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Multiplicative Lie triple derivations on standard operator algebras

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Abstract. Let $\mathcal{X}$ be a Banach space of dimension $n > 1$ and $\mathfrak{A} \subset \mathcal{B}(\mathcal{X})$ be a standard operator algebra. In the present paper it is shown that if a mapping $d : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) satisfies

$$d([[U,V],W]) = [[d(U),V],W] + [[U,d(V)],W] + [[U,V],d(W)]$$

for all $U,V,W \in \mathfrak{A}$, then $d = \psi + \tau$, where ψ is an additive derivation of $\mathfrak{A}$ and $\tau : \mathfrak{A} \to \mathbb{F}I$ vanishes at second commutator $[[U,V],W]$ for all $U,V,W \in \mathfrak{A}$. Moreover, if d is linear and satisfies the above relation, then there exists an operator $S \in \mathfrak{A}$ and a linear mapping τ from $\mathfrak{A}$ into $\mathbb{F}I$ satisfying $\tau([[U,V],W]) = 0$ for all $U,V,W \in \mathfrak{A}$, such that $d(U) = SU - US + \tau(U)$ for all $U \in \mathfrak{A}$.

1 Introduction

Let $\mathfrak{A}$ be an associative algebra over a field $\mathbb{F}$. Recall that a linear mapping $d : \mathfrak{A} \to \mathfrak{A}$ is said to be a derivation if $d(UV) = d(U)V + Ud(V)$ holds for all $U,V \in \mathfrak{A}$. If the condition of linearity is replaced by additivity in the above definition, then d is said to be an additive derivation. In particular, derivation d is called an inner derivation if there exists some $X \in \mathfrak{A}$ such that $d(U) = UX - XU$ for all $U \in \mathfrak{A}$. A linear mapping $d : \mathfrak{A} \to \mathfrak{A}$ is called a Lie (resp. Lie triple) derivation if $d([U,V]) = [d(U),V] + [U,d(V)]$ (resp. $d([[U,V],W]) = [[d(U),V],W] + [[U,d(V)],W] + [[U,V],d(W)]$) holds for all $U,V,W \in \mathfrak{A}$, where $[U,V] = UV - VU$ is the usual Lie product. If the condition of linearity is dropped from the above definition, then the corresponding Lie derivation and Lie triple derivation are called multiplicative Lie derivation and multiplicative Lie triple derivation respectively.

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Obviously, every derivation is a Lie derivation and every Lie derivation is a Lie triple derivation. However, the converse statements are not true in general.

There has been a great interest in the study of characterization of Lie derivations and Lie triple derivations for many years. The first quite surprising result is due to Martindale who proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive (see [13]). Miers [14] initially established that every Lie derivation d on a von Neumann algebra $\mathfrak{A}$ can be uniquely written as the sum $d = \delta + \tau$ where δ is an inner derivation of $\mathfrak{A}$ and τ is a linear mapping from $\mathfrak{A}$ into its center $Z(\mathfrak{A})$ vanishing on each commutator. Furthermore, Miers [15] obtained an analogous decomposition for Lie triple derivations of von Neumann algebras with no abelian summands. Yu and Zhang [18] proved that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map from triangular algebra into its center sending commutators to zero. Ji, Liu and Zhao [6] proved the similar result for nonlinear Lie triple derivation of triangular algebras. Zhang, Wu and Cao [19] studied Lie triple derivation on nest algebras. Mathieu and Villena [12] gave the characterizations of Lie derivations on C^*-algebras. In addition, the characterization of Lie derivations and Lie triple derivations on various algebras are considered in [1], [2],[3], [5],[6],[9], [7], [10], [16],[17], [20].

It is the objective of this article is to investigate multiplicative Lie triple derivations on Banach space standard operator algebras. Motivated by the work of F. Lu and B. Liu [11], in Section 2, we study the characterization of multiplicative Lie triple derivations on standard operator algebras.

2 Multiplicative Lie Triple derivations

Throughout this paper, $\mathcal{X}$ represents a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. By $\mathcal{X}^*$ and $\mathcal{B}(\mathcal{X})$ we denote the topological dual space of $\mathcal{X}$ and the algebra of all linear bounded operators on $\mathcal{X}$, respectively. If $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, then rank one operator is $x \otimes f$ is defined by $y \mapsto f(y)x$ for $y \in \mathcal{X}$. A subalgebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{X})$ is called a *standard operator algebra* if all the bounded finite rank operators are contained in $\mathfrak{A}$. An algebra $\mathfrak{A}$ is said to be prime if $A\mathfrak{A}B = 0$ implies either $A = 0$ or $B = 0$. It is to be noted that every standard operator algebra is prime. Motivated by the work of Jing [11], we have obtained the following main result.

Theorem 1. *Let $\mathcal{X}$ be a Banach space of dimension $n > 1$ and $\mathfrak{A} \subset \mathcal{B}(\mathcal{X})$ be a standard operator algebra. Suppose that a map $d : \mathfrak{A} \to \mathfrak{A}$ satisfies*

$$d([[U,V],W]) = [[d(U),V],W] + [[U,d(V)],W] + [[U,V],d(W)]\,, \tag{1}$$

for all $U, V, W \in \mathfrak{A}$. Then $d = \psi + \tau$, where ψ is an additive derivation and τ is a mapping from $\mathfrak{A}$ into $\mathbb{F}I$ satisfying $\tau([[U,V],W]) = 0$ for all $U, V, W \in \mathfrak{A}$.

In particular, if d is linear and satisfies equation (1), then there exist an operator $S \in \mathfrak{A}$ and a linear mapping τ from $\mathfrak{A}$ into $\mathbb{F}I$ that vanishes at second commutators $[[U,V],W]$, such that $d(U) = SU - US + \tau(U)$ for all $U \in \mathfrak{A}$.

For the convenience, in the sequel, take $x_0 \in \mathcal{X}$, $f_0 \in \mathcal{X}^*$ satisfying $f_0(x_0) = 1$. Let $P = x_0 \otimes f_0$ and $Q = I - P$ be idempotent of $\mathfrak{A}$, it is obvious that $PQ = QP = 0$.

Then $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{21} + \mathfrak{A}_{22}$, where $\mathfrak{A}_{11} = P\mathfrak{A}P$, $\mathfrak{A}_{12} = P\mathfrak{A}Q$, $\mathfrak{A}_{21} = Q\mathfrak{A}P$ and $\mathfrak{A}_{22} = Q\mathfrak{A}Q$. We facilitate our discussion with the following known results.

Lemma 1. *[4, Problem 230] Suppose $\mathcal{A}$ is a Banach algebra with the identity I. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{F}$ are such that $[A, B] = \lambda I$, then $\lambda = 0$.*

Lemma 2. *[8, Lemma 2 (ii)] For $U = U_{11} + U_{12} + U_{21} + U_{22} \in \mathfrak{A}$. If $U_{ij}V_{jk} = 0$ for every $U_{ij} \in \mathfrak{A}_{ij}$, $1 \le i, j, k \le 2$, then $V_{jk} = 0$. Dually, if $V_{ki}U_{ij} = 0$ for every $U_{ij} \in \mathfrak{A}_{ij}$, $1 \le i, j, k \le 2$, then $V_{ki} = 0$.*

Now we shall use the hypothesis of Theorem 1 freely without any specific mention in proving the following lemmas.

Lemma 3. *Let $U_{ii} \in \mathfrak{A}_{ii}$, $i = 1, 2$. If $U_{11}V_{12} = V_{12}U_{22}$ for all $V_{12} \in \mathfrak{A}_{12}$, then $U_{11} + U_{22} \in \mathbb{F}I$.*

Proof. For any $V_{11} \in \mathfrak{A}_{11}$ and $V_{12} \in \mathfrak{A}_{12}$, we get

$$U_{11}V_{11}V_{12} = V_{11}V_{12}U_{22} = V_{11}U_{11}V_{12}$$

for all $V_{12} \in \mathfrak{A}_{12}$. As $\mathfrak{A}$ is prime, we have $U_{11}V_{11} = V_{11}U_{11}$.

For any $V_{12} \in \mathfrak{A}_{12}$ and $V_{22} \in \mathfrak{A}_{22}$, we get

$$V_{12}V_{22}U_{22} = U_{11}V_{12}V_{22} = V_{12}U_{22}V_{22}$$

for all $V_{12} \in \mathfrak{A}_{12}$. It follows by the primeness of $\mathfrak{A}$ that $V_{22}U_{22} = U_{22}V_{22}$.

For any $V_{12} \in \mathfrak{A}_{12}$ and $V_{21} \in \mathfrak{A}_{21}$, we get

$$U_{22}V_{21}V_{12} = V_{21}V_{12}U_{22} = V_{21}U_{11}V_{12}$$

for all $V_{12} \in \mathfrak{A}_{12}$. It follows that $U_{22}V_{21} = V_{21}U_{22}$.

For any $V \in \mathfrak{A}$, we have

$$\begin{aligned}(U_{11} + U_{22})V &= (U_{11} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22}) \\ &= U_{11}V_{11} + U_{11}V_{12} + U_{22}V_{21} + U_{22}V_{22} \\ &= V_{11}U_{11} + V_{12}U_{11} + V_{21}U_{22} + V_{22}U_{22} \\ &= (V_{11} + V_{12} + V_{21} + V_{22})(U_{11} + U_{22}) \\ &= V(U_{11} + U_{22}).\end{aligned}$$

Hence it follows that $U_{11} + U_{22} \in \mathbb{F}I$. □

Lemma 4. $d(0) = 0$.

Proof.

$$d(0) = d([[0, 0], 0]) = \big[[d(0), 0], 0\big] + \big[[0, d(0)], 0\big] + \big[[0, 0], d(0)\big] = 0\,.$$

□

Lemma 5. *$Pd(P)P + Qd(P)Q \in \mathbb{F}I$.*

Proof. Let $x \in \mathcal{X}$, $f \in \mathcal{X}^*$. Then

$$\begin{aligned} d(Px \otimes Q^* f) = d([[Px \otimes Q^* f, P], P]) &= [[d(Px \otimes Q^* f), P], P] \\ &\quad + [[Px \otimes Q^* f, d(P)], P] + [[Px \otimes Q^* f, P], d(P)] \\ &= Qd(Px \otimes Q^* f)P + Pd(Px \otimes Q^* f)Q - Px \otimes Q^* f d(P)Q \\ &\quad + Pd(P)Px \otimes Q^* f - Px \otimes Q^* f d(P) + d(P)Px \otimes Q^* f. \end{aligned}$$

Multiplying the above identity from the left by P and from the right by Q, we arrive at

$$Px \otimes Q^* f d(P)Q = Pd(P)Px \otimes Q^* f.$$

Equivalently,

$$Px \otimes fQd(P)Q = Pd(P)Px \otimes fQ.$$

It follows that $Pd(P)P = \lambda P$ and $Qd(P)Q = \lambda Q$ for some $\lambda \in \mathbb{C}$. Hence $Pd(P)P + Qd(P)Q = \lambda I$. □

In the sequel, we define $\phi : \mathfrak{A} \to \mathfrak{A}$ by

$$\phi(U) = d(U) + d_{Pd(P)Q - Qd(P)P}(U) \quad \text{for all } U \in \mathfrak{A}$$

where $d_{Pd(P)Q-Qd(P)P}$ is the inner derivation determined by $Pd(P)Q - Qd(P)P$. It is easy to verify that

$$\phi([[U, V], W]) = \big[[\phi(U), V], W\big] + \big[[U, \phi(V)], W\big] + \big[[U, V], \phi(W)\big]$$

holds for all $U, V, W \in \mathfrak{A}$. Moreover, by Lemma 5, we have

$$\begin{aligned} \phi(P) &= d(P) - Pd(P)Q - Qd(P)P \\ &= d(P)P + d(P)Q - Pd(P)Q - Qd(P)P \\ &= Pd(P)P + Qd(P)Q \\ &= \lambda I. \end{aligned}$$

Thus $\phi(P) \in \mathbb{F}I$.

Lemma 6. *$\phi(PUQ + QUP) = P\phi(U)Q + Q\phi(U)P$ for all $U \in \mathfrak{A}$.*

Proof. Since $[[U, P], P] = PU - 2PUP + UP = PUQ + QUP$, it follows that

$$\begin{aligned} \phi(PUQ + QUP) = \phi([[U, P], P]) &= [[\phi(U), P], P] \\ &= P\phi(U)Q + Q\phi(U)P. \end{aligned}$$

□

Lemma 7. *$\phi(Q) \in \mathbb{F}I$.*

Proof. Using the similar arguments as that used in the proof of Lemma 5, we get

$$P\phi(Q)P + Q\phi(Q)Q \in \mathbb{F}I.$$

Since $\phi(Q) = P\phi(Q)P + P\phi(Q)Q + Q\phi(Q)P + Q\phi(Q)Q$, by Lemma 6, we have

$$P\phi(Q)Q + Q\phi(Q)P = \phi(PQQ + QQP) = 0.$$

Consequently, we get $\phi(Q) = P\phi(Q)P + Q\phi(Q)Q \in \mathbb{F}I$. □

Lemma 8. *If $[U, V] \in \mathbb{F}I$ for any $U, V \in \mathfrak{A}$, then $[\phi(U), V] + [U, \phi(V)] \in \mathbb{F}I$.*

Proof. For $[U, V] \in \mathbb{F}I$, we have $[[U, V], W] = 0$ for all $W \in \mathfrak{A}$.

$$\begin{aligned} 0 = \phi(0) = \phi[[U, V], W] &= [[\phi(U), V], W] + [[U, \phi(V)], W] \\ &= [[\phi(U), V] + [U, \phi(V)], W] \end{aligned}$$

for all $W \in \mathfrak{A}$. Thus $[\phi(U), V] + [U, \phi(V)] \in \mathbb{F}I$. □

Lemma 9. $\phi(U_{ij}) \subseteq \mathfrak{A}_{ij},\ 1 \leq i \neq j \leq 2$.

Proof. For $U_{12} \in \mathfrak{A}_{12}$, we have $U_{12} = [[U_{12}, P], P]$. Thus

$$\phi(U_{12}) = \phi([[U_{12}, P], P]) = [[\phi(U_{12}), P], P] = P\phi(U_{12})Q + Q\phi(U_{12})P,$$

and hence we see that $P\phi(U_{12})P = Q\phi(U_{12})Q = 0$. Now for $U_{12}, V_{12} \in \mathfrak{A}_{12}$, by Lemma 8, we have

$$[\phi(U_{12}), V_{12}] + [U_{12}, \phi(V_{12})] = \lambda I \in \mathbb{F}I. \tag{2}$$

Since $U_{12} = [P, U_{12}]$, by using (2), we find that

$$\begin{aligned} [\phi(U_{12}), V_{12}] = [\phi([P, U_{12}]), V_{12}] &= \lambda I - [[P, U_{12}], \phi(V_{12})] \\ &= \lambda I - \phi([[P, U_{12}], V_{12}]) + [[\phi(P), U_{12}], V_{12}] + [[P, \phi(U_{12})], V_{12}] \\ &= \lambda I + [[P, \phi(U_{12})], V_{12}]. \end{aligned}$$

This implies that

$$\begin{aligned} [P\phi(U_{12})Q + Q\phi(U_{12})P, V_{12}] &= \lambda I + [[P, P\phi(U_{12})Q + Q\phi(U_{12})P], V_{12}] \\ &= \lambda I + [P\phi(U_{12})Q - Q\phi(U_{12})P, V_{12}]. \end{aligned}$$

Hence $[Q\phi(U_{12})P, V_{12}] = \frac{1}{2}\lambda I \in \mathbb{F}I$. It follows from Lemma 1 that $[Q\phi(U_{12})P, V_{12}] = 0$. Thus $Q\phi(U_{12})V_{12} = 0$ and hence by Lemma 2, we have $Q\phi(U_{12})P = 0$. So $\phi(U_{12}) = P\phi(U_{12})Q \in \mathfrak{A}_{12}$ for each $U_{12} \in \mathfrak{A}_{12}$. This implies that $\phi(\mathfrak{A}_{12}) \subseteq \mathfrak{A}_{12}$.

Similarly, $\phi(U_{21}) = Q\phi(U_{21})P \in \mathfrak{A}_{21}$ for each $U_{21} \in \mathfrak{A}_{21}$ and therefore $\phi(\mathfrak{A}_{21}) \subseteq \mathfrak{A}_{21}$. □

Lemma 10. *There is a functional $f_i : \mathfrak{A}_{ii} \to \mathbb{F}I$ such that $\phi(U_{ii}) - f_i(U_{ii})I \in \mathfrak{A}_{ii}$ for all $U_{ii} \in \mathfrak{A}_{ii}$, $i = 1, 2$.*

Proof. For $U_{11} \in \mathfrak{A}_{11}$, by Lemma 6, we have

$$P\phi(U_{11})Q + Q\phi(U_{11})P = \phi(PU_{11}Q + QU_{11}P) = 0.$$

Thus, it can be assumed that $\phi(U_{11}) = A_{11} + A_{22}$ and $\phi(U_{22}) = B_{11} + B_{22}$, here $A_{ii}, B_{ii} \in \mathfrak{A}_{\rm ii}, i = 1, 2$. Since $[U_{11}, U_{22}] = 0$, then by Lemma 1, we have $[\phi(U_{11}), U_{22}] + [U_{11}, \phi(U_{22})] = \lambda I \in \mathbb{F}I$. Multiplying both sides by Q, we arrive at $[Q\phi(U_{11})Q, U_{22}] = \lambda Q$. Consequently, by Lemma 1, $[Q\phi(U_{11})Q, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$. Similarly $[U_{11}, P\phi(U_{22})P] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$.

Equivalently, $[A_{22}, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$ and $[U_{11}, B_{11}] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$. Therefore, there exist scalars $f_1(U_{11})$ and $f_2(U_{22})$ such that $A_{22} = f_1(U_{11})Q$ and $B_{11} = f_2(U_{22})P$. Hence $\phi(U_{11}) - f_1(U_{11})I \in \mathfrak{A}_{11}$ and $\phi(U_{22}) - f_2(U_{22})I \in \mathfrak{A}_{22}$. □

Our next aim is to show that ϕ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.

Lemma 11. *Let $U_{ii} \in \mathfrak{A}_{\rm ii}$ and $U_{ij} \in \mathfrak{A}_{\rm ij}$, $1 \leq i \neq j \leq 2$. Then $\phi(U_{ii} + U_{ij}) - \phi(U_{ii}) - \phi(U_{ij}) \in \mathbb{F}I$.*

Proof. Let $U_{11} \in \mathfrak{A}_{11}$, $U_{12} \in \mathfrak{A}_{12}$. We have

$$\begin{aligned}\phi([[U_{11} + U_{12}, P], P]) &= [[\phi(U_{11} + U_{12}), P], P] + [[U_{11} + U_{12}, \phi(P)], P] \\ &\quad + [[U_{11} + U_{12}, \phi(P)], \phi(P)] \\ &= [[\phi(U_{11} + U_{12}), P], P].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi([[U_{11} + U_{12}, P], P]) &= \phi([[U_{11}, P], P]) + \phi([[U_{12}, P], P]) \\ &= [[\phi(U_{12}), P], P] + [[\phi(U_{12}), P], P].\end{aligned}$$

Combining the above two identities, we get $[[\phi(U_{11}+U_{12}) - \phi(U_{12}) - \phi(U_{12}), P], P] = 0$, that is

$$\begin{aligned}0 = P(\phi(U_{11} + U_{12}) - \phi(U_{12}) - \phi(U_{12}))Q \\ + Q(\phi(U_{11} + U_{12}) - \phi(U_{12}) - \phi(U_{12}))P. \quad (3)\end{aligned}$$

Now, for any $V_{12} \in \mathfrak{A}_{12}$ and by Lemma 5, we have

$$\phi([[U_{11} + U_{12}, V_{12}], P]) = \big[[\phi(U_{11} + U_{12}), V_{12}], P\big] + \big[[U_{11} + U_{12}, \phi(V_{12})], P\big].$$

On the other hand, we have

$$\begin{aligned}\phi\Big(\big[[U_{11} + U_{12}, V_{12}], P\big]\Big) &= \phi\Big(\big[[U_{11}, V_{12}], P\big]\Big) + \phi\Big(\big[[U_{12}, V_{12}], P\big]\Big) \\ &= \big[[\phi(U_{11}), V_{12}], P\big] + \big[[U_{11}, \phi(V_{12})], P\big] \\ &\quad + \big[[\phi(U_{12}), V_{12}], P\big] + \big[[U_{12}, \phi(V_{12})], P\big].\end{aligned}$$

Combining the above two identities, we arrive at

$$\big[[\phi(U_{11} + U_{12}) - \phi(U_{11}) - \phi(U_{12}), V_{12}], P\big] = 0\,.$$

In other words,

$$P\phi(U_{11}+U_{12})-\phi(U_{11})-\phi(U_{12})PV_{12} \\ = V_{12}Q\phi(U_{11}+U_{12})-\phi(U_{11})-\phi(U_{12})Q. \quad (4)$$

Equations (3) and (4), together with Lemma 3, gives that

$$\phi(U_{11}+U_{12})-\phi(U_{11})-\phi(U_{12})\in \mathbb{F}I\,.$$

Similarly, one can easily prove the other part. □

Lemma 12. *ϕ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.*

Proof. Let $U_{12}, V_{12}\in\mathfrak{A}_{12}$. By Lemmas 5, 7 and 11, we see that

$$\begin{aligned}\phi(U_{12}+V_{12}) &= \phi\Big(\big[[P+U_{12},Q+V_{12}],Q\big]\Big)\\ &= \big[[\phi(P+U_{12}),Q+V_{12}],Q\big]+\big[[P+U_{12},\phi(Q+V_{12})],Q\big]\\ &\quad+\big[[P+U_{12},Q+V_{12}],\phi(Q)\big]\\ &= \big[[\phi(P)+\phi(U_{12}),Q+V_{12}],Q\big]+\big[[P+U_{12},\phi(Q)+\phi(V_{12})],Q\big]\\ &= \phi(U_{12})+\phi(V_{12}).\end{aligned}$$

Hence ϕ is additive on $\mathfrak{A}_{12}$. Similarly ϕ is additive on $\mathfrak{A}_{21}$. □

Now for any $U\in\mathfrak{A}$, define

$$\Delta(U)=\phi(PUP)+\phi(PUQ)+\phi(QUP)+\phi(QUQ)-(f_1(PUP)+f_2(QUQ))I\,.$$

By Lemmas 9 and 10, we have

Lemma 13. *Let $U_{ij}\in\mathfrak{A}_{\mathrm{ij}}$, $1\le i\ne j\le 2$. Then*

(*i*) $\Delta(U_{ij})\in\mathfrak{A}_{\mathrm{ij}}$, $1\le i\ne j\le 2$,

(*ii*) $\Delta(U_{12})=\phi(U_{12})$ and $\Delta(U_{21})=\phi(U_{21})$,

(*iii*) $\Delta(U_{11}+U_{12}+U_{21}+U_{22})=\Delta(U_{11})+\Delta(U_{12})+\Delta(U_{21})+\Delta(U_{22})$.

Now, we shall show that Δ is an additive derivation. First, we shall prove the additivity of Δ.

By Lemma 12 and (*ii*) part of Lemma 13, we immediately get the following result.

Lemma 14. *Δ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.*

Lemma 15. *Let $U_{ii}\in\mathfrak{A}_{\mathrm{ii}}$, $U_{ij}\in\mathfrak{A}_{\mathrm{ij}}$, $1\le i\ne j\le 2$. Then*

(*i*) $\Delta(U_{ii}V_{ij})=\Delta(U_{ii})V_{ij}+U_{ii}\Delta(V_{ij})$,

(*ii*) $\Delta(V_{ij}U_{jj})=\Delta(V_{ij})U_{jj}+V_{ij}\Delta(U_{jj})$.

Proof. Since $U_{11}V_{12} = [[U_{11}, V_{12}], Q]$, by Lemmas 7 and 13, we have

$$\begin{aligned}\Delta(U_{11}V_{12}) &= \phi(U_{11}V_{12}) = \phi([[U_{11}, V_{12}], Q]) \\ &= [[\phi(U_{11}), V_{12}], Q] + [[U_{11}, \phi(V_{12})], Q] + [[U_{11}, V_{12}], \phi(Q)] \\ &= [[\Delta(U_{11}), V_{12}], Q] + [[U_{11}, \Delta(V_{12})], Q] \\ &= \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12}).\end{aligned}$$

Similarly, it is easy to prove the other identities. □

Lemma 16. *Δ is additive on $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$.*

Proof. Let $U_{11}, V_{11} \in \mathfrak{A}_{11}$. For any $W_{12} \in \mathfrak{A}_{12}$, by Lemma 15, we have

$$\Delta((U_{11} + V_{11})W_{12}) = \Delta(U_{11} + V_{11})W_{12} + (U_{11} + V_{11})\Delta(W_{12})\,.$$

On the other hand, by Lemmas 14 and 15, we have

$$\begin{aligned}\Delta((U_{11} + V_{11})W_{12}) &= \Delta(U_{11}W_{12} + V_{11}W_{12}) = \Delta(U_{11}W_{12}) + \Delta(V_{11}W_{12}) \\ &= \Delta(U_{11})W_{12} + U_{11}\Delta(W_{12}) + \Delta(V_{11})W_{12} + V_{11}\Delta(W_{12}).\end{aligned}$$

Comparing the above two identities, we get

$$(\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))W_{12} = 0\,.$$

In other words

$$(\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))P\mathfrak{A}Q = 0\,.$$

Since $\mathfrak{A}$ is prime, it follows that

$$(\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))P = 0\,.$$

Hence, $\Delta(U_{11} + V_{11}) = \Delta(U_{11}) + \Delta(V_{11})$ as $\Delta(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Similarly, Δ is additive on $\mathfrak{A}_{22}$. □

Lemma 17. *Δ is additive.*

Proof. Let $U = \sum_{i,j=1}^{2} U_{ij}$, $V = \sum_{i,j=1}^{2} V_{ij}$ be in $\mathfrak{A}$. By Lemmas, 13, 14 and 16, we have

$$\begin{aligned}\Delta(U + V) &= \Delta\Big(\sum_{i,j=1}^{2} (U_{ij} + V_{ij})\Big) \\ &= \sum_{i,j=1}^{2} \Delta(U_{ij} + V_{ij}) = \sum_{i,j=1}^{2} \Big(\Delta(U_{ij}) + \Delta(V_{ij})\Big) \\ &= \Delta\Big(\sum_{i,j=1}^{2} U_{ij}\Big) + \Delta\Big(\sum_{i,j=1}^{2} V_{ij}\Big) = \Delta(U) + \Delta(V).\end{aligned}$$

□

In the sequel, we shall prove that Δ is a derivation.

Lemma 18. *Let $U_{ii}, V_{ii} \in \mathfrak{A}_{\mathrm{ii}}, i = 1, 2$. Then $\Delta(U_{ii}V_{ii}) = \Delta(U_{ii})V_{ii} + U_{ii}\Delta(V_{ii})$*

Proof. For any $U_{11}, V_{11} \in \mathfrak{A}_{11}$ and $W_{12} \in \mathfrak{A}_{12}$, we have by Lemma 15 that

$$\Delta(U_{11}V_{11}W_{12}) = \Delta(U_{11}V_{11})W_{12} + U_{11}V_{11}\Delta(W_{12})\,.$$

On the other hand we have,

$$\begin{aligned}\Delta(U_{11}V_{11}W_{12}) &= \Delta(U_{11})V_{11}W_{12} + U_{11}\Delta(V_{11}W_{12})\\ &= \Delta(U_{11})V_{11}W_{12} + U_{11}\Delta(V_{11})W_{12} + U_{11}V_{11}\Delta(W_{12}).\end{aligned}$$

Comparing the above two identities, we get

$$\big(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11})\big)W_{12} = 0\,.$$

In other words

$$\big(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11})\big)P\mathfrak{A}Q = 0\,.$$

Since $\mathfrak{A}$ is prime, it follows that

$$\big(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11})\big)P = 0\,.$$

Hence, $\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11})$ as $\Delta(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Similarly,

$$\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22})\,. \qquad \square$$

Lemma 19. *Let $U_{11} \in \mathfrak{A}_{11}$ and $V_{22} \in \mathfrak{A}_{22}$. Then*

$$\phi(U_{11} + V_{22}) - \Delta(U_{11}) - \Delta(V_{22}) \in \mathbb{F}I\,.$$

Proof. For any $U_{11} \in \mathfrak{A}_{11}$ and $V_{22} \in \mathfrak{A}_{22}$, we have

$$\phi([[U_{11} + V_{22}, Q], Q]) = [[\phi(U_{11} + V_{22}), Q], Q]$$

On the other hand, we have

$$\begin{aligned}\phi\Big([[U_{11} + V_{22}, Q], Q]\Big) &= \phi\Big([[U_{11}, Q], Q]\Big) + \phi\Big([[V_{22}, Q], Q]\Big)\\ &= \big[[\phi(U_{11}), Q], Q\big] + \big[[\phi(V_{22}), Q], Q\big]\\ &= \big[[\Delta(U_{11}) + f_1(U_{11}), Q], Q\big] + \big[[\Delta(V_{22}) + f_2(U_{22}), Q], Q\big]\\ &= \big[[\Delta(U_{11}), Q], Q\big] + \big[[\Delta(V_{22}), Q], Q\big].\end{aligned}$$

On combining the above two identities, we get

$$\big[[\phi(U_{11} + V_{22}) - \Delta(U_{11}) - \Delta(V_{22}), Q], Q\big] = 0\,,$$

that is

$$0 = P(\phi(U_{11}+V_{22}) - \Delta(U_{11}) - \Delta(V_{22}))Q + Q(\phi(U_{11}+V_{22}) - \Delta(U_{11}) - \Delta(V_{22}))P. \quad (5)$$

Now for any $W_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}\phi([U_{11}+V_{22}, W_{12}]) &= \phi(U_{11}W_{12} - W_{12}V_{22}) \\ &= \phi(U_{11}W_{12}) - \phi(W_{12}V_{22}) = \Delta(U_{11}W_{12}) - \Delta(W_{12}V_{22}) \\ &= \Delta\Big(\big[[U_{11}, W_{12}], Q\big]\Big) - \Delta\Big(\big[[W_{12}, V_{22}], Q\big]\Big) \\ &= \big[[\Delta(U_{11}), W_{12}], Q\big] + \big[[U_{11}, \Delta(W_{12})], Q\big] \\ &\quad - \big[[\Delta(W_{12}), V_{22}], Q\big] - \big[[W_{12}, \Delta(V_{22})], Q\big] \\ &= \big[[\Delta(U_{11}), W_{12}], Q\big] + \big[U_{11}, \Delta(W_{12})\big] \\ &\quad + [V_{22}, \Delta(W_{12})] + \big[[\Delta(V_{22}), W_{12}], Q\big].\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}\phi\Big(\big[U_{11}+V_{22}, W_{12}\big]\Big) &= \phi\Big(\big[[U_{11}+V_{22}, W_{12}], Q\big]\Big) \\ &= \big[[\phi(U_{11}+V_{22}), W_{12}], Q\big] + \big[[U_{11}+V_{22}, \Delta(W_{12})], Q\big] \\ &= \big[[\phi(U_{11}+V_{22}), W_{12}], Q\big] + \big[U_{11}, \Delta(W_{12})\big] + \big[V_{22}, \Delta(W_{12})\big].\end{aligned}$$

Comparing the above two identities, we obtain

$$\big[[\phi(U_{11}+V_{22}) - \Delta(U_{11}) - \Delta(V_{22}), W_{12}], Q\big] = 0\,.$$

In other words, we get

$$P\big(\phi(U_{11}+V_{22}) - \Delta(U_{11}) - \Delta(V_{22})\big)W_{12} = W_{12}\big(\phi(U_{11}+V_{22}) - \Delta(U_{11}) - \Delta(V_{22})\big)Q. \quad (6)$$

Equations (5) and (6), together with Lemma 3, yield that

$$\phi(U_{11}+U_{22}) - \Delta(U_{11}) - \Delta(U_{22}) \in \mathbb{F}I\,.$$ □

Lemma 20. *Let $U_{12} \in \mathfrak{A}_{12}$ and $V_{21} \in \mathfrak{A}_{21}$. Then*

$$\Delta(U_{12}V_{21}) = \Delta(U_{12})V_{21} + U_{12}\Delta(V_{21})$$

and

$$\Delta(U_{21}V_{12}) = \Delta(U_{21})V_{12} + U_{21}\Delta(V_{12})\,.$$

Proof. For any $U_{12} \in \mathfrak{A}_{12}$ and $V_{21} \in \mathfrak{A}_{21}$,compute

$$\begin{aligned}\phi([U_{12}, V_{21}]) - \Delta([U_{12}, V_{21}]) &= \phi([[P, U_{12}], V_{21}]) - \Delta(U_{12}V_{21} - V_{21}U_{12}) \\ &= \big[[P, \phi(U_{12})], V_{21}\big] + \big[[P, U_{12}], \phi(V_{21})\big] \\ &\quad - \Delta(U_{12}V_{21}) + \Delta(V_{21}U_{12}) \\ &= \Delta(U_{12})V_{21} + U_{12}\Delta(V_{21}) - \Delta(U_{12}V_{21}) \\ &\quad - \Delta(V_{21})U_{12} - V_{21}\Delta(U_{12}) + \Delta(V_{21}U_{12}).\end{aligned}$$

Since

$$\phi([U_{12},V_{21}])-\Delta([U_{12},V_{21}])=\phi(U_{12}V_{21}-V_{21}U_{12})-\Delta(U_{12}V_{21}-V_{21}U_{12}),$$

by Lemma 19, we have

$$\begin{aligned}\Delta(U_{12})V_{21}+U_{12}\Delta(V_{21})-\Delta(U_{12}V_{21})-\Delta(V_{21})U_{12}-V_{21}\Delta(U_{12})+\Delta(V_{21}U_{12})\\ =\lambda I\in\mathbb{F}I.\end{aligned}$$

From the later relation we obtain the two identities

$$\Delta(U_{12}V_{21})=\Delta(U_{12})V_{21}+U_{12}\Delta(V_{21})-\lambda P, \tag{7}$$

and

$$\Delta(V_{21}U_{12})=\Delta(V_{21})U_{12}+V_{21}\Delta(U_{12})+\lambda Q. \tag{8}$$

Now it is sufficient to show that $\lambda=0$. Assume $\lambda\neq 0$. Then by using equations (7) and (8) together with Lemma 15, we have

$$\begin{aligned}\Delta(U_{12}V_{21}U_{12})&=\Delta(U_{12})V_{21}U_{12}+U_{12}\Delta(V_{21}U_{12})\\&=\Delta(U_{12})V_{21}U_{12}+U_{12}\Delta(V_{21})U_{12}+U_{12}V_{21}\Delta(U_{12})+\lambda U_{12},\end{aligned}$$

and

$$\begin{aligned}\Delta(U_{12}V_{21}U_{12})&=\Delta(U_{12}V_{21})U_{12}+U_{12}V_{21}\Delta(U_{12})\\&=\Delta(U_{12})V_{21}U_{12}+U_{12}\Delta(V_{21})U_{12}+U_{12}V_{21}\Delta(U_{12})-\lambda U_{12}.\end{aligned}$$

Comparing the above two identities, we obtain $\lambda U_{12}=0$. Since $\mathbb{F}$ is a field, we have $U_{12}=0$, a contradiction. Consequently,

$$\Delta(U_{12}V_{21})=\Delta(U_{12})V_{21}+U_{12}\Delta(V_{21})$$

and

$$\Delta(U_{21}V_{12})=\Delta(U_{21})V_{12}+U_{21}\Delta(V_{12}).$$

□

Thus, we have shown that Δ is an additive derivation.

Proof of Theorem 1. Let us define $\tau:\mathfrak{A}\to\mathfrak{A}$ by $\tau(U)=\phi(U)-\Delta(U)$ for $U\in\mathfrak{A}$. For $i=j$, $\tau(U_{ij})=f_i(U_{ij})I$; otherwise $\tau(U_{ij})=0$. We shall show that $\tau(U)\in\mathbb{F}I$ for all $U\in\mathfrak{A}$. For $T_{12}\in\mathfrak{A}_{12}$ and $U\in\mathfrak{A}$. Since

$$[[U,T_{12}],P]=[UT_{12}-T_{12}U,P]=T_{12}QUQ-PUPT_{12},$$

it follows

$$\begin{aligned}\phi(T_{12}QUQ-PUPT_{12})&=\phi\Big([[U,T_{12}],P\Big])\\&=\big[[\phi(U),T_{12}\big],P\big]+\big[[U,\phi(T_{12})],P\big]\\&=\phi(T_{12})QUQ-PUP\phi(T_{12})+T_{12}Q\phi(U)Q-P\phi(U)PT_{12}.\end{aligned}$$

On the other hand by Lemma 12, we have

$$\begin{aligned}\phi(T_{12}QUQ - PUPT_{12}) &= \phi(T_{12}QUQ) - \phi(PUPT_{12}) \\ &= \phi\Big([[P,T_{12}],QUQ]\Big) - \phi\Big([[T_{12},P],PUP]\Big) \\ &= \big[[P,\phi(T_{12})],QUQ\big] + \big[[P,T_{12}],\phi(QUQ)\big] \\ &\quad - \big[[\phi(T_{12}),P],PUP\big] - \big[[T_{12},P],\phi(PUP)\big] \\ &= \phi(T_{12})QUQ + T_{12}\phi(QUQ) - \phi(QUQ)T_{12} - \phi(PUP)T_{12} \\ &\quad + T_{12}\phi(PUP) - PUP\phi(T_{12}).\end{aligned}$$

Comparing the above two identities, we obtain

$$\big(P\phi(U)P - \phi(PUP) - \phi(QUQ)\big)T_{12} = T_{12}\big(Q\phi(U)Q - \phi(PUP) - \phi(QUQ)\big).$$

Hence for all $T_{12} \in \mathfrak{A}_{12}$,

$$\begin{aligned}&\big(P\phi(U)P - Q\phi(U)Q - \phi(PUP) - \phi(QUQ)\big)T_{12} \\ &\qquad = T_{12}\big(Q\phi(U)Q + Q\phi(U)Q - \phi(PUP) - \phi(QUQ)\big)\end{aligned}$$

By using the Lemma 3, we get the desired result.

$$P\phi(U)P + Q\phi(U)Q - \phi(PUP) - \phi(QUQ) \in \mathbb{F}I. \tag{9}$$

Now by Lemmas 9 and 12, we have

$$P\phi(U)Q + Q\phi(U)P = \phi(PUQ + QUP) = \phi(PUQ) + \phi(QUP),$$

and hence

$$\begin{aligned}&\phi(U) - \phi(PUP) + \phi(PUQ) + \phi(QUP) + \phi(QUQ) \\ &\quad = P\phi(U)P + Q\phi(U)Q + P\phi(U)Q + Q\phi(U)P \\ &\quad - \phi(PUP) - \phi(QUQ) - \phi(PUQ) - \phi(QUP) \\ &\qquad = P\phi(U)P + Q\phi(U)Q - \phi(PUP) - \phi(QUQ) \in \mathbb{F}I.\end{aligned}$$

By equation (9) and by the definition of Δ and τ, we see that $\tau(U) \in \mathbb{F}I$ for all $U \in \mathfrak{A}$. Since Δ is an additive Lie triple derivation, it follows that for all $U, V, W \in \mathfrak{A}$

$$\begin{aligned}\tau([[U,V],W]) &= \phi\big([[U,V],W]\big) - \Delta\big([[U,V],W]\big) \\ &= \big[[\phi(U),V],W\big] + \big[[U,\phi(V)],W\big] + \big[[U,V],\phi(W)\big] - \Delta\big([[U,V],W]\big) \\ &= \big[[\Delta(U),V],W\big] + \big[[U,\Delta(V)],W\big] + \big[[U,V],\Delta(W)\big] - \Delta\big([[U,V],W]\big) \\ &= 0.\end{aligned}$$

Finally, let us define $\psi(U) = \Delta(U) - (TU - UT)$ for all $U \in \mathfrak{A}$, where $T = Pd(P)Q - Qd(P)P$. It is easy to check that ψ is an additive derivation on $\mathfrak{A}$. By the definitions of Δ and ϕ, we have $d(U) = \psi(U) + \tau(U)$ for all $U \in \mathfrak{A}$.

Furthermore, if d is linear , then ψ and τ are also linear. As any linear derivation on $\mathfrak{A}$ is inner, then there exists an operator $S \in \mathfrak{A}$ such that $\psi(U) = SU - US$ for all $U \in \mathfrak{A}$. Hence $d(U) = SU - US + \tau(U)$. This completes the proof. □

References

[1] W. Cheung: Lie derivations of triangular algebras. *Linear Multilinear Algebra* 51 (2003) 299–310.

[2] L. Chen, J.H. Zhang: Nonlinear Lie derivations on upper triangular matrices. *Linear Multilinear Algebra* 56 (6) (2008) 725–730.

[3] M.N. Daif: When is a multiplicative derivation additive?. *International Journal of Mathematics and Mathematical Sciences* 14 (3) (1991) 615–618.

[4] P. Halmos: *A Hilbert space Problem Book, 2nd ed.*. Springer-Verlag, New York (1982).

[5] W. Jing, F. Lu: Lie derivable mappings on prime rings. *Linear Multilinear Algebra* 60 (2012) 167–180.

[6] P. Ji, R. Liu and Y. Zhao: Nonlinear Lie triple derivations of triangular algebras. *Linear Multilinear Algebra* 60 (2012) 1155–1164.

[7] P.S. Ji, L. Wang: Lie triple derivations of TUHF algebras. *Linear Algebra Appl.* 403 (2005) 399–408.

[8] F. Lu: Additivity of Jordan maps on standard operator algebras. *Linear Algebra Appl.* 357 (2002) 123–131.

[9] F. Lu: Lie triple derivations on nest algebras. *Math. Nachr.* 280 (8) (2007) 882–887.

[10] F. Lu, W. Jing: Characterizations of Lie derivations of $\mathcal{B}(\mathcal{X})$. *Linear Algebra Appl.* 432 (1) (2010) 89–99.

[11] F. Lu, B. Liu: Lie derivable maps on $\mathcal{B}(\mathcal{X})$. *Journal of Mathematical Analysis and Applications* 372 (2010) 369–376.

[12] M. Mathieu, A. R. Villena: The structure of Lie derivations on C^*-algebras. *J. Funct. Anal.* 202 (2003) 504–525.

[13] W.S. Martindale III: When are multiplicative mappings additive?. *Proc. Amer. Math. Soc.* 21 (1969) 695–698.

[14] C.R. Mires: Lie derivations of von Neumann algebras. *Duke Math. J.* 40 (1973) 403–409.

[15] C.R. Mires: Lie triple derivations of von Neumann algebras. *Proc. Am. Math. Soc.* 71 (1978) 57–61.

[16] P. Šemrl: Additive derivations of some operator algebras. *llinois J. Math.* 35 (1991) 234–240.

[17] A.R. Villena: Lie derivations on Banach algebras. *J. Algebra* 226 (2000) 390–409.

[18] W. Yu, J. Zhang: Nonlinear Lie derivations of triangular algebras. *Linear Algebra Appl.* 432 (11) (2010) 2953–2960.

[19] J.H. Zhang, B.W. Wu, H.X. Cao: Lie triple derivations of nest algebras. *Linear Algebra Appl.* 416 (2-3) (2006) 559–567.

[20] F. Zhang, J. Zhang: Nonlinear Lie derivations on factor von Neumann algebras. *Acta Mathematica Sinica. (Chin. Ser)* 54 (5) (2011) 791–802.

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