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Remarks on Ramanujan's inequality concerning the prime counting function

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Abstract. In this paper we investigate Ramanujan's inequality concerning the prime counting function, asserting that $\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\pi\left(\frac{x}{\mathrm{e}}\right)$ for x sufficiently large. First, we study its sharpness by giving full asymptotic expansions of its left and right hand sides expressions. Then, we discuss the structure of Ramanujan's inequality, by replacing the factor $\frac{x}{\log x}$ on its right hand side by the factor $\frac{x}{\log x - h}$ for a given h, and by replacing the numerical factor e by a given positive α. Finally, we introduce and study inequalities analogous to Ramanujan's inequality.

1 Introduction

Among several conjectures and results concerning the distribution of prime numbers, on page 310 of his second notebook [11], Ramanujan asserts that the inequality

$$\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right), \tag{1}$$

holds for x sufficiently large. Here, as usual, $\pi(x)$ denotes the number of primes not exceeding x. To confirm (1) for x sufficiently large, we note that the prime number theorem with error term gives the expansion

$$\pi(x) = x\sum_{k=0}^{n}\frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{2}$$

for any integer $n \geqslant 0$. Using (2) with $n = 4$ implies

$$\pi(x)^2 - \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right) = -\frac{x^2}{\log^6 x} + O\left(\frac{x^2}{\log^7 x}\right),$$

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and this proves (1) for x sufficiently large. We refer the interested reader to pages 111–137 of [2] and pages 22–47 of [5] for more details on Ramanujan's ideas concerning the analytic theory of primes.

The most important studies regarding Ramanujan's inequality (1) ask about the positive integer $x_{\mathcal{R}}$ for which (1) holds if $x \geqslant x_{\mathcal{R}}$ and fails for $x < x_{\mathcal{R}}$. In 2012 the author [7] approximated $x_{\mathcal{R}}$ under the assumption of the existence of some very good bounds for the function $\pi(x)$. In 2015 Dudek and Platt [3], based on the sharp bounds due to Trudgian, which appeared some months after their work in [12], obtained such very good bounds for $\pi(x)$ implying that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9658}$. Dudek and Platt note, on page 292, that using a result by Mossinghoff and Trudgian [8] they can prove that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9394}$. In 2018 Axler [1] proved that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9032}$. He also proved that (1) holds unconditionally for every x satisfying $38{,}358{,}837{,}683 \leqslant x \leqslant 10^{19}$.

Under the assumption that the Riemann hypothesis is true, the author [7] proved that $x_{\mathcal{R}} \leqslant 138{,}766{,}146{,}692{,}471{,}228$. Dudek and Platt [3] refined this conditional result by showing that $x_{\mathcal{R}} \leqslant 1.15 \times 10^{16}$. Moreover, they proved that assuming the Riemann hypothesis, the largest integer counterexample to Ramanujan's inequality (1) is at $x = 38{,}358{,}837{,}682$.

In 2013 the author [6] studied the following generalization of (1) for a given positive integer n,

$$\pi(x)^{2^n} < \frac{\mathrm{e}^n}{\prod_{k=1}^{n}\left(1-\frac{k-1}{\log x}\right)^{2^{n-k}}}\left(\frac{x}{\log x}\right)^{2^n-1}\pi\left(\frac{x}{\mathrm{e}^n}\right).$$

For $n = 1$ the above generalization coincides with (1).

In this paper we are motivated by some questions, including "How sharp is Ramanujan's inequality (1)?" and "How subtle is the form of Ramanujan's inequality (1)?". We will consider these questions in Sections 2 and 3, respectively. In Section 4 we consider some inequalities analogous to Ramanujan's inequality. We end the paper with a computational observation concerning the constant $x_{\mathcal{R}}$.

2 Sharpness of Ramanujan's inequality

To work on the sharpness of Ramanujan's inequality we give full asymptotic expansions of its left and right hand sides expressions, providing some very nice corollaries.

Theorem 1. *Let* $\ell_k = \sum_{j=0}^{k} j!(k-j)!$ *and* $r_k = \sum_{j=0}^{k} j!\binom{k}{j}$. *Then, for a given integer* $n \geqslant 0$ *we have*

$$\pi(x)^2 = x^2 \sum_{k=0}^{n} \frac{\ell_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right), \tag{3}$$

and

$$\frac{\mathrm{e}\,x}{\log x}\pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=0}^{n} \frac{r_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right). \tag{4}$$

As an immediate corollary, we obtain full asymptotic expansions of

$$\pi(x)^2 - \frac{\mathrm{e}\,x}{\log x}\pi\left(\frac{x}{\mathrm{e}}\right) \text{ as } x \to \infty.$$

Corollary 1. *For a given integer $n \geqslant 4$ we have*

$$\pi(x)^2 - \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=4}^{n} \frac{d_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right), \tag{5}$$

where $d_k = \ell_k - r_k = \sum_{j=0}^{k} j!\big((k-j)! - \binom{k}{j}\big)$.

Note that $d_0 = d_1 = d_2 = d_3 = 0$, and some more initial values of d_k are $d_4 = -1$, $d_5 = -14$, $d_6 = -145$, $d_7 = -1412$, $d_8 = -13985$ and so on. Since $d_k < 0$ for any $k \geqslant 4$ (see Remark 1 below) we obtain the following refinement of Ramanujan's inequality (1).

Corollary 2. *Let $m \geqslant 4$ be a fixed integer. Then, for x sufficiently large*

$$\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right) + x^2 \sum_{k=4}^{m} \frac{d_k}{\log^{k+2} x}.$$

Remark 1. The sequences $(\ell_n)_{n\geqslant 0}$ and $(r_n)_{n\geqslant 0}$ are known in the OEIS as A003149 and A000522, respectively. The recurrence $\ell_n = n! + \frac{n+1}{2}\ell_{n-1}$ holds for each $n \geqslant 1$. Thus, by induction we obtain

$$2\,n! \leqslant \ell_n \leqslant \frac{8}{3}\,n! \tag{6}$$

for each $n \geqslant 1$. Moreover, the above recurrence implies that $\ell_n \sim 2\,n!$ as $n \to \infty$. On the other hand, we observe that if $\mathrm{e}_n = \sum_{k=0}^{n} \frac{1}{k!}$ then

$$0 < \mathrm{e} - \mathrm{e}_n = \sum_{k=1}^{\infty} \frac{1}{(n+k)!} = \frac{1}{n!} \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{n+j} < \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n\,n!}.$$

Thus, for each $n \geqslant 1$ we obtain $r_n = n!\,\mathrm{e}_n = \lfloor \mathrm{e}\,n! \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Meanwhile, for $n \geqslant 4$ we have $n! \geqslant 24 > 1/(\mathrm{e} - \frac{8}{3})$. Thus, $\frac{8}{3}n! < \mathrm{e}\,n! - 1$, and considering the right hand side of (6) we obtain $\ell_n < r_n$ for each $n \geqslant 4$.

3 Structure of Ramanujan's inequality

To study the structure of Ramanujan's inequality (1), we make observations concluding that he was quite clever when he was creating his inequality on the prime counting function. Our method is mainly to use the expansion (2) with $n = 1$ and $n = 4$, and to simplify the expressions following methods introduced in the proof of Theorem 1.

The first observation concerns the factor $\frac{x}{\log x}$ on the right hand side of (1). This factor basically comes from the true size of $\pi(x)$, which is $\frac{x}{\log x}$. Although we know that $\frac{x}{\log x - 1}$ is a much closer approximation for $\pi(x)$, the following result asserts that the factor $\frac{x}{\log x}$ on the right hand side of (1) is the best choice among several approximations of the form $\frac{x}{\log x - h}$.

Theorem 2. *Let h be a given real number. If $h \geqslant 0$ then for x sufficiently large*

$$\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x - h}\,\pi\left(\frac{x}{\mathrm{e}}\right).$$

If $h < 0$ then the above inequality reverses.

Remark 2. More precisely, the expansion (2) with $n = 4$ gives

$$\pi(x)^2 = x^2\left(\frac{1}{\log^2 x} + \frac{2}{\log^3 x} + \frac{5}{\log^4 x} + \frac{16}{\log^5 x} + \frac{64}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right), \tag{7}$$

and

$$\frac{\mathrm{e}\,x}{\log x - h}\,\pi\left(\frac{x}{\mathrm{e}}\right) =$$
$$x^2\left(\frac{1}{\log^2 x} + \frac{2 + A_1(h)}{\log^3 x} + \frac{5 + B_1(h)}{\log^4 x} + \frac{16 + C_1(h)}{\log^5 x} + \frac{65 + D_1(h)}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

where

$$\begin{aligned} A_1(h) &= h, \\ B_1(h) &= h(h+2), \\ C_1(h) &= h(h^2 + 2h + 5), \\ D_1(h) &= h(h^3 + 2h^2 + 5h + 16). \end{aligned}$$

A comparison of the coefficients shows that $h = 0$ is the most critical choice, as in Ramanujan's inequality (1).

The second observation on the structure of Ramanujan's inequality (1) concerns the factor e. The following result is a formulation of Theorem 1.3 of [7]; however, in the present paper we give a much simpler proof.

Theorem 3. *If $\alpha \geqslant \mathrm{e}$ then for x sufficiently large*

$$\pi(x)^2 < \frac{\alpha\,x}{\log x}\,\pi\left(\frac{x}{\alpha}\right).$$

If $0 < \alpha < \mathrm{e}$ then the above inequality reverses.

Remark 3. Assume that $\alpha > 0$, and let $u = \log\alpha$. Then the expansion (2) with $n = 4$ gives

$$\frac{\alpha\,x}{\log x}\,\pi\left(\frac{x}{\alpha}\right) = x^2\left(\frac{1}{\log^2 x} + \frac{A_2(u)}{\log^3 x} + \frac{B_2(u)}{\log^4 x} + \frac{C_2(u)}{\log^5 x} + \frac{D_2(u)}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

where

$$\begin{aligned} A_2(u) &= u + 1, \\ B_2(u) &= u^2 + 2u + 2, \\ C_2(u) &= u^3 + 3u^2 + 6u + 6, \\ D_2(u) &= u^4 + 4u^3 + 12u^2 + 24u + 24. \end{aligned}$$

Note that for $\alpha = \mathrm{e}$ one has $A_2(1) = 2$, $B_2(1) = 5$, $C_2(1) = 16$ and $D_2(1) = 65$. Comparing these values and the coefficients in (7) shows that $\alpha = \mathrm{e}$ is the most critical choice, as in Ramanujan's inequality (1).

As another observation on the structure of Ramanujan's inequality, in (1) we replace x by $\mathrm{e}\,x$. Thus Ramanujan's inequality takes the following equivalent form:

$$\pi(\mathrm{e}\,x)^2 < \frac{\mathrm{e}^2\,x}{1+\log x}\,\pi(x). \tag{8}$$

The expansion (2) with $n = 4$ gives

$$\pi(\mathrm{e}\,x)^2 = \mathrm{e}^2\,x^2\left(\frac{1}{\log^2 x} + \frac{2}{\log^4 x} + \frac{4}{\log^5 x} + \frac{19}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

and

$$\frac{\mathrm{e}^2\,x}{1+\log x}\,\pi(x) = \mathrm{e}^2\,x^2\left(\frac{1}{\log^2 x} + \frac{2}{\log^4 x} + \frac{4}{\log^5 x} + \frac{20}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right).$$

Thus we obtain (8) for x sufficiently large. Here, the point is missing the factor $\frac{1}{\log^3 x}$ in both expansions; to see the reason, for a given h we have

$$\frac{\mathrm{e}^2\,x}{h+\log x}\,\pi(x) = \mathrm{e}^2\,x^2\left(\frac{1}{\log^2 x} + \frac{A_3(h)}{\log^3 x} + \frac{B_3(h)}{\log^4 x} + \frac{C_3(h)}{\log^5 x} + \frac{D_3(h)}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

where

$$\begin{aligned}
A_3(h) &= -h + 1,\\
B_3(h) &= h^2 - h + 2,\\
C_3(h) &= -h^3 + h^2 - 2h + 6,\\
D_3(h) &= h^4 - h^3 + 2h^2 - 6h + 24.
\end{aligned}$$

Comparing the coefficients shows that $h = 1$ is the most critical choice, as in (8). Moreover, we get the following result.

Theorem 4. *Let h be a given real number. If $h \leqslant 1$ then for x sufficiently large*

$$\pi(\mathrm{e}\,x)^2 < \frac{\mathrm{e}^2\,x}{h+\log x}\,\pi(x).$$

If $h > 1$ then the above inequality reverses.

4 Inequalities analogous to Ramanujan's inequality

Not only is beautiful and curious, Ramanujan's inequality compels us to improve our knowledge on the distribution of prime numbers. In search of analogues of Ramanujan's inequality, first we try to consider $\pi(x)^3$ instead of $\pi(x)^2$, and we compare it with expressions analogous to the right hand side of (1). Our observations end in the following double sided inequality.

Theorem 5. *For x sufficiently large*

$$\left(\frac{\sqrt{\mathrm{e}}\,x}{\log x}\right)^2 \pi\left(\frac{x}{\mathrm{e}}\right) < \pi(x)^3 < \frac{\mathrm{e}^2\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right)^2. \tag{9}$$

Finally, we propose the question of studying inequalities of Ramanujan type concerning other functions of prime numbers.

1. Ramanujan's inequality concerning the Chebyshev functions. Since $\psi(x) \sim x$ and $\theta(x) \sim x$ as $x \to \infty$, we may ask about the validity of the following inequalities for x sufficiently large:

$$\psi(x)^2 < \mathrm{e}\,x\,\psi\left(\frac{x}{\mathrm{e}}\right), \quad \text{and} \quad \theta(x)^2 < \mathrm{e}\,x\,\theta\left(\frac{x}{\mathrm{e}}\right). \tag{10}$$

2. Ramanujan's inequality concerning the n-th prime. Since $p_n \sim n\log n$ as $n \to \infty$, an analogue of Ramanujan's inequality concerning the n-th prime reads as

$$p_n^2 < \mathrm{e}\,n\log n\,p_{\lfloor \frac{n}{\mathrm{e}} \rfloor},$$

or equivalently, by replacing n by $\mathrm{e}\,n$, as

$$p_{\lfloor \mathrm{e}\,n \rfloor^2} < \mathrm{e}^2\,n\,(1+\log n)\,p_n.$$

3. Ramanujan's inequality concerning $\pi(x;q,a)$. For given positive integer q and integer a with $\gcd(a,q) = 1$ let $\pi(x;q,a)$ denote the number of primes p not exceeding x with $p \equiv a \pmod q$. The prime number theorem for arithmetic progressions asserts that $\pi(x;q,a) \sim \frac{1}{\varphi(q)}\frac{x}{\log x}$. Hence, analogous to Ramanujan's inequality concerning the number of primes in arithmetic progressions, we may ask about the validity of the following inequality for x sufficiently large:

$$\pi(x;q,a)^2 < \varphi(q)\,\frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}};q,a\right).$$

5 Proofs

Proof of Theorem 1. For a given integer $n \geqslant 0$, multiplying twice (2) gives (3). Also, (2) gives

$$\pi\left(\frac{x}{\mathrm{e}}\right) = \frac{x}{\mathrm{e}}\sum_{k=0}^{n}\frac{k!}{(\log x - 1)^{k+1}} + O\left(\frac{x}{\log^{n+2} x}\right).$$

Let $c_0 = 1$ and $c_m = \frac{1}{m!}\prod_{i=1}^{m}(k+i)$ for $m \geqslant 1$. By using the binomial expansion we get $(1-t)^{-(k+1)} = \sum_{m=0}^{n} c_m\,t^m + O(t^{n+1})$ as $t \to 0$. Hence

$$\begin{aligned}\sum_{k=0}^{n}\frac{k!}{(\log x - 1)^{k+1}} &= \sum_{k=0}^{n}\frac{k!}{\log^{k+1} x}\left(1 - \frac{1}{\log x}\right)^{-(k+1)} \\ &= \sum_{k=0}^{n}\frac{k!}{\log^{k+1} x}\sum_{m=0}^{n}\frac{c_m}{\log^m x} + O\left(\frac{1}{\log^{n+2} x}\right).\end{aligned}$$

Collecting the diagonal terms of the above double sum gives

$$\sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} \sum_{m=0}^{n} \frac{c_m}{\log^m x} = \sum_{k=0}^{n} \frac{r_k}{\log^{k+1} x} + O\left(\frac{1}{\log^{n+2} x}\right),$$

where $r_k = \sum_{j=0}^{k} j! c_{k-j}$. Note that $c_m = \binom{k+m}{m}$ for $m \geqslant 0$. Thus we obtain (4). □

Proof of Theorem 2. For a given h, by using (2) with $n = 1$ we obtain

$$\pi(x)^2 = x^2 \left(\frac{1}{\log^2 x} + \frac{2}{\log^3 x} + O\Big(\frac{1}{\log^4 x}\Big)\right),$$

and

$$\frac{\mathrm{e}\, x}{\log x - h}\, \pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \left(\frac{1}{\log^2 x} + \frac{2+h}{\log^3 x} + O\Big(\frac{1}{\log^4 x}\Big)\right).$$

Comparing the coefficients of the term $\frac{1}{\log^3 x}$ gives the result for $h \neq 0$. The case $h = 0$ is Ramanujan's inequality (1). □

Proof of Theorem 3. For a given $\alpha > 0$, by using (2) with $n = 1$ we obtain

$$\pi(x)^2 = x^2 \left(\frac{1}{\log^2 x} + \frac{2}{\log^3 x} + O\Big(\frac{1}{\log^4 x}\Big)\right),$$

and

$$\frac{\alpha\, x}{\log x}\, \pi\left(\frac{x}{\alpha}\right) = x^2 \left(\frac{1}{\log^2 x} + \frac{1+\log \alpha}{\log^3 x} + O\Big(\frac{1}{\log^4 x}\Big)\right).$$

Comparing the coefficients of the term $\frac{1}{\log^3 x}$ gives the result for $\alpha \neq \mathrm{e}$. The case $\alpha = \mathrm{e}$ is Ramanujan's inequality (1). □

Proof of Theorem 4. For $h \neq 1$ we consider the expansion (2) with $n = 1$, to get

$$\pi(\mathrm{e}\, x)^2 = \mathrm{e}^2 x^2 \left(\frac{1}{\log^2 x} + O\Big(\frac{1}{\log^4 x}\Big)\right),$$

and

$$\frac{\mathrm{e}^2 x}{h + \log x}\, \pi(x) = \mathrm{e}^2 x^2 \left(\frac{1}{\log^2 x} + \frac{-h+1}{\log^3 x} + O\Big(\frac{1}{\log^4 x}\Big)\right).$$

The case $h = 1$ has been considered in (8). □

Proof of Theorem 5. We use the expansion (2) with $n = 1$ to obtain

$$\pi(x)^3 = x^3 \left(\frac{1}{\log^3 x} + \frac{3}{\log^4 x} + O\Big(\frac{1}{\log^5 x}\Big)\right),$$

and similar expansions for the left and right hand sides of (9) with factors 2 and 4 instead of 3 for the term $\frac{1}{\log^4 x}$, respectively, hence concluding the proof. □

6 A Computational Remark

We conclude the paper with another observation, of computational nature, on Ramanujan's inequality. If we let

$$\mathcal{R}(x) := \frac{\pi(x)}{x},$$

then (1) is equivalent to

$$\mathcal{R}(x)^2 < \frac{1}{\log x}\,\mathcal{R}\left(\frac{x}{\mathrm{e}}\right). \tag{11}$$

The main point of this equivalent form is the missing term x in the expansion of both sides. Hence, in numerical investigations, by taking $z = \log x$ we arrive to equivalent inequalities only in terms of z, in logarithmic scale and much smaller to work with. To make this point clear, we rework $x_{\mathcal{R}}$ by using sharp bounds of Trudgian [12] for $\pi(x)$, who proved unconditionally

$$|\pi(x) - \mathrm{li}(x)| \leqslant x\,\mathcal{T}(x) \quad (\text{for } x \geqslant 229), \tag{12}$$

with

$$\mathcal{T}(x) = \frac{0.2795}{(\log x)^{\frac{3}{4}}} \exp\left(-\sqrt{\frac{\log x}{6.455}}\right).$$

Also, it is known [9] that for a given $n \geqslant 0$,

$$|\mathrm{li}(x) - x\,S_n(n)| \leqslant 3\sqrt{n}\,(n+1)!\,\frac{x}{\log^{n+2} x} \quad (\text{for } x > 1), \tag{13}$$

where

$$S_n(x) = \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x}.$$

By combining (12) and (13) we get $L_n(x) \leqslant \mathcal{R}(x) \leqslant U_n(x)$ for $x \geqslant 229$ with

$$L_n(x) = S_n(x) - \frac{3\sqrt{n}\,(n+1)!}{\log^{n+2} x} - \mathcal{T}(x),$$

and

$$U_n(x) = S_n(x) + \frac{3\sqrt{n}\,(n+1)!}{\log^{n+2} x} + \mathcal{T}(x).$$

Thus, (11) (and consequently Ramanujan's inequality (1)) holds if $U_n(x)^2 \log x < L_n(\frac{x}{\mathrm{e}})$ for some integer $n \geqslant 0$, and letting $z = \log x$, this is equivalent to $f_n(z) > 0$, with

$$f_n(z) = \sum_{k=0}^{n} \frac{k!}{(z-1)^{k+1}} - \frac{3\sqrt{n}\,(n+1)!}{(z-1)^{n+2}} - \frac{0.2795}{(z-1)^{\frac{3}{4}}} \exp\left(-\sqrt{\frac{z-1}{6.455}}\right)$$
$$- z\left(\sum_{k=0}^{n} \frac{k!}{z^{k+1}} + \frac{3\sqrt{n}\,(n+1)!}{z^{n+2}} + \frac{0.2795}{z^{\frac{3}{4}}} \exp\left(-\sqrt{\frac{z}{6.455}}\right)\right)^2.$$

Running some careful computations in Maple, taking $n = 5$, we observe that $f_n(z) > 0$ for $z \geqslant 9048.857$. Hence we obtain $x_{\mathcal{R}} \leqslant e^{9048.857}$, unconditionally. Regarding to this computational observation, we mention that Platt and Trudgian [10] proved that $x_{\mathcal{R}} \leqslant e^{4041}$. Also, they proved that Ramanujan's inequality (1) holds unconditionally for every x satisfying $38{,}358{,}837{,}683 \leqslant x \leqslant e^{56}$.

A recent conditional bound concerning $\pi(x)$ is due to Dusart [4], asserting that if the Riemann hypothesis is true then

$$|\pi(x) - \mathrm{li}(x)| \leqslant x\,\mathcal{D}(x) \quad (\text{for } x \geqslant 5639), \tag{14}$$

with

$$\mathcal{D}(x) = \frac{\log x - \log\log x}{8\pi\sqrt{x}}.$$

We combine (13) and (14) to get $\bar{L}_n(x) \leqslant \mathcal{R}(x) \leqslant \bar{U}_n(x)$ for $x \geqslant 5639$ under the assumption that the Riemann hypothesis is true, with

$$\bar{L}_n(x) = S_n(x) - \frac{3\sqrt{n}\,(n+1)!}{\log^{n+2} x} - \mathcal{D}(x),$$

and

$$\bar{U}_n(x) = S_n(x) + \frac{3\sqrt{n}\,(n+1)!}{\log^{n+2} x} + \mathcal{D}(x).$$

Thus, (11) and consequently (1) hold assuming the Riemann hypothesis is true, if $\bar{U}_n(x)^2 \log x < \bar{L}_n(\frac{x}{e})$ for some integer $n \geqslant 0$, and letting $z = \log x$, this is equivalent to $g_n(z) > 0$, with

$$g_n(z) = \sum_{k=0}^{n} \frac{k!}{(z-1)^{k+1}} - \frac{3\sqrt{n}\,(n+1)!}{(z-1)^{n+2}} - \frac{z-1-\log(z-1)}{8\pi e^{\frac{z-1}{2}}} - z\left(\sum_{k=0}^{n} \frac{k!}{z^{k+1}} + \frac{3\sqrt{n}\,(n+1)!}{z^{n+2}} + \frac{z-\log z}{8\pi e^{\frac{z}{2}}}\right)^2.$$

Computations on Maple with $n = 12$ show that $g_n(z) > 0$ for $z \geqslant 39.18$. Hence we obtain $x_{\mathcal{R}} \leqslant e^{39.18} \approxeq 1.037 \times 10^{17}$ under the assumption that the Riemann hypothesis is true. Note that Dudek and Platt [3] obtained $x_{\mathcal{R}} \leqslant 1.15 \times 10^{16}$ under the same condition.

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