\right]e_3 \\
&= \frac{1}{4}\left(\sum_{r=0}^{3} B_{n+2+r}e_r - \sum_{r=0}^{3} B_{n+1+r} - \sum_{r=0}^{3} P_{2r+1}e_r\right) \\
&= \frac{1}{4}\left(QB_{n+2} - QB_{n+1} - \sum_{r=0}^{3} P_{2r+1}e_r\right),
\end{aligned}$$

which completes the proof. □

The following result gives a general relation concerning balancing and Lucas--balancing quaternions.

Theorem 6. *For $m, n \geq 0$,*

$$QB_{mn} = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} B_m^r B_{m-1}^{n-r} QB_r\,,$$

and

$$QC_{mn} = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} C_m^r C_{m-1}^{n-r} QC_r\,.$$

Proof. Using the identity

$$B_{km+n} = \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} B_k^r B_{k-1}^{m-r} B_{r+n} \qquad \text{([12, Eq. (11)])},$$

we have

$$\begin{aligned} QB_{mn} &= \sum_{l=0}^{3} B_{mn+l} e_l \\ &= \sum_{l=0}^{3} \left(\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} B_m^r B_{m-1}^{n-r} B_{r+l} \right) e_l \\ &= \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} B_m^r B_{m-1}^{n-r} \left(\sum_{l=0}^{3} B_{r+l} e_l \right), \end{aligned}$$

which completes the proof of the first part. The second part can be obtained similarly using the identity

$$C_{km+n} = \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} B_k^r B_{k-1}^{m-r} C_{r+n} \qquad ([12, \text{Theorem } 2.1]). \qquad \square$$

The next result follows directly from Theorem 6 by setting $m = 2$.

Corollary 2. *For $n \geq 0$,*

$$QB_{2n} = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} 6^r QB_r\,,$$

and

$$QC_{2n} = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} 6^r QC_r\,.$$

The balancing and Lucas-balancing sums involving binomial coefficients were studied in [13]. The following are analogous to the identities studied in Theorem 4.1, [13].

Theorem 7. *For any positive integers m and k with $m > k \geq 0$, we have*

$$\sum_{r=0}^{n} QB_{mr+k} = \frac{(QB_k - QB_{mn+m+k}) + (QB_{mn+k} - QB_{k-m})}{2(1 - QC_m)}$$

and

$$\sum_{r=0}^{n} QC_{mr+k} = \frac{(QC_k - QC_{mn+m+k}) + (QC_{mn+k} - QC_{k-m})}{2(1 - QC_m)}\,.$$

Proof. Using the Binet formula for balancing quaternions, we obtain

$$\begin{aligned}
\sum_{r=0}^{n} QB_{mr+k} &= \sum_{r=0}^{n} \frac{A\lambda_1^{mr+k} - B\lambda_2^{mr+k}}{\lambda_1 - \lambda_2} \\
&= \frac{1}{\lambda_1 - \lambda 2}\left(A\lambda_1^k \sum_{r=0}^{n} \lambda_1^{mr} - B\lambda_2^k \sum_{r=0}^{n} \lambda_2^{mr}\right) \\
&= \frac{1}{\lambda_1 - \lambda 2}\left[A\lambda_1^k\left(\frac{\lambda_1^{mn+m} - 1}{\lambda_1^m - 1}\right) - B\lambda_2^k\left(\frac{\lambda_2^{mn+m} - 1}{\lambda_2^m - 1}\right)\right] \\
&= \frac{1}{\lambda_1 - \lambda 2}\left[A\left(\frac{\lambda_1^{mn+m+k} - \lambda_1^k}{\lambda_1^m - 1}\right) - B\left(\frac{\lambda_2^{mn+m+k} - \lambda_2^k}{\lambda_2^m - 1}\right)\right] \\
&= \frac{1}{\lambda_1 - \lambda 2}\left[\frac{(A\lambda_1^k - B\lambda_2^k)}{2 - (\lambda_1^m + \lambda_2^m)} - \frac{(A\lambda_1^{mn+m+k} - B\lambda_2^{mn+m+k})}{2 - (\lambda_1^m + \lambda_2^m)}\right] \\
&\quad + \left[\frac{(A\lambda_1^{mn+k} - B\lambda_2^{mn+k})}{2 - (\lambda_1^m + \lambda_2^m)} - \frac{(A\lambda_1^{k-m} - B\lambda_2^{k-m})}{2 - (\lambda_1^m + \lambda_2^m)}\right] \\
&= \frac{(QB_k - QB_{mn+m+k}) + (QB_{mn+k} - QB_{k-m})}{2(1 - QC_m)}\,.
\end{aligned}$$

The proof for the Lucas-balancing quaternions is similar. □

The following result is an immediate consequence of the above result.

Corollary 3. *For* $m \geq 0$, *we have*

$$\sum_{r=0}^{n} QB_{mr} = \frac{(QB_0 - QB_{mn+m}) + (QB_{mn} + QB_m)}{2(1 - QC_m)}\,,$$

and

$$\sum_{r=0}^{n} QB_r = \frac{(QB_0 - QB_{n+1}) + (QB_n + QB_1)}{2(1 - QC_1)}\,.$$

4 Generating functions for balancing and Lucas-balancing quaternions

Generating functions are used to solve linear recurrence relations with constant coefficients. Recall that a generating function for a sequence $\{a_n\}$ of real numbers is defined by

$$L(s) = \sum_{n=0}^{\infty} a_n s^n\,.$$

The generating function for the balancing sequence is given by

$$G(s) = \frac{s}{1 - 6s + s^2} \qquad \text{([2, Theorem 6.1])}$$

whereas that for the Lucas-balancing sequence is

$$g(s) = \frac{1-3s}{1-6s+s^2} \qquad \text{([3, Proposition 4.4])}.$$

In order to find the generating functions for both QB_n and QC_n, we need the following result.

Theorem 8. *For any natural numbers k and m with $k > m \geq 0$ and $n \in \mathbb{N}$, the generating functions of QB_{kn+m} and QC_{kn+m} are respectively*

$$\sum_{n=0}^{\infty} QB_{kn+m} s^n = \frac{QB_m - QB_{m-k}s}{1-2C_k s + s^2}$$

and

$$\sum_{n=0}^{\infty} QC_{kn+m} s^n = \frac{QC_m - QC_{m-k}s}{1-2C_k s + s^2}\,.$$

Proof. Using the Binet formula for QB_n, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} QB_{kn+m} s^n &= \sum_{n=0}^{\infty} \left(\frac{A\lambda_1^{kn+m} - B\lambda_2^{kn+m}}{\lambda_1 - \lambda_2} \right) s^n \\
&= \frac{1}{\lambda_1 - \lambda_2} \left(\frac{A\lambda_1^m}{1-\lambda_1^k s} - \frac{B\lambda_2^m}{1-\lambda_2^k s} \right) \\
&= \frac{1}{\lambda_1 - \lambda_2} \left[\frac{(A\lambda_1^m - B\lambda_2^m) - (A\lambda_1^{m-k} - B\lambda_2^{m-k})s}{1-(\lambda_1^k + \lambda_2^k)s + s^2} \right] \\
&= \frac{QB_m - QB_{m-k}s}{1-2C_k s + s^2}\,,
\end{aligned}$$

which is the desired result. For QC_{kn+m} the proof is similar as QB_{kn+m}. □

The following results are direct consequences of Theorem 8.

Corollary 4. *The generating function $G_Q(s)$ for balancing quaternions and $g_Q(s)$ for Lucas-balancing quaternions are respectively*

$$G_Q(s) = \frac{se_0 + e_1 + (6-s)e_2 + (35-6s)e_3}{1-6s+s^2}$$

and

$$g_Q(s) = \frac{(1-3s)e_0 + (3-s)e_1 + (17-3s)e_2 + (99-17s)e_3}{1-6s+s^2}\,.$$

Proof. Let the generating function for QB_n be

$$G_Q(s) = \sum_{n=0}^{\infty} QB_n s^n\,.$$

By putting $k = 1$ and $m = 0$, Theorem 8 becomes

$$\begin{aligned} G_Q(s) &= \frac{1}{1-6s+s^2}\sum_{r=0}^{3}(B_r - B_{r-1}s)e_r \\ &= \frac{se_0 + e_1 + (6-s)e_2 + (35-6s)e_3}{1-6s+s^2}, \end{aligned}$$

which completes the proof. The proof is similar for Lucas-balancing quaternions. □

The next results demonstrate the exponential generating functions and Poission generating functions for both balancing and Lucas-balancing quaternions.

Theorem 9. *For $m, n \in \mathbb{N}$, the exponential generating functions of the quaternions QB_{m+n} and QC_{m+n} are respectively*

$$\sum_{n=0}^{\infty}\frac{QB_{m+n}}{n!}s^n = \frac{A\lambda_1^m e^{\lambda_1 s} - B\lambda_2^m e^{\lambda_2 s}}{\lambda_1 - \lambda_2}$$

and

$$\sum_{n=0}^{\infty}\frac{QC_{m+n}}{n!}s^n = \frac{A\lambda_1^m e^{\lambda_1 s} + B\lambda_2^m e^{\lambda_2 s}}{2}.$$

Proof. Using Binet's formula for QB_{m+n}, we have

$$\begin{aligned} \sum_{n=0}^{\infty} QB_{m+n}\frac{s^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{A\lambda_1^{m+n} - B\lambda_2^{m+n}}{\lambda_1-\lambda_2}\right)\frac{s^n}{n!} \\ &= \frac{A\lambda_1^m}{\lambda_1-\lambda_2}\sum_{n=0}^{\infty}\frac{(\lambda_1 s)^n}{n!} - \frac{B\lambda_2^m}{\lambda_1-\lambda_2}\sum_{n=0}^{\infty}\frac{(\lambda_2 s)^n}{n!} \\ &= \left(\frac{A\lambda_1^m}{\lambda_1-\lambda_2}\right)e^{\lambda_1 s} - \left(\frac{B\lambda_2^m}{\lambda_1-\lambda_2}\right)e^{\lambda_2 s}, \end{aligned}$$

and the result follows. Further simplification gives

$$\sum_{n=0}^{\infty}\frac{QB_{m+n}}{n!}s^n = QC_m\left(\sum_{n=0}^{\infty}B_n\frac{s^n}{n!}\right) + QB_m\left(\sum_{n=0}^{\infty}C_n\frac{s^n}{n!}\right).$$

The proof for the Lucas-balancing quaternions is similar. □

Corollary 5. *The exponential generating functions for balancing and Lucas-balancing quaternions are respectively*

$$\sum_{n=0}^{\infty}\frac{QB_n}{n!}s^n = \frac{Ae^{\lambda_1 s} - Be^{\lambda_2 s}}{\lambda_1-\lambda_2} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{QC_n}{n!}s^n = \frac{Ae^{\lambda_1 s} + Be^{\lambda_2 s}}{2}.$$

The following result relating to Poisson generating functions is an immediate consequence of Theorem 9 because Poisson generating functions for balancing and Lucas-balancing quaternions can be obtained by multiplying e^{-s} to the exponential generating functions for both these quaternions.

Corollary 6. *The Poisson generating function for balancing and Lucas-balancing quaternions are*

$$\sum_{n=0}^{\infty} \frac{QB_n}{n!} s^n e^{-s} = \frac{Ae^{\lambda_1 s} - Be^{\lambda_2 s}}{e^s(\lambda_1 - \lambda_2)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{QC_n}{n!} s^n e^{-s} = \frac{Ae^{\lambda_1 s} + Be^{\lambda_2 s}}{2e^s},$$

respectively.

The various generating functions discussed above are applied to derive the following identities.

Lemma 3. *For any natural number n, $QB_{n+r+1} - QB_{n+r-1} = 2QC_{n+r}$.*

Proof. Using Proposition 5, we get

$$\begin{aligned}\sum_{n=0}^{\infty} QB_{n+r+1}s^n - \sum_{n=0}^{\infty} QB_{n+r-1}s^n &= \frac{QB_{r+1} - QB_r s}{1-6s+s^2} - \frac{QB_{r-1} - QB_{r-2}s}{1-6s+s^2} \\ &= \frac{(QB_{r+1} - QB_{r-1}) - (QB_r - QB_{r-2})s}{1-6s+s^2} \\ &= \frac{2QC_r - 2QC_{r-1}s}{1-6s+s^2} \\ &= 2\sum_{n=0}^{\infty} QC_{n+r}s^n,\end{aligned}$$

which completes the proof. □

Lemma 4. *For any natural number n,*

$$\sum_{n=0}^{\infty} \frac{QB_n}{n!} s^n = \frac{e^{3s}}{\sqrt{8}}\left[QC_0 \sinh(\sqrt{8}s) + \sqrt{8}QB_0 \cosh(\sqrt{8}s)\right]$$

and

$$\sum_{n=0}^{\infty} \frac{QC_n}{n!} s^n = e^{3s}\left[\sqrt{8}QB_0 \sinh(\sqrt{8}s) + QC_0 \cosh(\sqrt{8}s)\right].$$

Proof. For any natural number n, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{QB_n}{n!}s^n &= \frac{Ae^{\lambda_1 s}-Be^{\lambda_2 s}}{\lambda_1-\lambda_2}\\
&= \frac{e^{3s}}{2\sqrt{8}}(Ae^{\sqrt{8}s}-Be^{-\sqrt{8}s})\\
&= \frac{e^{3s}}{2\sqrt{8}}\left[\left(\frac{QC_r+\sqrt{8}QB_r}{\lambda_1^r}\right)e^{\sqrt{8}s}-\left(\frac{QC_r-\sqrt{8}QB_r}{\lambda_2^r}\right)e^{-\sqrt{8}s}\right]\\
&= \frac{e^{3s}}{\sqrt{8}}\left[C_rQC_r\left(\frac{e^{\sqrt{8}s}-e^{-\sqrt{8}s}}{2}\right)-\sqrt{8}B_rQC_r\left(\frac{e^{\sqrt{8}s}+e^{-\sqrt{8}s}}{2}\right)\right.\\
&\quad\left.+\sqrt{8}C_rQB_r\left(\frac{e^{\sqrt{8}s}+e^{-\sqrt{8}s}}{2}\right)-8B_rQB_r\left(\frac{e^{\sqrt{8}s}-e^{-\sqrt{8}s}}{2}\right)\right]\\
&= \frac{e^{3s}}{\sqrt{8}}\left[\sinh(\sqrt{8}s)\,(C_rQC_r-8B_rQB_r)\right.\\
&\quad\left.+\sqrt{8}\cosh(\sqrt{8}s)(C_rQB_r-B_rQC_r)\right].
\end{aligned}$$

Using the Proposition 4 in the above expression we get the desired result. □

Binet's formulas for balancing and Lucas-balancing quaternions was already shown in Theorem 1. However, these formulas can also be derived by applying generating functions for both balancing and Lucas-balancing quaternions as follows.

By virtue of Corollary 4, we have

$$G_Q(s)=\frac{1}{1-6s+s^2}\sum_{r=0}^{3}(B_r-B_{r-1}s)e_r\,.$$

Further simplification using partial fractions reduces the above identity to

$$\begin{aligned}
G_Q(s) &= \frac{1}{\lambda_1-\lambda_2}\left[\frac{QB_1-\lambda_2QB_0}{1-s\lambda_1}-\frac{QB_1-\lambda_1QB_0}{1-s\lambda_2}\right]\\
&= \frac{1}{\lambda_1-\lambda_2}\left[\sum_{s=0}^{3}(B_{s+1}-\lambda_2B_s)e_s\sum_{n=0}^{\infty}\lambda_1^n s^n\right.\\
&\quad\left.\sum_{s=0}^{3}(B_{s+1}-\lambda_1B_s)e_s\sum_{n=0}^{\infty}\lambda_2^n s^n\right]\\
&= \frac{1}{\lambda_1-\lambda_2}\left[\sum_{s=0}^{3}\lambda_1^s e_s\sum_{n=0}^{\infty}\lambda_1^n s^n-\sum_{s=0}^{3}\lambda_2^s e_s\sum_{n=0}^{\infty}\lambda_2^n s^n\right].
\end{aligned}$$

That is,

$$G_Q(s)=\sum_{n=0}^{\infty}QB_ns^n=\sum_{n=0}^{\infty}\left(\frac{A\lambda_1^n-B\lambda_2^n}{\lambda_1-\lambda_2}\right)s^n\,,$$

and hence the Binet formula for QB_n is obtained. Similarly, the Binet formula for QC_n can also be obtained by using the generating function for Lucas-balancing quaternions.

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