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Symmetric identity for polynomial sequences satisfying $A'_{n+1}(x) = (n+1)A_n(x)$

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Abstract. Using umbral calculus, we establish a symmetric identity for any sequence of polynomials satisfying $A'_{n+1}(x) = (n+1)A_n(x)$ with $A_0(x)$ a constant polynomial. This identity allows us to obtain in a simple way some known relations involving Apostol-Bernoulli polynomials, Apostol--Euler polynomials and generalized Bernoulli polynomials attached to a primitive Dirichlet character.

1 Introduction and preliminaries

With each formal power series $S(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!} \in \mathbb{C}[[z]]$ we can associate a polynomial sequence $(A_k(x))_{k\geq 0}$ defined by

$$\sum_{k=0}^{\infty} A_k(x)\frac{z^k}{k!} = S(z)\mathrm{e}^{xz}, \tag{1}$$

or equivalently by $A_k(x) = \sum_{i=0}^{k} \binom{k}{i} a_i x^{k-i}$. Clearly

$$A'_k(x) = kA_{k-1}(x), \quad k \geq 1, \quad A'_0(x) = 0. \tag{2}$$

Reciprocally, it is not difficult to prove that if a polynomial sequence $(A_k(x))_{k\geq 0}$ satisfies (2) then (1) holds for

$$S(z) = \sum_{k=0}^{\infty} A_k(0)\frac{z^k}{k!}\,.$$

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Apostol-Bernoulli polynomials $(\mathcal{B}_k(x;\lambda))_{k\geq 0}$ and Apostol-Euler polynomials $(\mathcal{E}_k(x;\lambda))_{k\geq 0}$ in the variable x defined for $\lambda \in \mathbb{C}$ by

$$\frac{z}{\lambda \mathrm{e}^z - 1}\mathrm{e}^{xz} = \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\frac{z^k}{k!}$$

and for $\lambda \in \mathbb{C} \setminus \{-1\}$ by

$$\frac{2}{\lambda \mathrm{e}^z + 1}\mathrm{e}^{xz} = \sum_{k=0}^{\infty} \mathcal{E}_k(x;\lambda)\frac{z^k}{k!}$$

provide an example of such polynomials. If χ is a primitive Dirichlet character with conductor $f = f_\chi$, then $B_{n,\chi}(x)$, $n = 0, 1, 2, \ldots$, the generalized Bernoulli polynomials attached to χ are defined by the generating function [1]

$$F_\chi(t,x) = \frac{t}{\mathrm{e}^{ft} - 1}\sum_{a=1}^{f}\chi(a)\mathrm{e}^{at}\mathrm{e}^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!}\,, \qquad |t| < \frac{2\pi}{f}.$$

The polynomial sequence $(B_{n,\chi}(x))_{n\geq 0}$ is another example of polynomial sequence satisfying (1). Furthermore, if $a_0 \neq 0$, the polynomial sequence $(A_k(x))_{k\geq 0}$ is called an Appell sequence [2]. Which is the case for the classical Bernoulli and Euler polynomials $B_k(x)$ and $E_k(x)$ defined respectively by

$$\frac{z}{\mathrm{e}^z - 1}\mathrm{e}^{xz} = \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}$$

and

$$\frac{2}{\mathrm{e}^z + 1}\mathrm{e}^{xz} = \sum_{n=0}^{\infty} E_k(x)\frac{z^k}{k!}\,.$$

In 2003, motivated by the work of Kaneko [8], Momiyama [10] and Wu et al. [17], Sun [14] derived a general combinatorial identity in terms of polynomials with dual sequences of coefficients. More precisely, he considered the two following polynomial sequences

$$R_k(x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^i u_i x^{k-i} \quad \text{and} \quad R_k^*(x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^i u_i^* x^{k-i}\,, \tag{3}$$

where $(u_n)_{n\geq 0}$ is any complex sequence and $(u_n^*)_{n\geq 0}$ is its dual sequence defined for $k \geq 0$ by $u_k^* = \sum_{i=0}^{k}\binom{k}{i}(-1)^i u_i$, and proved [14, Theorem 1.1], for any integers $n, m \geq 0$ and $x + y + z = 1$, the following identities

$$(-1)^{n-1}\sum_{k=0}^{m}\binom{m}{k}x^{m-k}\frac{R_{n+k+1}(y)}{n+k+1} + (-1)^{m+1}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\frac{R^*_{m+k+1}(z)}{m+k+1} = \frac{m!n!x^{m+n+1}u_0}{(m+n+1)!}\,, \tag{4}$$

$$(-1)^{n-1}\sum_{k=0}^{m}\binom{m}{k}x^{m-k}R_{n+k}(y)+(-1)^{m}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}R^{*}_{m+k}(z)=0\,, \tag{5}$$

$$\begin{aligned}(-1)^{n}\Big\{\sum_{k=0}^{m}\binom{m+1}{k}x^{m+1-k}(n+1+k)R_{n+k}(y)&\\ +(n+m+2)R_{n+m+1}(y)\Big\}&\\ +(-1)^{m}\Big\{\sum_{k=0}^{n}\binom{n+1}{k}x^{n+1-k}(m+1+k)R^{*}_{m+k}(z)&\\ +(n+m+2)R^{*}_{m+n+1}(z)\Big\}&=0\,. \end{aligned} \tag{6}$$

Which allowed him to derive various known identities involving Bernoulli numbers.

Chen and Sun [3], by applying the extended Zeilberger's algorithm, established several recurrence relations for Bernoulli numbers and polynomials which generalize the relations of Momiyama [10], Gessel [6] and Gelfand [5]. Prévost [11], by using the Padé approximation of the exponential function, extended Chen and Sun's results and obtained numerous recurrence relations involving Apostol-Bernoulli polynomials $\mathcal{B}_k(x;\lambda)$ or Apostol-Euler polynomials $\mathcal{E}_k(x;\lambda)$. Recently, Agoh [1] found some shortened recurrence relations for generalized Bernoulli polynomials attached to a primitive Dirichlet character χ which allowed him to reprove and generalize several identities on classical Bernoulli numbers and polynomials such as Saalschütz-Gelfand [5], von Ettingshausen-Seidel-Stern-Kaneko's [15], [16], [13], [8] and Chen and Sun [3] formulas. In particular, he showed [1, Theorem 2.1] that for any integers $n, m \geq 0$ and $s \geq 1$, we have

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\frac{(sf)^{m-k}B_{n+1+k,\chi}(x)}{n+1+k}-\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{(sf)^{n-k}B_{m+1+k,\chi}(x)}{m+1+k}&\\ =\sum_{a=1}^{f}\chi(a)\sum_{r=0}^{s-1}(a+x+rf)^{m}(a+x+(r-s)f)^{n}&\\ +\frac{(-1)^{n+1}}{m+n+1}\binom{n+m}{n}^{-1}(sf)^{n+m+1}B_{0,\chi}(x)\,.&\end{aligned} \tag{7}$$

With the use of the well-known relation [1, Eq. (2.4)]

$$B_{n,\chi}(x+sf)=B_{n,\chi}(x)+\sum_{k=0}^{s-1}\sum_{a=1}^{f}\chi(a)n(a+x+kf)^{n-1}, \tag{8}$$

where n and s are non-negative integers, identity (7) can be deduced from the following relation, due to He and Zhang [7, Theorem 1.2]

$$\sum_{k=0}^{m}\binom{m}{k}\frac{\alpha^{m-k}A_{n+k+q}(x)}{(n+q+k)^{\underline{q}}}-\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{\alpha^{n-k}A_{m+q+k}(x+\alpha)}{(m+q+k)^{\underline{q}}}$$
$$=\frac{(-1)^{n+1}\alpha^{n+m+1}}{(q-1)!}\int_0^1(1-t)^m t^n A_{q-1}(x+\alpha t)\,\mathrm{d}t\,, \quad (9)$$

where $(A_n(x))_{n\geq 0}$ is any sequence of polynomials defined by relation (1), α a complex number, and $n, m \geq 0$, $q \geq 1$ any integers. Recall that the falling factorial $z^{\underline{q}}$ is a polynomial in z defined by $z^{\underline{0}} = 1$ and

$$z^{\underline{q}} = \prod_{j=0}^{q-1}(z-j)\,,$$

for $q \geq 1$.

It is the main aim of this paper to establish in Theorem 1 a Saalschütz-Gelfand type identity for the polynomial sequence $(A_n(x))_{n\geq 0}$ satisfying (2). This identity allows us to reprove some known identities for generalized Bernoulli polynomials attached to a primitive Dirichlet character due to Agoh [1] and Apostol-Bernoulli polynomials due to Prévost [11, Theorem 2] in a simple way.

2 Main result

Before giving our main result it is convenient to have the following lemma.

Lemma 1. *For any non-negative integers n, m and q, we have the following identity*

$$\sum_{k=0}^{m}\binom{m}{k}\frac{z^{n+k+q}}{(n+k+q)^{\underline{q}}}-\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{(1+z)^{m+k+q}}{(m+k+q)^{\underline{q}}}$$
$$=\frac{(-1)^{n+1}}{(q-1)!}\sum_{k=0}^{q-1}\binom{q-1}{k}\frac{m!(n+q-k-1)!}{(m+n+q-k)!}z^k\,. \quad (10)$$

Proof. Consider the polynomial $P(z) = z^n(1+z)^m$ and let $Q(z)$ denote the q-fold primitive of $P(z)$ defined by

$$Q(z)=\int_0^z\int_0^{z_1}\cdots\int_0^{z_{q-1}}P(t)\,\mathrm{d}t\,\mathrm{d}z_{q-1}\cdots\mathrm{d}z_1\,. \quad (11)$$

On the one hand, by the Cauchy well-known formula [4, p. 115] for repeated integration, we have

$$Q(z)=\frac{1}{(q-1)!}\int_0^z(z-t)^{q-1}P(t)\,\mathrm{d}t\,. \quad (12)$$

Changing variable t into $u-1$ in the right-hand side of (12) gives

$$\begin{aligned} Q(z) = \frac{1}{(q-1)!} \int_0^{z+1} ((z+1)-u)^{q-1} u^m (u-1)^n \,\mathrm{d}u \\ - \frac{1}{(q-1)!} \int_0^1 (z+(1-u))^{q-1} u^m (u-1)^n \,\mathrm{d}u \\ = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{(q-1)!} \int_0^{z+1} ((z+1)-u)^{q-1} u^{m+k} \,\mathrm{d}u \\ - \frac{1}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} (-1)^n z^k \int_0^1 u^m (1-u)^{n+q-k-1} \,\mathrm{d}u \,. \end{aligned}$$

Recall that for any non-negative integers i and j,

$$\int_0^1 u^i (1-u)^j \,\mathrm{d}u = \frac{i!j!}{(i+j+1)!}$$

and for any non-negative integer k, from the Cauchy formula (12) we have

$$\frac{1}{(q-1)!} \int_0^z (z-u)^{q-1} u^k \,\mathrm{d}u = \frac{z^{k+q}}{(k+q)^{\underline{q}}}. \tag{13}$$

From this, we can deduce that

$$\begin{aligned} Q(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(z+1)^{m+k+q}}{(m+k+q)^{\underline{q}}} \\ - \frac{1}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} (-1)^n \frac{m!(n+q-k-1)!}{(m+n+q-k)!} z^k \,. \end{aligned} \tag{14}$$

On the other hand, expanding $P(t)$ in (11) and using the well-known identity

$$\int_0^z \int_0^{z_1} \cdots \int_0^{z_{q-1}} t^{n+k} \,\mathrm{d}t \,\mathrm{d}z_{q-1} \cdots \mathrm{d}z_1 = \frac{z^{n+k+q}}{(n+k+q)^{\underline{q}}} \,,$$

where k is any non-negative integer, give

$$Q(z) = \sum_{k=0}^{m} \binom{m}{k} \frac{z^{n+k+q}}{(n+k+q)^{\underline{q}}} \,. \tag{15}$$

Finally, by equalling both expressions (14) and (15) of $Q(z)$ we obtain (10). □

Theorem 1. *Let $S(z) \in \mathbb{C}[[z]]$ be a formal power series and $(A_k(x))_{k\geq 0}$ be a sequence of polynomials given by*

$$\sum_{k=0}^{+\infty} A_k(x) \frac{z^k}{k!} = S(z) \mathrm{e}^{xz} \,. \tag{16}$$

For any complex number α and any integers n, $m \geq 0$ and $q \geq 1$, we have

$$\sum_{k=0}^{m}\binom{m}{k}\frac{\alpha^{m-k}A_{n+q+k}(x)}{(n+q+k)^{\underline{q}}}-\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{\alpha^{n-k}A_{m+q+k}(x+\alpha)}{(m+q+k)^{\underline{q}}}$$
$$=\frac{(-1)^{n+1}}{(q-1)!}\sum_{k=0}^{q-1}\binom{q-1}{k}\alpha^{n+m+q-k}\frac{m!(n+q-k-1)!}{(m+n+q-k)!}A_k(x)\,. \quad (17)$$

Proof. Let

$$S(z)=\sum_{k=0}^{\infty}a_k\frac{z^k}{k!}\,.$$

Then, from (16) we have, for $r \geq 0$,

$$A_r(x)=\sum_{k=0}^{r}\binom{r}{k}a_{r-k}x^k\,.$$

Inspired by the work of Rota [12], let us consider the linear map L from $\mathbb{C}[x,y]$ to $\mathbb{C}[x]$ defined for any non-negative integers r and s by

$$L(x^ry^s)=a_sx^r\,. \quad (18)$$

For $\alpha = 0$, it is obvious that equation (17) holds. Assume that $\alpha \neq 0$. As for any $r \geq 0$ we have

$$L((x+y)^r)=\sum_{i=0}^{r}\binom{r}{i}a_{r-i}x^i=A_r(x),$$
$$L((x+\alpha+y)^r)=\sum_{i=0}^{r}\binom{r}{i}a_{r-i}(x+\alpha)^i=A_r(x+\alpha)\,,$$

replacing z with $(\frac{x+y}{\alpha})$ in (10) and applying L we get the desired relation (17). □

Remark 1. If I and J denote respectively the right-hand sides of relations (9) and (17), then it is not difficult to show that $I = J$. In fact, for each fixed x and α, the Taylor expansion of the polynomial $A_{q-1}(x+\alpha t)$ in power of t is given by

$$A_{q-1}(x+\alpha t)=\sum_{k=0}^{q-1}\binom{q-1}{k}t^{q-1-k}\alpha^{q-1-k}A_k(x)\,.$$

Thus,

$$J=\frac{\alpha^{n+m+1}}{(q-1)!}\sum_{k=0}^{q-1}\binom{q-1}{k}\alpha^{q-1-k}A_k(x)\int_0^1 t^{n+q-1-k}(1-t)^m\,\mathrm{d}t=I\,.$$

Which leads to another proof of Theorem 1.2 of He and Zhang [7].

3 Corollaries

Using Theorem 1, we can give an alternative proof of He and Zhang's theorem [7, Theorem 1.1 and Corollary 1.2] as follows.

Corollary 1. *Let $S(z) \in \mathbb{C}[[z]]$ be a formal power series and $(A_k(x))_{k\geq 0}$ be a sequence of polynomials given by*

$$\sum_{k=0}^{+\infty} A_k(x)\frac{z^k}{k!} = S(z)\mathrm{e}^{xz}\,.$$

For any complex number α and any non-negative integers n, m and q, we have

$$\sum_{k=0}^{m}\binom{m}{k}(n+k)^{\underline{q}}\alpha^{m-k}A_{n-q+k}(x) - \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(m+k)^{\underline{q}}\alpha^{n-k}A_{m-q+k}(x+\alpha) = 0\,. \quad (19)$$

Proof. From (2) we have for each $k \geq 1$, $A'_k(x) = kA_{k-1}(x)$ and $A'_0(x) = 0$. Taking $q = 1$ in relation (17) and then differentiating the obtained relation $q+1$ times with respect to x gives (19). □

Remark 2. Relation (19) is clearly satisfied for $\alpha = 0$. Note that in the case $\alpha \neq 0$ relation (19) can also be obtained by using the same method as in the proof of Theorem 1. Indeed, by differentiating the following identity

$$\sum_{k=0}^{m}\binom{m}{k}z^{n+k} = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(1+z)^{m+k}$$

q times with respect to z, we obtain

$$\sum_{k=0}^{m}\binom{m}{k}(n+k)^{\underline{q}}z^{n-q+k} - \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(m+k)^{\underline{q}}(1+z)^{m-q+k} = 0\,.$$

Replacing z with $(\frac{x+y}{\alpha})$ and applying the linear map L defined by (18) we get (19).

Next applying Theorem 1 and Corollary 1 for $\alpha = sf$ to the sequence

$$(B_{n,\chi}(x))_{n\geq 0}\,,$$

with the help of relation (8) we get the following two corollaries.

Corollary 2. *For integers $q, s \geq 1$, and $n, m \geq 0$, we have*

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\frac{(sf)^{m-k}B_{n+q+k,\chi}(x)}{(n+q+k)^{\underline{q}}} &- \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{(sf)^{n-k}B_{m+q+k,\chi}(x)}{(m+q+k)^{\underline{q}}} \\ &= \sum_{i=0}^{s-1}\sum_{a=1}^{f}\chi(a)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{(sf)^{n-k}(a+x+if)^{m+q-1+k}}{(m+q-1+k)^{\underline{q-1}}} \\ &+ \frac{(-1)^{n+1}}{(q-1)!}\sum_{k=0}^{q-1}\binom{q-1}{k}\frac{(sf)^{n+m+q-k}m!(n+q-k-1)!}{(m+n+q-k)!}B_{k,\chi}(x)\,. \quad (20)\end{aligned}$$

With the help of relation (13), the right hand-side of (20) can also be written, for $q \geq 2$, as

$$\frac{1}{(q-2)!}\sum_{i=0}^{s-1}\sum_{a=1}^{f}\chi(a)\int_0^{a+x+if}(a+x+if-u)^{q-2}u^m(u-sf)^n\,\mathrm{d}u + \frac{(-1)^{n+1}}{(q-1)!}\sum_{k=0}^{q-1}\binom{q-1}{k}\frac{(sf)^{n+m+q-k}m!(n+q-k-1)!}{(m+n+q-k)!}B_{k,\chi}(x)\,.$$

Taking $q = 1$ in (20) gives immediately Theorem 2.1 of Agoh [1].

Corollary 3. *For integers $s \geq 1$, and $n, m, q \geq 0$, we have*

$$\sum_{k=0}^{m}\binom{m}{k}(n+k)^{\underline{q}}(sf)^{m-k}B_{n-q+k,\chi}(x) - \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(m+k)^{\underline{q}}(sf)^{n-k}B_{m-q+k,\chi}(x)$$
$$= (q+1)!\sum_{a=1}^{f}\chi(a)\sum_{r=0}^{s-1}\sum_{j=0}^{q+1}\binom{m}{q+1-j}\binom{n}{j}(a+x+rf)^{m-q-1+j}(a+x+(r-s)f)^{n-j}\,.$$

One observes that the left hand side of the previous identity can be expressed as

$$\frac{\mathrm{d}^{q+1}}{\mathrm{d}x^{q+1}}\sum_{a=1}^{f}\chi(a)\sum_{r=0}^{s-1}(a+x+rf)^m(a+x+(r-s)f)^n\,.$$

By taking respectively $q = 0$ and $q = 1$ in Corollary 3, we get Theorems 2.3 and 2.4 of Agoh [1].

The following corollary enables us to obtain Sun's result by giving different values to q.

Corollary 4. *Let $(R_k(x))_{n\geq 0}$ and $(R_k^*(x))_{n\geq 0}$ be the polynomial sequences defined by (3), then for any non-negative integers n, m and q and for any complex number α, we have*

$$(-1)^{n-1}\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}\frac{R_{n+k+q}(x)}{(n+k+q)^{\underline{q}}} + (-1)^{m+q}\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\frac{R^*_{m+k+q}(1-x-\alpha)}{(m+k+q)^{\underline{q}}}$$
$$= \frac{1}{(q-1)!}\sum_{k=0}^{q-1}\binom{q-1}{k}\frac{m!(n+q-k-1)!}{(m+n+q-k)!}\alpha^{m+n+q-k}R_k(x)\,, \quad (21)$$

$$(-1)^{n-1}\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}(n+k)^{\underline{q}}R_{n-q+k}(x) + (-1)^{m-q}\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}(m+k)^{\underline{q}}R^*_{m-q+k}(1-x-\alpha) = 0\,. \quad (22)$$

Proof. If

$$F(z) = \sum_{k=0}^{\infty} u_k \frac{z^k}{k!}$$

denotes the exponential generating function for the sequence u, then it is easy to see that the polynomial sequences $(R_k(x))_{k\geq 0}$ and $(R^*_k(x))_{k\geq 0}$ satisfy

$$\sum_{k=0}^{+\infty} R_k(x)\frac{z^k}{k!} = F(-z)\mathrm{e}^{zx} \qquad \text{and} \qquad \sum_{k=0}^{+\infty} R^*_k(x)\frac{z^k}{k!} = F(z)\mathrm{e}^{z(x-1)}\,.$$

Thus, it comes immediately that

$$R_k(x) = (-1)^k R^*_k(1-x)\,.$$

With this, a direct application of Theorem 1 and Corollary 1 for the polynomial sequence $(R_k(x))_{k\geq 0}$ gives (21) and (22). □

For $q = 1$ in (21) and $q \in \{0, 1\}$ in (22) we obtain respectively Sun's results (4), (5), and (6).

To conclude this section, we deal with Apostol-Bernoulli polynomials $(\mathcal{B}_k(x;\lambda))_{k\geq 0}$ defined [9] for any complex number λ, by means of the following generating function

$$\frac{z\mathrm{e}^{xz}}{\lambda\mathrm{e}^{z}-1} = \sum_{k=0}^{\infty}\mathcal{B}_k(x;\lambda)\frac{z^n}{n!}\,,$$

with $|z + \ln\lambda| < 2\pi$. Let ℓ be a positive integer. Expanding the both sides of the functional relation

$$\frac{\lambda^{\ell} z\mathrm{e}^{(\ell+x)z}}{\lambda\mathrm{e}^{z}-1} = \frac{z\mathrm{e}^{xz}}{\lambda\mathrm{e}^{z}-1} + \sum_{i=0}^{\ell-1}\lambda^i z\mathrm{e}^{iz}\mathrm{e}^{xz}$$

into power series of z and then equating the coefficients of $\frac{z^k}{k!}$ $(k \geq 0)$, yields

$$\lambda^{\ell}\mathcal{B}_0(x+\ell;\lambda) = \mathcal{B}_0(x;\lambda)$$

and

$$\lambda^{\ell}\mathcal{B}_k(x+\ell;\lambda) = \mathcal{B}_k(x;\lambda) + \sum_{i=0}^{\ell-1}\lambda^i k(x+i)^{k-1}$$

for $k \geq 1$. One can see that the previous equality also holds true even when $k = 0$. Therefore, for any positive integer ℓ and any non-negative integer k

$$\lambda^{\ell}\mathcal{B}_k(x+\ell;\lambda) = \mathcal{B}_k(x;\lambda) + \sum_{i=0}^{\ell-1}\lambda^i k(x+i)^{k-1}\,. \quad (23)$$

Theorem 1 enables us to obtain simply the following corollary due to Prévost [11, Theorem 2].

Corollary 5. *For any integers $n, m, p \geq 0$ and $\ell \geq 1$, if $p \geq n+m+1$ then*

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\lambda^{\ell}\frac{(n+m-k)!\mathcal{B}_{p-k}(x;\lambda)}{(p-k)!\ell^{p-k}}-\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\frac{(n+m-k)!\mathcal{B}_{p-k}(x;\lambda)}{(p-k)!\ell^{p-k}}\\=\sum_{i=0}^{\ell-1}\lambda^{i}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\frac{(n+m-k)!(x+i)^{p-k-1}}{(p-k-1)!\ell^{p-k}}\\-\lambda^{\ell}\frac{(-1)^{n}n!m!}{p!}\sum_{k=0}^{p-m-n-1}\binom{p}{k}\binom{p-m-k-1}{n}\frac{\mathcal{B}_{k}(x;\lambda)}{\ell^{k}}, \quad (24)\end{aligned}$$

and if $p \leq n+m$ then

$$\begin{aligned}\sum_{k=0}^{\min\{m,p\}}\binom{m}{k}\lambda^{\ell}\frac{(n+m-k)!\mathcal{B}_{p-k}(x;\lambda)}{(p-k)!\ell^{p-k}}\\-\sum_{k=0}^{\min\{n,p\}}\binom{n}{k}(-1)^{k}\frac{(n+m-k)!\mathcal{B}_{p-k}(x;\lambda)}{(p-k)!\ell^{p-k}}\\=\sum_{i=0}^{\ell-1}\lambda^{i}\sum_{k=0}^{\min(p-1,n)}\binom{n}{k}(-1)^{k}\frac{(n+m-k)!(x+i)^{p-k-1}}{(p-k-1)!\ell^{p-k}}. \quad (25)\end{aligned}$$

Proof. Applying Theorem 1 and Corollary 1 for $S(z)=\frac{z}{\lambda e^{z}-1}$ and $\alpha=\ell$, multiplying the whole by λ^{ℓ} and using identity (23) give

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\frac{\lambda^{\ell}\mathcal{B}_{n+q+k}(x;\lambda)}{(n+q+k)^{\underline{q}}\ell^{n+q+k}}-\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{\mathcal{B}_{m+q+k}(x;\lambda)}{(m+q+k)^{\underline{q}}\ell^{m+q+k}}\\=\sum_{i=0}^{\ell-1}\lambda^{i}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{(m+q+k)(x+i)^{m+q+k-1}}{(m+q+k)^{\underline{q}}\ell^{m+q+k}}\\+\lambda^{\ell}\frac{(-1)^{n+1}}{(q-1)!}\sum_{k=0}^{q-1}\binom{q-1}{k}\frac{m!(n+q-k-1)!}{(n+m+q-k)!}\frac{\mathcal{B}_{k}(x;\lambda)}{\ell^{k}},\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}(n+k)^{\underline{q}}\frac{\lambda^{\ell}\mathcal{B}_{n-q+k}(x;\lambda)}{\ell^{n-q+k}}-\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(m+k)^{\underline{q}}\frac{\mathcal{B}_{m-q+k}(x;\lambda)}{\ell^{m-q+k}}\\=\sum_{i=0}^{\ell-1}\lambda^{i}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(m+k)^{\underline{q}}\frac{(m-q+k)(x+i)^{m-q+k-1}}{\ell^{m-q+k}}.\end{aligned}$$

Reverse the order of summations in the two previous relations to obtain

$$\sum_{k=0}^{m} \binom{m}{k} \frac{\lambda^{\ell} \mathcal{B}_{n+m+q-k}(x;\lambda)}{(n+m+q-k)^{\underline{q}} \ell^{n+m+q-k}} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\mathcal{B}_{n+m+q-k}(x;\lambda)}{(n+m+q-k)^{\underline{q}} \ell^{m+n+q-k}}$$
$$= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{(n+m+q-k)(x+i)^{n+m+q-k-1}}{(n+m+q-k)^{\underline{q}} \ell^{n+m+q-k}}$$
$$+ \lambda^{\ell} \frac{(-1)^{n+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{m!(n+q-k-1)!}{(n+m+q-k)!} \frac{\mathcal{B}_k(x;\lambda)}{\ell^k}, \quad (26)$$

$$\sum_{k=0}^{m} \binom{m}{k} (n+m-k)^{\underline{q}} \frac{\lambda^{\ell} \mathcal{B}_{n+m-q-k}(x;\lambda)}{\ell^{n+m-q-k}} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n+m-k)^{\underline{q}} \frac{\mathcal{B}_{n+m-q-k}(x;\lambda)}{\ell^{n+m-q-k}}$$
$$= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n+m-k)^{\underline{q}} \frac{(n+m-q-k)(x+i)^{n+m-q-k-1}}{\ell^{n+m-q+k}}. \quad (27)$$

Suppose first that $p \geq n+m+1$. Substituting q by $p-n-m$ in (26) gives

$$\sum_{k=0}^{m} \binom{m}{k} \lambda^{\ell} \frac{(n+m-k)! \mathcal{B}_{p-k}(x;\lambda)}{(p-k)! \ell^{p-k}} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{(n+m-k)! \mathcal{B}_{p-k}(x;\lambda)}{(p-k)! \ell^{p-k}}$$
$$= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{(n+m-k)!(x+i)^{p-k-1}}{(p-k-1)! \ell^{p-k}}$$
$$- \lambda^{\ell} \frac{(-1)^n n! m!}{p!} \sum_{k=0}^{p-n-m-1} \binom{p}{k} \binom{p-m-k-1}{n} \frac{\mathcal{B}_k(x;\lambda)}{\ell^k}.$$

Thus, we obtain relation (24). Suppose now that $p \leq n+m$. Taking $q = n+m-p$ in (27) gives

$$\sum_{k=0}^{m} \binom{m}{k} (n+m-k)^{\underline{n+m-p}} \frac{\lambda^{\ell} \mathcal{B}_{p-k}(x;\lambda)}{\ell^{p-k}} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n+m-k)^{\underline{n+m-p}} \frac{\mathcal{B}_{p-k}(x;\lambda)}{\ell^{p-k}}$$
$$= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n+m-k)^{\underline{n+m-p}} \frac{(p-k)(x+i)^{p-k-1}}{\ell^{p+k}}. \quad (28)$$

Noting that the factor $(n+m-k)^{\underline{n+m-p}}$ in the left hand side vanishes when $k > p$, and the factor $(n+m-k)^{\underline{n+m-p}}(p-k)$ in the right hand side vanishes when $k > p-1$, relation (28) leads to relation (25). □

Remark 3. In a similar way, by making use of

$$\lambda^\ell \mathcal{E}_k(x+\ell;\lambda) = (-1)^\ell \mathcal{E}_k(x;\lambda) + 2(-1)^{\ell-1}\sum_{i=0}^{\ell-1}(-1)^i\lambda^i(x+i)^k\,,$$

we obtain an identity due to Prévost [11, Theorem 2], for Apostol-Euler polynomials $(\mathcal{E}_k(x;\lambda))_{k\geq 0}$ defined [9] for $\lambda \in \mathbb{C}\setminus\{-1\}$ by means of the following generating function

$$\frac{2}{\lambda e^z+1}e^{xz} = \sum_{n=0}^{\infty}\mathcal{E}_n(x;\lambda)\frac{z^n}{n!} \qquad (|z+\ln\lambda| < \pi).$$

References

[1] T. Agoh: Shortened recurrence relations for generalized Bernoulli numbers and polynomials. *Journal of Number Theory* 176 (2017) 149–173.

[2] P. Appell: Sur une classe de polynômes. *Annales Scientifiques de l'École Normale Supérieure* 9 (1880) 119–144.

[3] W.Y.C. Chen,L.H. Sun: Extended Zeilberger's algorithm for identities on Bernoulli and Euler polynomials. *Journal of Number Theory* 129 (9) (2009) 2111–2132.

[4] I.M. Gel'fand, G.E. Shilov: *Generalized Functions, Vol. 1: Properties and Operations.* Acad. Press, New York (1964).

[5] M.B. Gelfand: A note on a certain relation among Bernoulli numbers. *Baškir. Gos. Univ. Ucen. Zap. Vyp* 31 (1968) 215–216.

[6] I.M. Gessel: Applications of the classical umbral calculus. *Algebra Univers.* 49 (4) (2003) 397–434.

[7] Y. He, W. Zhang: Some symmetric identities involving a sequence of polynomials. *The Electronic Journal of Combinatorics* 17 (N7) (2010) 7pp.

[8] M. Kaneko: A recurrence formula for the Bernoulli numbers. *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 71 (8) (1995) 192–193.

[9] Q.-M. Luo, H.M. Srivastava: Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials. *Journal of Mathematical Analysis and Applications* 308 (1) (2005) 290–302.

[10] H. Momiyama: A new recurrence formula for Bernoulli numbers. *Fibonacci Quarterly* 39 (3) (2001) 285–288.

[11] M. Prévost: Padé approximation and Apostol-Bernoulli and Apostol-Euler polynomials. *Journal of computational and applied mathematics* 233 (11) (2010) 3005–3017.

[12] G.-C. Rota: The number of partitions of a set. *The American Mathematical Monthly* 71 (5) (1964) 498–504.

[13] M. Stern: Beiträge zur Theorie der Bernoullischen und Eulerschen Zahlen. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen* 26 (1880) 3–46.

[14] Z.-W. Sun: Combinatorial identities in dual sequences. *European Journal of Combinatorics* 24 (6) (2003) 709–718.

[15] A. von Ettingshausen: *Vorlesungen über die höhere Mathematik*. C. Gerold, Vienna (1827).

[16] P.L. von Seidel: Uber eine einfache Entstehungsweise der Bernoullischen Zahlen und einiger verwandten Reihen, Sitzungsberichte der Münch. *Akad. Math. Phys. Classe* 7 (1877) 157–187.

[17] K.-J. Wu, Z.-W. Sun, H. Pan: Some identities for Bernoulli and Euler polynomials. *Fibonacci Quarterly* 42 (4) (2004) 295–299.

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