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# $(\phi, \varphi)$-derivations on semiprime rings and Banach algebras

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**Abstract.** Let $\mathcal{R}$ be a semiprime ring with unity $e$ and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. In this paper it is shown that if $\mathcal{R}$ satisfies

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1})$$

for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$, then $\mathcal{D}$ is an $(\phi, \varphi)$-derivation. Moreover, this result makes it possible to prove that if $\mathcal{R}$ admits an additive mappings $\mathcal{D}, \mathcal{G} \colon \mathcal{R} \to \mathcal{R}$ satisfying the relations

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}),$$
$$2\mathcal{G}(x^n) = \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$, then $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations under some torsion restriction. Finally, we apply these purely ring theoretic results to semi-simple Banach algebras.

## 1 Introduction and Results

Throughout this paper $\mathcal{R}$ will denote an associative ring with the center $\mathcal{Z}(\mathcal{R})$. Recall that a ring $\mathcal{R}$ is said to be prime if for any $a, b \in \mathcal{R}$, $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$, and $\mathcal{R}$ is semiprime if for any $a \in \mathcal{R}$, $a\mathcal{R}a = \{0\}$ implies $a = 0$. A ring $\mathcal{R}$ is said to be $n$-torsion free, where $n > 1$ is an integer, if $nx = 0$ implies $x = 0$ for all $x \in \mathcal{R}$. For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ will denote the commutator $xy - yx$. By a Banach algebra $\mathfrak{B}$ we mean an algebra equipped with a norm $\|\cdot\|$ that makes it into a Banach space and additionally satisfies the inequality $\|uv\| \leq \|u\|\|v\|$ for all $u, v \in \mathfrak{B}$ (see [3]). The Jacobson radical of $\mathfrak{B}$, denoted by $\mathrm{rad}(\mathfrak{B})$, is the intersection of all the primitive ideals of $\mathfrak{B}$. An algebra $\mathfrak{B}$ is called semi-simple Banach algebra if $\mathrm{rad}(\mathfrak{B}) = 0$.

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An additive mapping $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is said to be a derivation (resp. Jordan derivation)on $\mathcal{R}$ if

$$\mathcal{D}(xy) = \mathcal{D}(x)y + x\mathcal{D}(y)$$

(resp. $\mathcal{D}(x^2) = \mathcal{D}(x)x + x\mathcal{D}(x)$) holds for all $x, y \in \mathcal{R}$. A derivation $\mathcal{D}$ is inner if there exists $a \in \mathcal{R}$ such that $\mathcal{D}(x) = [a, x]$ holds for all $x \in \mathcal{R}$. It is easy to verify that every derivation is a Jordan derivation but the converse is not true in general. A classical result of Herstein [10] states that every Jordan derivation is a derivation on a prime ring of characteristic different from two. A brief proof of Herstein's result can be found in [6]. An additive mapping $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is called a Jordan triple derivation if

$$\mathcal{D}(xyx) = \mathcal{D}(x)yx + x\mathcal{D}(y)x + xy\mathcal{D}(x)$$

holds for all $x, y \in \mathcal{R}$. Obviously, every derivation is a Jordan triple derivation but not conversely. Brešar [5, Theorem 4.3], established that a Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Motivated by the above result, Vukman [17] recently showed that if $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ an additive mapping on a 2-torsion free semiprime ring $\mathcal{R}$ satisfying either

$$\mathcal{D}(xyx) = \mathcal{D}(xy)x + xy\mathcal{D}(x)$$

for all pairs $x, y \in \mathcal{R}$ or

$$\mathcal{D}(xyx) = \mathcal{D}(x)yx + x\mathcal{D}(yx)$$

for all pairs $x, y \in \mathcal{R}$, then $\mathcal{D}$ is a derivation. In 2016 Širovnik [16] generalized the above result. In fact, he established that if $\mathcal{D}, \mathcal{G}\colon \mathcal{R} \to \mathcal{R}$ are two additive mappings on a 2-torsion free semiprime ring $\mathcal{R}$ satisfying either

$$\mathcal{D}(xyx) = \mathcal{D}(xy)x + xy\mathcal{G}(x)$$

and

$$\mathcal{G}(xyx) = \mathcal{G}(xy)x + xy\mathcal{D}(x)$$

for all pairs $x, y \in \mathcal{R}$ or

$$\mathcal{D}(xyx) = \mathcal{D}(x)yx + x\mathcal{G}(yx)$$

and

$$\mathcal{G}(xyx) = \mathcal{G}(x)yx + x\mathcal{D}(yx)$$

for all pairs $x, y \in \mathcal{R}$, then $\mathcal{D}$ and $\mathcal{G}$ are derivations and $\mathcal{D} = \mathcal{G}$. Following the same line, a number of results have been obtained by several authors (see [1], [2], [4], [8], [12], [13], [14], [15], [18], [19], [20]), where further references can be found.

Let $\phi$, $\varphi$ be any two mappings on $\mathcal{R}$. An additive mapping $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is said to be an ($\phi$, $\varphi$)-derivation (resp. Jordan ($\phi$, $\varphi$)-derivation) on $\mathcal{R}$ if

$$\mathcal{D}(xy) = \mathcal{D}(x)\phi(y) + \varphi(x)\mathcal{D}(y)$$

(resp. $\mathcal{D}(x^2) = \mathcal{D}(x)\phi(x) + \varphi(x)\mathcal{D}(x)$) holds for all $x, y \in \mathcal{R}$. An additive mapping $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is called a Jordan triple $(\phi,\ \varphi)$-derivation if

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(y)\phi(x) + \varphi(xy)\mathcal{D}(x)$$

holds for all $x, y \in \mathcal{R}$. Obviously, every $(\phi,\ \varphi)$-derivation is a Jordan $(\phi,\ \varphi)$-derivation and a Jordan triple $(\phi,\ \varphi)$-derivation, but not conversely. Brešar and Vukman [7] obtained that every Jordan $(\phi,\ \varphi)$-derivation is a $(\phi,\ \varphi)$-derivation on a prime ring of characteristic different from two . For these kind of results we refer the reader to ([9], [11]), where further references can be found. Liu and Shiue [11] have recently generalized the above result to 2-torsion free semiprime rings. Moreover in the same paper they showed that every Jordan triple $(\phi,\ \varphi)$-derivation is a $(\phi,\ \varphi)$-derivation on a 2-torsion free semiprime ring.

In view of the above results we begin our investigation by extending the results of Vukman [17] to $(\phi,\ \varphi)$-derivations. In fact, we have shown that an additive mapping $\mathcal{D}$ on a semiprime ring $\mathcal{R}$ which satisfies either of the identities

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x)$$

or

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx)$$

for all $x, y \in \mathcal{R}$ is a $(\phi,\ \varphi)$-derivation. Further, it is also shown that if the additive mapping $\mathcal{D}$ on $\mathcal{R}$ satisfies

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1})$$

for all $x \in \mathcal{R}$, then $\mathcal{D}$ is a $(\phi,\ \varphi)$-derivation. Finally, we have shown that under what conditions and additive mapping $\mathcal{D}$ on $\mathcal{R}$ satisfying

$$\mathcal{D}(x^n) = \sum_{j=1}^{n} \phi(x^{n-j})\mathcal{D}(x)\varphi(x^{j-1}) \text{ for all } x \in \mathcal{R}$$

is an $(\phi,\ \varphi)$-derivations.

## 2 Main Results

We facilitate our investigation with the following theorem:

**Theorem 1.** *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is an additive mapping such that either*

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x) \quad \text{for all } x, y \in \mathcal{R}, \tag{1}$$

*or*

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx) \quad \text{for all } x, y \in \mathcal{R}. \tag{2}$$

*Then $\mathcal{D}$ is a $(\phi,\ \varphi)$-derivation.*

For developing the proof of our theorem, we need the following Lemma.

**Lemma 1.** *Let $\mathcal{R}$ be a semiprime ring and $\phi$ be an automorphism of $\mathcal{R}$. Suppose $f\colon \mathcal{R} \to \mathcal{R}$ is an additive mapping such that either $f(x)\phi(x) = 0$ holds for all $x \in \mathcal{R}$ or $\phi(x)f(x) = 0$ holds for all $x \in \mathcal{R}$, then $f = 0$.*

*Proof.* Since, we have

$$f(x)\phi(x) = 0 \quad \text{for all } x \in \mathcal{R}. \tag{3}$$

The linearization of the above relation gives

$$f(x)\phi(y) + f(y)\phi(x) = 0 \text{ for all } x, y \in \mathcal{R}. \tag{4}$$

Replace $y$ by $y^2$ in the above equation, we see that

$$f(x)\phi(y^2) + f(y^2)\phi(x) = 0 \text{ for all } x, y \in \mathcal{R}. \tag{5}$$

Right multiplication of (4) by $\phi(y)$ gives

$$f(x)\phi(y^2) + f(y)\phi(x)\phi(y) = 0 \text{ for all } x, y \in \mathcal{R}. \tag{6}$$

By comparing (5) and (6), we obtain

$$f(y^2)\phi(x) - f(y)\phi(x)\phi(y) = 0 \text{ for all } x, y \in \mathcal{R}. \tag{7}$$

Since $\phi$ is an automorphism, we have

$$f(y^2)z - f(y)z\phi(y) = 0 \text{ for all } y, z \in \mathcal{R}.$$

Replace $z$ by $\phi(x)f(y)$ in the above relation, and use (3), we obtain,

$$f(y^2)\phi(x)f(y) = 0 \text{ for all } x, y \in \mathcal{R}.$$

In view of the above relation right multiplication of (7) by $f(y)$ yields

$$f(y)\phi(x)\phi(y)f(y) = 0$$

for all $x, y \in \mathcal{R}$, which leads to $\phi(y)f(y)\phi(x)\phi(y)f(y) = 0$ for all $x, y \in \mathcal{R}$. Hence we have

$$\phi(y)f(y) = 0 \text{ for all } y \in \mathcal{R}. \tag{8}$$

Right multiplication of (4) by $f(x)$ and using (8), we find that

$$f(x)\phi(y)f(x) = 0 \text{ for all } x, y \in \mathcal{R}.$$

Since $\mathcal{R}$ is semiprime, it follows that $f = 0$, which completes the proof. □

*Proof.* [Proof of Theorem 1] We will restrict our attention on the relation (1), the proof in case when $\mathcal{R}$ satisfies the relation (2) is similar and will therefore be omitted. Linearize the relation (1), we see that

$$\mathcal{D}(xyz + zyx) = \mathcal{D}(xy)\phi(z) + \mathcal{D}(zy)\phi(x) + \varphi(xy)\mathcal{D}(z) + \varphi(zy)\mathcal{D}(x),$$

for all $x, y, z \in \mathcal{R}$. In particular for $z = x^2$, the above relation gives

$$\mathcal{D}(xyx^2 + x^2yx) = \mathcal{D}(xy)\phi(x^2) + \mathcal{D}(x^2y)\phi(x) + \varphi(xy)\mathcal{D}(x^2) + \varphi(x^2y)\mathcal{D}(x), \quad (9)$$

for all $x, y, \in \mathcal{R}$. Putting $xy + yx$ for $y$ in (1) and applying the relation (1), we obtain

$$\begin{aligned} \mathcal{D}(xyx^2 + x^2yx) &= \mathcal{D}(x^2y + xyx)\phi(x) + \varphi(x^2y + xyx)\mathcal{D}(x) \\ &= \mathcal{D}(x^2y)\phi(x) + \mathcal{D}(xy)\phi(x^2) + \varphi(xy)\mathcal{D}(x)\phi(x) \\ &\quad + \varphi(x^2y)\mathcal{D}(x) + \varphi(xyx)\mathcal{D}(x), \end{aligned} \quad (10)$$

for all $x, y \in \mathcal{R}$. By comparing (9) and (10), we have

$$\varphi(x)\varphi(y)A(x) = 0, \text{ for all } x, y \in \mathcal{R}, \quad (11)$$

where $A(x)$ stands for $\mathcal{D}(x^2) - \mathcal{D}(x)\phi(x) - \varphi(x)\mathcal{D}(x)$. Since $\varphi$ is surjective, we have

$$\varphi(x)zA(x) = 0, \text{ for all } x, z \in \mathcal{R}. \quad (12)$$

Right multiplication of (12) by $\varphi(x)$ and left multiplication by $A(x)$ gives,

$$A(x)\varphi(x)zA(x)\varphi(x) = 0, \text{ for all } x, z \in \mathcal{R}.$$

By the semiprimeness of $\mathcal{R}$, it follows that

$$A(x)\varphi(x) = 0, \text{ for all } x \in \mathcal{R}. \quad (13)$$

The substitution of $A(x)y\varphi(x)$ for $z$ in the relation (12), gives

$$\varphi(x)A(x)y\varphi(x)A(x) = 0$$

for all pairs $x, y \in \mathcal{R}$. Hence, we obtain

$$\varphi(x)A(x) = 0, \text{ for all } x \in \mathcal{R}. \quad (14)$$

The linearization of the relation (13) gives

$$B(x, y)\varphi(x) + A(x)\varphi(y) + B(x, y)\varphi(y) + A(y)\varphi(x) = 0$$

for all pairs $x, y \in \mathcal{R}$, where $B(x, y)$ denotes

$$\mathcal{D}(xy + yx) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y) - \mathcal{D}(y)\phi(x) - \varphi(y)\mathcal{D}(x).$$

Putting in the above relation $-x$ for $x$ and comparing the relation so obtained with the above relation one obtains

$$B(x, y)\varphi(x) + A(x)\varphi(y) = 0, \text{ for all } x, y \in \mathcal{R}.$$

In view of the relation (14), right multiplication by $A(x)$ gives, $A(x)\varphi(y)A(x) = 0$ for all pairs $x, y \in \mathcal{R}$. Hence it follows that $A(x) = 0$ for all $x \in \mathcal{R}$. In other words,

$\mathcal{D}$ is a Jordan ($\phi$, $\varphi$)-derivation. By [11, Corollary 1] one can conclude that $\mathcal{D}$ is a ($\phi$, $\varphi$)-derivation. It is our aim to show that Theorem 1 can be proved without using [11, Corollary 1]. From the fact that $\mathcal{D}$ is a Jordan ($\phi$, $\varphi$)-derivation, it follows that $\mathcal{D}$ is a Jordan triple ($\phi$, $\varphi$)-derivation. Now, comparing the relation $\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(y)\phi(x) + \varphi(xy)\mathcal{D}(x)$, for all $x, y \in \mathcal{R}$, with the relation (1), we get

$$\big(\mathcal{D}(xy) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y)\big)\phi(x) = 0, \text{ for all } x, y \in \mathcal{R}.$$

For any fixed $y \in \mathcal{R}$, we have an additive mapping $x \mapsto \mathcal{D}(xy) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y)$ on $\mathcal{R}$. Thus from the above relation and Lemma 1 it follows that $\mathcal{D}(xy) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y) = 0$ for all pairs $x, y \in \mathcal{R}$. In other words, $\mathcal{D}$ is a ($\phi$, $\varphi$)-derivation. This completes the proof.

□

**Remark 1.** It is to be noted that if $\phi$ and $\varphi$ are the identity automorphisms on $\mathcal{R}$, then the above result reduces to the [17, Theorem 2].

**Theorem 2.** *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is an additive mapping such that either*

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) - \varphi(xy)\mathcal{D}(x) \textit{ for all } x, y \in \mathcal{R}, \tag{15}$$

*or*

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) - \varphi(x)\mathcal{D}(yx) \textit{ for all } x, y \in \mathcal{R}. \tag{16}$$

*Then $\mathcal{D} = 0$.*

*Proof.* We will restrict our attention on the relation (15), the proof in the other case is similar. Linearization of the relation (15) gives

$$\mathcal{D}(xyz + zyx) = \mathcal{D}(xy)\phi(z) + \mathcal{D}(zy)\phi(x) - \varphi(xy)\mathcal{D}(z) - \varphi(zy)\mathcal{D}(x),$$

for all $x, y, z \in \mathcal{R}$. Following the same procedure as used in the above theorem we get, $A(x) = 0$ for all pairs $x, y \in \mathcal{R}$, where $A(x)$ stands for $\mathcal{D}(x^2) - \mathcal{D}(x)\phi(x) - \varphi(x)\mathcal{D}(x)$. Thus $\mathcal{D}$ is a Jordan ($\phi$, $\varphi$)-derivation and hence it follows that $\mathcal{D}$ is a Jordan triple ($\phi$, $\varphi$)-derivation. Now, comparing the relation $\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(y)\phi(x) + \varphi(xy)\mathcal{D}(x)$, for all $x, y \in \mathcal{R}$, with the relation (15), one obtains

$$\varphi(x)\varphi(y)\mathcal{D}(x) = 0, \text{ for all } x, y \in \mathcal{R}. \tag{17}$$

Since $\varphi$ is surjective, we have

$$\varphi(x)z\mathcal{D}(x) = 0, \text{ for all } x, z \in \mathcal{R}. \tag{18}$$

Right multiplication of (18) by $\varphi(x)$ and left multiplication by $\mathcal{D}(x)$ gives

$$\mathcal{D}(x)\varphi(x)z\mathcal{D}(x)\varphi(x) = 0, \text{ for all } x, z \in \mathcal{R}.$$

By the semiprimeness of $\mathcal{R}$ it follows that

$$\mathcal{D}(x)\varphi(x) = 0, \text{ for all } x \in \mathcal{R}. \tag{19}$$

The substitution of $\mathcal{D}(x)y\varphi(x)$ for $z$ in the relation (18), gives

$$\varphi(x)\mathcal{D}(x)y\varphi(x)\mathcal{D}(x) = 0$$

for all pairs $x, y \in \mathcal{R}$. Hence, we obtain

$$\varphi(x)\mathcal{D}(x) = 0, \text{ for all } x, y \in \mathcal{R}. \tag{20}$$

The linearization of the relation (19) gives

$$\mathcal{D}(x)\varphi(y) + \mathcal{D}(y)\varphi(x) = 0, \text{ for all } x, y \in \mathcal{R}.$$

In view of the relation (20), right multiplication by $\mathcal{D}(x)$ gives,

$$\mathcal{D}(x)\varphi(y)\mathcal{D}(x) = 0, \text{ for all } x, y \in \mathcal{R}.$$

Hence it follows that $\mathcal{D} = 0$, which completes the proof. □

**Corollary 1.** *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}, \mathcal{G} \colon \mathcal{R} \to \mathcal{R}$ is an additive mappings such that either*

$$\begin{aligned} \mathcal{D}(xyx) &= \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{G}(x), \\ \mathcal{G}(xyx) &= \mathcal{G}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x) \quad \textit{for all } x, y \in \mathcal{R}, \end{aligned} \tag{21}$$

*or*

$$\begin{aligned} \mathcal{D}(xyx) &= \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{G}(yx), \\ \mathcal{G}(xyx) &= \mathcal{G}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx) \quad \textit{for all } x, y \in \mathcal{R}. \end{aligned} \tag{22}$$

*Then $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations and $\mathcal{D} = \mathcal{G}$.*

*Proof.* We will restrict our attention on the relations (21), the proof in case we have the relations (22) is similar and will therefore be omitted. Thus the relations are

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{G}(x), \text{ for all } x, y \in \mathcal{R}, \tag{23}$$

$$\mathcal{G}(xyx) = \mathcal{G}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x), \text{ for all } x, y \in \mathcal{R}. \tag{24}$$

Combining the relations (24) and (23), gives

$$T(xyx) = T(xy)\phi(x) - \varphi(xy)T(x), \text{ for all } x, y \in \mathcal{R}, \tag{25}$$

where $T = \mathcal{D} - \mathcal{G}$. By applying Theorem 2 one obtains that $\mathcal{D} = \mathcal{G}$. Thus relation (21) reduces to

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x), \text{ for all } x, y \in \mathcal{R}.$$

Using Theorem 1, it follows that $\mathcal{D}$ is a $(\phi, \varphi)$-derivation, which completes the proof. □

Disadvantage of Theorem 1 is that in identities (1) and (2) there is no symmetry. Therefore, Theorem 1, together with the desire for symmetry leads to the following conjecture.

**Conjecture 1.** Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is an additive mapping such that

$$2\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x) + \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx), \tag{26}$$

holds for all pairs $x, y \in \mathcal{R}$. Then $\mathcal{D}$ is a $(\phi,\ \varphi)$-derivation.

Note that in case a ring has the identity element, the proof of the above conjecture is immediate. The substitution $y = e$ in the relation (26), where $e$ stands for the identity element, gives that $\mathcal{D}$ is a Jordan $(\phi,\ \varphi)$-derivation and then it follows from [11, Corollary 1] that $\mathcal{D}$ is a $(\phi,\ \varphi)$-derivation.

The substitution of $y = x^{n-2}$ in the relation (26) gives

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

which leads to the following conjecture.

**Conjecture 2.** Let $\mathcal{R}$ be a semiprime ring with a suitable torsion restriction and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is an additive mapping such that

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

holds for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$. Then $\mathcal{D}$ is a $(\phi,\ \varphi)$-derivation.

Now we prove the above conjecture in case a ring has the identity element.

**Theorem 3.** *Let $\mathcal{R}$ be a $(n-1)!$-torsion free semiprime ring with identity $e$ and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is an additive mapping such that*

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

*for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$. Then $\mathcal{D}$ is a $(\phi,\ \varphi)$-derivation.*

*Proof.* We have the relation

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}), \tag{27}$$

holds for all $x \in \mathcal{R}$. The substitution of $x = e$ in the relation (27) gives $\mathcal{D}(e) = 0$. Let $y$ be any element of the center $\mathcal{Z}(\mathcal{R})$. Putting $x + y$ for $x$ in the above relation, we obtain

$$\begin{aligned}2\sum_{i=0}^{n}\binom{n}{i}\mathcal{D}(x^{n-i}y^i) &= \Big(\sum_{i=0}^{n-1}\binom{n-1}{i}\mathcal{D}(x^{n-1-i}y^i)\Big)\phi(x+y)\\ &\quad+\Big(\sum_{i=0}^{n-1}\binom{n-1}{i}\varphi(x^{n-1-i}y^i)\Big)\mathcal{D}(x+y)\\ &\quad+\mathcal{D}(x+y)\Big(\sum_{i=0}^{n-1}\binom{n-1}{i}\phi(x^{n-1-i}y^i)\Big)\\ &\quad+\varphi(x+y)\Big(\sum_{i=0}^{n-1}\binom{n-1}{i}\mathcal{D}(x^{n-1-i}y^i)\Big).\end{aligned}$$

Using (27) in the above relation and rearranging it in sense of collecting together terms involving equal number of factors of $y$, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y)=0, \tag{28}$$

where $f_i(x,y)$ stands for the expression of terms involving $i$ factors of $y$. Replace $x$ by $x+2y$, $x+3y$, …, $x+(n-1)y$ in the relation (27) and expressing the resulting system of $(n-2)$ homogeneous equations of variables $f_i(x,y)$ for $i=1,2,\dots n-1$ together with (28), we see that the coefficient matrix of the system of $(n-1)$ homogenous equations is a Van-der Monde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1\\ 2 & 2^2 & \dots & 2^{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1}\end{pmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution. In particular, if $y$ is replaced with the identity element $e$, we obtain

$$\begin{aligned}f_{n-2}(x,e) &= 2\binom{n}{n-2}\mathcal{D}(x^2)-\binom{n-1}{n-2}\mathcal{D}(x)\phi(x)-\binom{n-1}{n-3}\mathcal{D}(x^2)\\ &\quad-\binom{n-1}{n-2}\varphi(x)\mathcal{D}(x)-\binom{n-1}{n-3}\varphi(x^2)\mathcal{D}(e)-\binom{n-1}{n-2}\mathcal{D}(x)\phi(x)\\ &\quad-\binom{n-1}{n-3}\mathcal{D}(e)\phi(x^2)-\binom{n-1}{n-3}\mathcal{D}(x^2)-\binom{n-1}{n-2}\varphi(x)\mathcal{D}(x).\end{aligned}$$

After few calculations and considering the relation $\mathcal{D}(e)=0$, we obtain

$$(n(n-1)-(n-1)(n-2))\mathcal{D}(x^2)=2(n-1)\big(\mathcal{D}(x)\phi(x)+\varphi(x)\mathcal{D}(x)\big).$$

Since $\mathcal{R}$ is $(n-1)!$-torsion free, it follows from the above relation that

$$\mathcal{D}(x^2)=\mathcal{D}(x)\phi(x)+\varphi(x)\mathcal{D}(x) \text{ for all } x\in\mathcal{R}.$$

Hence $\mathcal{D}$ is a Jordan $(\phi, \varphi)$-derivation. By [11, Corollary 1], $\mathcal{D}$ is a $(\phi, \varphi)$-derivation, which completes the proof. □

**Theorem 4.** *Let $\mathcal{R}$ be a $(n-1)!$-torsion free semiprime ring with identity $e$ and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose there exist additive mappings $\mathcal{D}, \mathcal{G}\colon \mathcal{R} \to \mathcal{R}$ satisfying the relations*

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}),$$

$$2\mathcal{G}(x^n) = \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

*for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$. Then $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations.*

*Proof.* We have

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}), \tag{29}$$

$$2\mathcal{G}(x^n) = \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}), \tag{30}$$

for all $x \in \mathcal{R}$, where $n \geq 2$ is a fixed integer. Subtracting the two relations of equation, we obtain

$$2T(x^n) = T(x^{n-1})\phi(x) - \varphi(x^{n-1})T(x) - T(x)\phi(x^{n-1}) - \varphi(x)T(x^{n-1}), \tag{31}$$

where $T = \mathcal{D} - \mathcal{G}$. We denote the identity element of the ring $\mathcal{R}$ by $e$. Putting $e$ for $x$ in the above relation gives

$$T(e) = 0. \tag{32}$$

Let $y$ be any element of the center $\mathcal{Z}(\mathcal{R})$. Putting $x + y$ for $x$ in the relation 31 and follow the same procedure as used in Theorem 3, we arrive at

$$\begin{aligned} f_{n-1}(x, e) &= 2\binom{n}{n-1}T(x) - \binom{n-1}{n-1}\Big(T(e)\varphi(x) + eT(x) + T(x)e + \phi(x)T(e)\Big) \\ &\quad - \binom{n-1}{n-2}\Big(T(x)e + \phi(x)T(e) + T(e)\varphi(x) + eT(x)\Big) \\ &= 0. \end{aligned}$$

Using 32 in the above identity, we obtain

$$2nT(x) = 2T(x) - 2(n-1)T(x)$$

Since $\mathcal{R}$ is $(n-1)!$-torsion free, it follows from the above relation that $T(x) = 0$ for all $x \in \mathcal{R}$. Therefore, we get $\mathcal{D} = \mathcal{G}$. Thus equations 29 and 30 reduces into one relation, which is

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}).$$

Using Theorem 3, we conclude that $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations. This completes the proof. □

Following are the immediate consequences of above theorems.

Since every semi-simple Banach algebra $\mathcal{B}$ is a semiprime ring (see [3] for details), we have the following results.

**Corollary 2.** *Let $\mathcal{B}$ be a semi-simple Banach algebra and $\phi$, $\varphi$ be automorphisms of $\mathcal{B}$. Suppose $\mathcal{D}, \mathcal{G}\colon \mathcal{B} \to \mathcal{B}$ are linear mappings such that either*

$$\mathcal{D}(uvu) = \mathcal{D}(uv)\phi(u) + \varphi(uv)\mathcal{G}(u),$$
$$\mathcal{G}(uvu) = \mathcal{G}(uv)\phi(u) + \varphi(uv)\mathcal{D}(u) \quad \text{for all } u, v \in \mathcal{B},$$

*or*

$$\mathcal{D}(uvu) = \mathcal{D}(u)\phi(vu) + \varphi(u)\mathcal{G}(vu),$$
$$\mathcal{G}(uvu) = \mathcal{G}(u)\phi(vu) + \varphi(u)\mathcal{D}(vu) \quad \text{for all } u, v \in \mathcal{B}.$$

*Then $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations and $\mathcal{D} = \mathcal{G}$.*

**Corollary 3.** *Let $\mathcal{B}$ be a semi-simple Banach algebra with identity $e$ and $\phi$, $\varphi$ be automorphisms of $\mathcal{B}$. Suppose $\mathcal{D}, \mathcal{G}\colon \mathcal{B} \to \mathcal{B}$ are additive mappings such that*

$$2\mathcal{D}(u^n) = \mathcal{D}(u^{n-1})\phi(u) + \varphi(u^{n-1})\mathcal{G}(u) + \mathcal{G}(u)\phi(u^{n-1}) + \varphi(u)\mathcal{G}(u^{n-1}),$$
$$2\mathcal{G}(u^n) = \mathcal{G}(u^{n-1})\phi(u) + \varphi(u^{n-1})\mathcal{D}(u) + \mathcal{D}(u)\phi(u^{n-1}) + \varphi(u)\mathcal{D}(u^{n-1}),$$

*holds for all $u \in \mathcal{B}$ and some fixed integer $n \geq 2$. Then $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations.*

Theorem 4 and Corollary 1 leads to the following conjectures. So, we conclude our paper by giving the following conjectures:

**Conjecture 3.** Let $\mathcal{R}$ be a semiprime ring with a suitable torsion restriction and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}, \mathcal{G}\colon \mathcal{R} \to \mathcal{R}$ are additive mappings satisfying the relations

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}),$$
$$2\mathcal{G}(x^n) = \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$. Then $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations.

**Conjecture 4.** Let $\mathcal{R}$ be a semiprime ring with a suitable torsion restriction and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}, \mathcal{G}\colon \mathcal{R} \to \mathcal{R}$ are additive mappings such that either

$$\mathcal{D}(x^3) = \mathcal{D}(x^2)\phi(x) + \varphi(x^2)\mathcal{G}(x), \tag{33}$$
$$\mathcal{G}(x^3) = \mathcal{G}(x^2)\phi(x) + \varphi(x^2)\mathcal{D}(x) \quad \text{for all } x, y \in \mathcal{R},$$

or

$$\mathcal{D}(x^3) = \mathcal{D}(x)\phi(x^2) + \varphi(x)\mathcal{G}(x^2), \tag{34}$$
$$\mathcal{G}(x^3) = \mathcal{G}(x)\phi(x^2) + \varphi(x)\mathcal{D}(x^2) \quad \text{for all } x, y \in \mathcal{R}.$$

Then $\mathcal{D}$ and $\mathcal{G}$ are $(\phi, \varphi)$-derivations and $\mathcal{D} = \mathcal{G}$.

**Conjecture 5.** Let $\mathcal{R}$ be a semiprime ring with a suitable torsion restriction and $\phi$, $\varphi$ be automorphisms of $\mathcal{R}$. Suppose $\mathcal{D}\colon \mathcal{R} \to \mathcal{R}$ is an additive mapping such that

$$\mathcal{D}(x^n) = \sum_{j=1}^{n} \phi(x^{n-j})\mathcal{D}(x)\varphi(x^{j-1}),$$

holds for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$. Then $\mathcal{D}$ is a $(\phi,\ \varphi)$-derivation.

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