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Gradedness of the set of rook placements in A_{n-1}

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Abstract. A rook placement is a subset of a root system consisting of positive roots with pairwise non-positive inner products. To each rook placement in a root system one can assign the coadjoint orbit of the Borel subgroup of a reductive algebraic group with this root system. Degenerations of such orbits induce a natural partial order on the set of rook placements. We study combinatorial structure of the set of rook placements in A_{n-1} with respect to a slightly different order and prove that this poset is graded.

1 Introduction

Denote by $G = \mathrm{GL}_n(\mathbb{C})$ the group of all invertible $n \times n$ matrices over the complex numbers. Let B be the Borel subgroup of G consisting of all invertible upper-triangular matrices, U be the unipotent radical of B (it consists of all upper-triangular matrices with 1's on the diagonal), and T be the subgroup of all invertible diagonal matrices (it is the maximal torus of G contained in B). Next, let $\mathfrak{b}$ and $\mathfrak{n}$ be the Lie algebras of B and U respectively.

Let Φ be the root system of G with respect to T, Φ^+ be the set of positive roots with respect to B, Δ be the set of simple roots, and W be the Weyl group of Φ (for basic facts on algebraic groups and root systems, see [3], [4] and [5]). The root system Φ is of type A_{n-1}; as usual, we identify the set of positive roots with the subset of the Euclidean space $\mathbb{R}^n$ of the form

$$A_{n-1}^+ = \{\epsilon_i - \epsilon_j,\ 1 \leqslant i < j \leqslant n\}.$$

Under this identification, Δ consists of the roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leqslant i \leqslant n-1$ ($\{\epsilon_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$).

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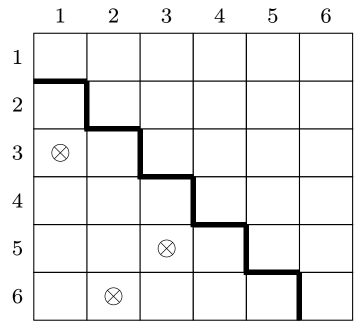
Definition 1. A *rook placement* is a subset $D \subseteq \Phi^+$ such that $(\alpha, \beta) \leqslant 0$ for all distinct α, $\beta \in D$. (Here $(\cdot, \cdot)$ denotes the standard inner product on $\mathbb{R}^n$.)

Example 1. Let $n = 6$. Below we draw the rook placement $D = \{\epsilon_1 - \epsilon_3,\ \epsilon_2 - \epsilon_6, \epsilon_3 - \epsilon_5\}$. If a root $\epsilon_i - \epsilon_j$ is contained in D, then we put the symbol $\otimes$ in the (j, i)th entry of the $n \times n$ chessboard. If we interpret these symbols as rooks, then it follows from the definition that the rooks do not hit each other.



We denote the set of all rook placement in A_{n-1} by $\mathcal{R}(n)$. Further, let $\mathcal{I}(n)$ be the set of all orthogonal rook placements. Below we describe two closely related partial orders on these sets.

The Lie algebra $\mathfrak{n}$ has the basis $\{e_\alpha,\ \alpha \in \Phi^+\}$ consisting of the root vectors: for $\alpha = \epsilon_i - \epsilon_j$, e_α is nothing but the elementary matrix $e_{i,j}$. Denote by $\{e^*_\alpha,\ \alpha \in \Phi^+\}$ the dual basis of the dual space $\mathfrak{n}^*$. Given a rook placement D, put

$$f_D = \sum_{\beta \in D} e^*_\beta \in \mathfrak{n}^*.$$

The group B acts on its Lie algebra $\mathfrak{b}$ by the adjoint action, and $\mathfrak{n}$ is an invariant subspace. Hence one has the dual action of the groups B and U on the space $\mathfrak{n}^*$; we call this action *coadjoint*. We say that the B-orbit $\Omega_D \subset \mathfrak{n}^*$ of the linear form f_D is *associated* with the rook placement D.

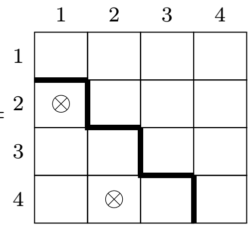
Such orbits play an important role in the A.A. Kirillov's orbit method [14], [15] describing representations of B and U. For $D \in \mathcal{I}(n)$, such orbits were studied by A.N. Panov in [18] and by me in [6]. One can define analogues of such orbits for other root systems, see [7], [8], [9] for the case of $\mathcal{I}(n)$. For arbitrary rook placements in $\mathcal{R}(n)$, such orbits were considered in [10]; see also [1], [2], where C. Andre and A. Neto used rook placements to construct so-called supercharacter theory for the group U. Note that in [16], [17], A. Melnikov studied the adjoint B-orbits of elements of the form $\sum_{\beta \in D} e_\beta$, $D \in \mathcal{I}(n)$.

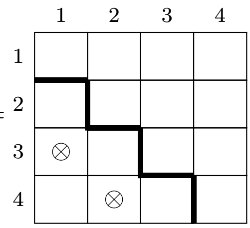
Given a subset $A \subseteq \mathfrak{n}^*$, we will denote by $\overline{A}$ its closure with respect to the Zarisski topology. There exists a natural partial order on the set $\mathcal{R}(n)$ induced by the degenerations of associated orbits: we will write $D_1 \leqslant_B D_2$ if $\Omega_{D_1} \subseteq \overline{\Omega}_{D_2}$. We need to introduce one more partial order on the set of rook placements. Namely, given an arbitrary $D \in \mathcal{R}(n)$, denote by R_D the $n \times n$ matrix defined by

$$(R_D)_{i,j} = \begin{cases} \#\{\epsilon_a - \epsilon_b \in D \mid a \leqslant j,\ b \geqslant i\}, & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

Put $D_1 \leqslant D_2$ if $(R_{D_1})_{i,j} \leqslant (R_{D_2})_{i,j}$ for all i, j.

Example 2. Let $n = 4$, $D_1 = \{\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_4\}$, $D_2 = \{\epsilon_1 - \epsilon_3,\ \epsilon_2 - \epsilon_4\}$. Then

$D_1 =$  , $\quad R_{D_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$

$D_2 =$  , $\quad R_{D_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

We conclude that $D_1 \leqslant D_2$. On the other hand, it is easy to check that $D_1 \nleqslant_B D_2$, see
[10, Remark 1.6 (iii)], so these two partial orders on $\mathcal{R}(n)$ do not coincide.

Nevertheless, it turns out that these orders are closely related to each other. Precisely, given rook placements D_1, $D_2 \in \mathcal{R}(n)$, it follows from $D_1 \leqslant_B D_2$ that $D_1 \leqslant D_2$ [10, Theorem 1.5]. Furthermore, if D_1, $D_2 \in \mathcal{I}(n)$ then the conditions $D_1 \leqslant_B D_2$ and $D_1 \leqslant D_2$ are equivalent [6, Theorem 1.7]. Besides, given a rook placement

$$D = \{\epsilon_{i_1} - \epsilon_{j_1},\ \ldots,\ \epsilon_{i_l} - \epsilon_{j_l}\},$$

we denote by $w_D \in S_n$ the permutation, which is equal to the product of transpositions

$$w_D = (i_1, j_1) \ldots (i_l, j_l).$$

Now, both of the conditions above (for orthogonal rook placements D_1, D_2) are equivalent to the condition that w_{D_1} is less or equal to w_{D_2} with respect to the Bruhat order [6, Theorem 1.1]. Similar facts are true for orthogonal rook placements in the root system C_n, see [7]. Note that these results are in some sense "dual" to A. Melnikov's results.

In the paper [12], F. Incitti studied the order on $\mathcal{I}(n)$ induced by the Bruhat order on the elements w_D, $D \in \mathcal{I}(n)$, from purely combinatorial point of view (see also [11] for other classical root systems). In particular, given an orthogonal rook placement D, he explicitly described the set of its immediate predecessors (it consists of $D' \in \mathcal{I}(n)$ such that there exists an edge from D' to D in the Hasse diagram of this poset). The set of immediate predecessors for the partial order $\leqslant$ on $\mathcal{I}(n)$ and $\mathcal{R}(n)$ was described by me in [6, Lemmas 3.6, 3.7, 3.8] and by A.S. Vasyukhin and me in [10, Theorem 3.3] respectively. (In the case of $\mathcal{I}(n)$, the set of immediate predecessors for $\leqslant$ coincides with the set described by F. Incitti, which implies that those two partial orders coincide.)

Furthermore, F. Incitti proved that the poset $\mathcal{I}(n)$ is graded and calculated its Möbius function. Recall that a finite poset X is called *graded* if it has the greatest and the lowest elements and all maximal chains in X have the same length. Gradedness is equivalent to the existence of a rank function. By definition, it is a (unique) function ρ on X, which value on the lowest element is zero, such that if x is an immediate predecessor of y then $\rho(y) = \rho(x) + 1$. In [12, Theorem 5.2], F. Incitti constructed the rank function on $\mathcal{I}(n)$. As the main result of this paper, we prove the gradedness of the poset $\mathcal{R}(n)$.

The main tool used in the proof is so-called Kerov placements (see [13]). To each rook placement $D \in \mathcal{R}(n)$ one can assign a certain orthogonal rook placement $K(D) \in \mathcal{I}(2n-2)$. We prove that if rook placements D_1 is an immediate predecessor of D_2 in $\mathcal{R}(n)$ then $K(D_1)$ is an immediate predecessor of $K(D_2)$ in $\mathcal{I}(2n-2)$ (and vice versa), see Theorem 3. As a corollary, we construct a rank function on $\mathcal{R}(n)$ and prove the gradedness of this poset, see Corollary 1.

The structure of the paper is as follows. In the next section we describe the set of immediate predecessors of a given rook placement for $\mathcal{I}(n)$ and $\mathcal{R}(n)$. In the third section we introduce the Kerov map

$$K\colon \mathcal{R}(n) \to \mathcal{I}(2n-2)$$

and show that it preserves the property "to be an immediate predecessor". This allows us to use F. Incitti's results to construct a rank function on $\mathcal{R}(n)$, which implies the gradedness of this poset.

2 Immediate predecessors

To prove that the set $\mathcal{R}(n)$ is graded with respect to the partial order introduced above, we need to describe the set of immediate predecessors of a given rook placement in $\mathcal{R}(n)$ and $\mathcal{I}(n)$. Such a description for $\mathcal{R}(n)$ was provided in [10], while for $\mathcal{I}(n)$ it was presented in F. Incitti's work [12]. Recall that a rook placement $D \in \mathcal{R}(n)$ is called an *immediate predecessor* of a rook placement $T \in \mathcal{R}(n)$ if $D < T$ and there are no $S \in \mathcal{R}(n)$ such that $D < S < T$. (As usual, $D < T$ means that $D \leqslant T$ and $D \neq T$.) In other words, there exists an oriented edge from D to T in the Hasse diagram of the poset $\mathcal{R}(n)$. The definition of immediate predecessors for $\mathcal{I}(n)$ is literally the same.

We denote the set of all immediate predecessors in $\mathcal{R}(n)$ (respectively, in $\mathcal{I}(n)$) of a rook placement $D \in \mathcal{R}(n)$ (respectively, of an orthogonal rook placement $D \in \mathcal{I}(n)$) by $L_{\mathcal{R}}(D)$ (respectively, by $L_{\mathcal{I}}(D)$). This set consists of rook placements of several types, which we will describe now. First, we will consider the set $L_{\mathcal{R}}(D)$ in details.

It is convenient to introduce the following notation. We will write simply (i, j) instead of $\epsilon_j - \epsilon_i$, $i > j$. Besides, for each k from 1 to n, we put

$$\mathcal{R}_k = \{(k, s) \in \Phi^+ \mid 1 \le s < k\},\ \mathcal{C}_k = \{(r, k) \in \Phi^+ \mid k < r \le n\}.$$

Definition 2. The sets $\mathcal{R}_k$, $\mathcal{C}_k$ are called the kth *row* and the kth *column* of Φ^+ respectively. We will write $\mathrm{row}(\alpha) = k$ and $\mathrm{col}(\alpha) = k$ if $\alpha \in \mathcal{R}_k$ and $\alpha \in \mathcal{C}_k$ respectively. Note that, for $D \in \mathcal{R}(n)$, one has

$$|D \cap \mathcal{R}_k| \le 1 \text{ and } |D \cap \mathcal{C}_k| \le 1 \text{ for all } 1 \le k \le n.$$

Furthermore, if $D \in \mathcal{I}(n)$ then

$$|D \cap (\mathcal{R}_k \cup \mathcal{C}_k)| \leq 1 \text{ for all } 1 \leq k \leq n.$$

There exists a natural partial order on the set of positive roots: given α, $\beta \in \Phi^+$, by definition, $\alpha \leq \beta$ if $\beta - \alpha$ is a (probably, empty) sum of positive roots. In the other words,

$$(a, b) \leq (c, d) \text{ if } c \geq a \text{ and } d \leq b.$$

Given a rook placement $D \in \mathcal{R}(n)$, denote by $\widetilde{M}(D)$ the set of minimal roots from D (with respect to $\leqslant$). Now, we set

$$\begin{aligned}
&M_{\mathcal{R}}(D) = \{(i, j) \in \widetilde{M}(D) \mid D \cap \mathcal{R}_k \neq \emptyset \text{ and } D \cap \mathcal{C}_k \neq \emptyset \text{ for all } j < k < i\},\\
&N_{\mathcal{R}}^{-}(D) = \{D_{(i,j)}^{-}, (i, j) \in M_{\mathcal{R}}(D)\},
\end{aligned}$$

where $D_{(i,j)}^{-} = D \setminus \{(i, j)\}$.

Next, fix a root $(i, j) \in D$. Denote

$$m = \min\{k \mid j < k < i \text{ and } D \cap \mathcal{C}_k = \emptyset\}.$$

Suppose that such a number m exists. Assume that $D \cap \mathcal{R}_k \neq \emptyset$ for all k from $j+1$ to m. Assume, in addition, that there are no $(p, q) \in D$ such that $(i, j) > (p, q)$ and $(i, m) \not> (p, q)$. The set of all roots $(i, j) \in D$ satisfying these conditions is denoted by $A_{\to}^{\mathcal{R}}(D)$; given $(i, j) \in A_{\to}^{\mathcal{R}}(D)$, we put

$$D_{(i,j)}^{\to,\mathcal{R}} = (D \setminus \{(i, j)\}) \cup \{(i, m)\}.$$

Similarly, suppose that there exists a number

$$m' = \max\{k \mid j < k < i \text{ and } D \cap \mathcal{R}_k = \emptyset\}.$$

Assume also that $D \cap \mathcal{C}_k \neq \emptyset$ for $m'+1 \leq k \leq i-1$ and that there are no $(p, q) \in D$ such that $(i, j) > (p, q)$ and $(m', j) \not> (p, q)$. Denote the set of all such (i, j)'s by $A_{\uparrow}^{\mathcal{R}}$; given $(i, j) \in A_{\uparrow}^{\mathcal{R}}$, we put

$$D_{(i,j)}^{\uparrow,\mathcal{R}} = (D \setminus \{(i, j)\}) \cup \{(m', j)\}.$$

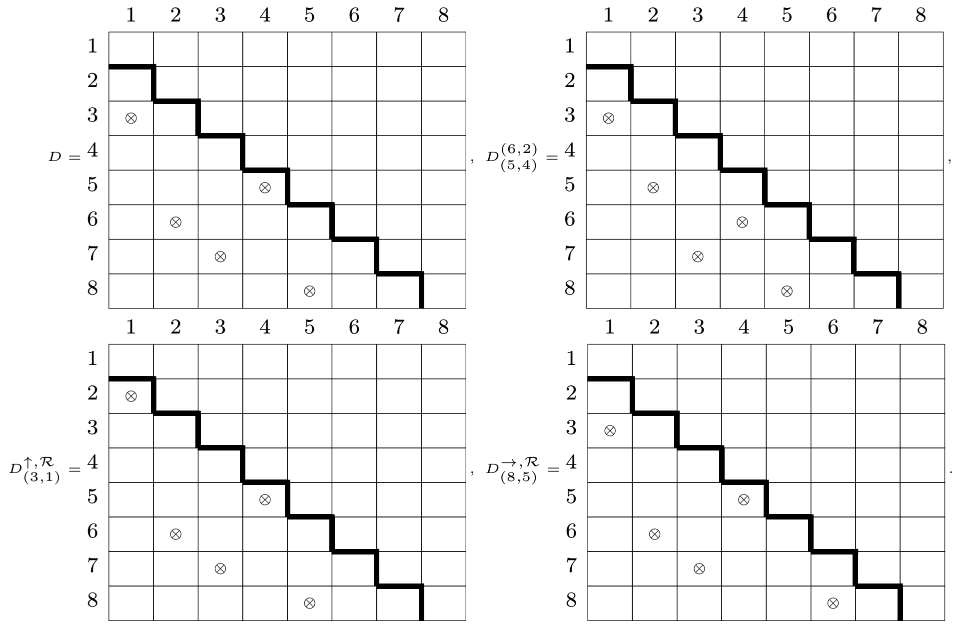
Now, let $B_{(i,j)}^{\mathcal{R}}(D)$ be the set of roots $(\alpha, \beta) \in D$ such that $(\alpha, \beta) > (i, j)$ and there are no $(p, q) \in D$ satisfying $(i, j) < (p, q) < (\alpha, \beta)$. For each $(\alpha, \beta) \in B_{(i,j)}^{\mathcal{R}}(D)$ we set

$$D_{(i,j)}^{(\alpha,\beta),\mathcal{R}} = (D \setminus \{(i, j), (\alpha, \beta)\}) \cup \{(i, \beta), (\alpha, j)\}.$$

By definition, let

$$\begin{aligned}
N_{\mathcal{R}}^{0}(D) = &\left\{D_{(i,j)}^{\uparrow,\mathcal{R}},\ (i, j) \in A_{\uparrow}^{\mathcal{R}}\right\} \cup \left\{D_{(i,j)}^{\to,\mathcal{R}},\ (i, j) \in A_{\to}^{\mathcal{R}}\right\}\\
&\cup \bigcup_{(i,j) \in D} \left\{D_{(i,j)}^{(\alpha,\beta),\mathcal{R}}, (\alpha, \beta) \in B_{(i,j)}^{\mathcal{R}}(D)\right\}.
\end{aligned}$$

Example 3. Let $n = 8$ and $D = \{(3,1),(6,2),(7,3),(5,4),(8,5)\}$. Clearly, $M_{\mathcal{R}}(D) = \{(5,4)\}$, $(8,5) \in A^{\mathcal{R}}_{\rightarrow}$, $(3,1) \in A^{\mathcal{R}}_{\uparrow}$ and $(6,2) \in B^{\mathcal{R}}_{(5,4)}(D)$. On the picture below we draw the rook placements D, $D^{(6,2),\mathcal{R}}_{(5,4)}$, $D^{\uparrow,\mathcal{R}}_{(3,1)}$ and $D^{\rightarrow,\mathcal{R}}_{(8,5)}$.



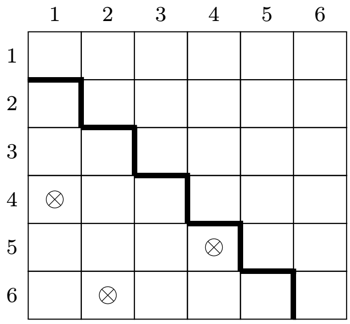
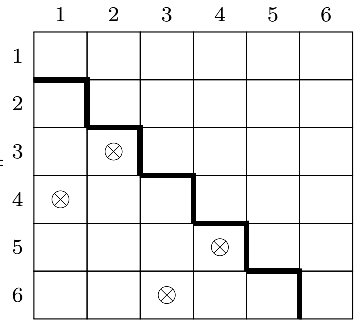
Next, fix a root $(i,j) \in D$, and consider a pair $(\alpha,\beta) \in \mathbb{Z} \times \mathbb{Z}$. Suppose that $i > \beta \geq \alpha > j$, $D \cap \mathcal{R}_{\alpha} = D \cap \mathcal{C}_{\beta} = \emptyset$, $D \cap \mathcal{R}_k \neq \emptyset$, $D \cap \mathcal{C}_k \neq \emptyset$ for all $\alpha < k < \beta$, and the conditions $(p,q) \in D$, $(i,j) > (p,q)$, $(\alpha,j) \ngtr (p,q)$ imply $(i,\beta) > (p,q)$. Moreover, assume that if $\alpha \neq \beta$ then $D \cap \mathcal{R}_{\beta} \neq \emptyset$ and $D \cap \mathcal{C}_{\alpha} \neq \emptyset$. Denote the set of all such pairs (α,β) by $C^{\mathcal{R}}_{(i,j)}(D)$. For an arbitrary pair $(\alpha,\beta) \in C^{\mathcal{R}}_{(i,j)}(D)$, we put

$$D^{\alpha,\beta,\mathcal{R}}_{(i,j)} = (D \setminus \{(i,j)\}) \cup \{(i,\beta),(\alpha,j)\}.$$

By definition, let

$$N^{+}_{\mathcal{R}}(D) = \bigcup_{(i,j)\in D} \left\{ D^{\alpha,\beta,\mathcal{R}}_{(i,j)}, (\alpha,\beta) \in C^{\mathcal{R}}_{(i,j)}(D) \right\}.$$

Example 4. Let $n = 6$ and $D = \{(4,1),(6,2),(5,4)\}$, then $(3,3) \in C^{\mathcal{R}}_{(6,2)}(D)$. On the picture below we draw the rook placements D and $D^{3,3,\mathcal{R}}_{(6,2)}$.

$D =$  , $D^{3,3}_{(6,2)} =$  .

Finally, we set

$$N_{\mathcal{R}}(D) = N^-_{\mathcal{R}}(D) \cup N^0_{\mathcal{R}}(D) \cup N^+_{\mathcal{R}}(D).$$

The set of immediate predecessors of a given rook placement from $\mathcal{R}(n)$ is described as follows.

Theorem 1 ([10, Theorem 3.3]). *Let $D \in \mathcal{R}(n)$. Then $L_{\mathcal{R}}(D) = N(D)$.*

Now we turn to the description of immediate predecessors for $\mathcal{I}(n)$. Given an orthogonal rook placement $D \in \mathcal{R}(n)$, put

$$\begin{aligned} M_{\mathcal{I}}(D) &= \{(i,j) \in \widetilde{M}(D) \mid D \cap (\mathcal{R}_k \cup \mathcal{C}_k) \neq \emptyset \text{ for all } j < k < i\}, \\ N^-_{\mathcal{I}}(D) &= \{D^-_{(i,j)}, (i,j) \in M_{\mathcal{I}}(D)\}, \end{aligned}$$

where $D^-_{(i,j)} = D \setminus \{(i,j)\}$, as above.

Let $D \in \mathcal{I}(n)$, $(i,j) \in D$. Denote

$$m = \min\{k \mid j < k < i \text{ and } D \cap \mathcal{C}_k = D \cap \mathcal{R}_k = \emptyset\}.$$

Suppose that such a number m exists. Assume that there are no $(p,q) \in D$ such that $(i,j) > (p,q)$ and $(i,m) \not> (p,q)$. The set of all $(i,j) \in D$ satisfying these conditions is denoted by $A^{\mathcal{I}}_{\to}(D)$; given $(i,j) \in A^{\mathcal{I}}_{\to}(D)$, we set

$$D^{\to,\mathcal{I}}_{(i,j)} = (D \setminus \{(i,j)\}) \cup \{(i,m)\}.$$

Similarly, suppose that there exists

$$m' = \max\{k \mid j < k < i \text{ and } D \cap \mathcal{R}_k = D \cap \mathcal{C}_k = \emptyset\},$$

and there are no $(p,q) \in D$ such that $(i,j) > (p,q)$ and $(m',j) \not> (p,q)$. The set of all such (i,j)'s is denoted by $A^{\mathcal{I}}_{\uparrow}$; given $(i,j) \in A^{\mathcal{I}}_{\uparrow}$, we set

$$D^{\uparrow,\mathcal{I}}_{(i,j)} = (D \setminus \{(i,j)\}) \cup \{(m',j)\}.$$

Next, let $B^{\mathcal{I}}_{(i,j)}(D)$ be the set of roots $(\alpha, \beta) \in D$ such that $j < \beta < i < \alpha$,

$$D \cap (\mathcal{R}_r \cup \mathcal{C}_r) \neq \emptyset$$

for all $\beta < r < i$ and there are no $(p, q) \in D$ for which $j < q < \beta < p < i$ or $\beta < q < i < p < \alpha$ (in other words, for which $(i, j) > (p, q)$ and $(\beta, j) \not> (p, q)$, or $(\alpha, \beta) > (p, q)$ and $(\alpha, i) \not> (p, q)$). To each $(\alpha, \beta) \in B^{\mathcal{I}}_{(i,j)}(D)$ we assign the set

$$D^{(\alpha,\beta),\mathcal{I}}_{(i,j)} = (D \setminus \{(i, j), (\alpha, \beta)\}) \cup \{(\beta, j), (\alpha, i)\}.$$

Now, let

$$N^0_{\mathcal{I}}(D) = \left\{D^{\uparrow,\mathcal{I}}_{(i,j)},\ (i,j) \in A^{\mathcal{I}}_{\uparrow}\right\} \cup \left\{D^{\rightarrow,\mathcal{I}}_{(i,j)},\ (i,j) \in A^{\mathcal{I}}_{\rightarrow}\right\}$$
$$\cup \bigcup_{(i,j)\in D} \left\{D^{(\alpha,\beta),\mathcal{R}}_{(i,j)}, (\alpha,\beta) \in B^{\mathcal{R}}_{(i,j)}(D)\right\} \cup \bigcup_{(i,j)\in D} \left\{D^{(\alpha,\beta),\mathcal{I}}_{(i,j)}, (\alpha,\beta) \in B^{\mathcal{I}}_{(i,j)}(D)\right\}.$$

Example 5. If $n = 8$, $D = \{(5, 1), (6, 2), (8, 4)\}$, then $(8, 4) \in B^{\mathcal{I}}_{6,2}(D)$, hence

$$D = \begin{array}{c|c|c|c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
1 & & & & & & & & \\ \hline
2 & & & & & & & & \\ \hline
3 & & & & & & & & \\ \hline
4 & & & & & & & & \\ \hline
5 & \otimes & & & & & & & \\ \hline
6 & & \otimes & & & & & & \\ \hline
7 & & & & & & & & \\ \hline
8 & & & & \otimes & & & & \\ \hline
\end{array}, \quad D^{(8,4),\mathcal{I}}_{(6,2)} = \begin{array}{c|c|c|c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
1 & & & & & & & & \\ \hline
2 & & & & & & & & \\ \hline
3 & & & & & & & & \\ \hline
4 & & \otimes & & & & & & \\ \hline
5 & \otimes & & & & & & & \\ \hline
6 & & & & & & & & \\ \hline
7 & & & & & & & & \\ \hline
8 & & & & & & \otimes & & \\ \hline
\end{array}.$$

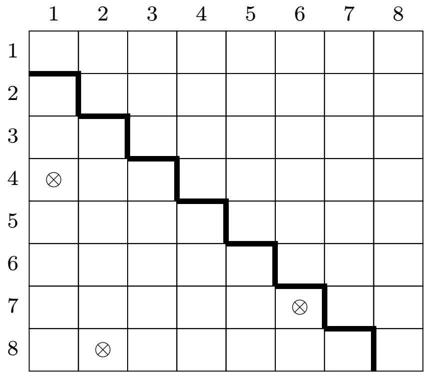
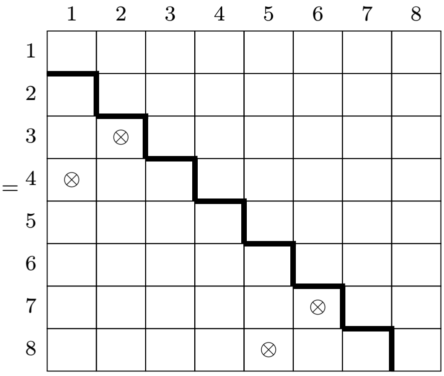
Besides, denote by $C^{\mathcal{I}}_{i,j}(D)$ the set of pairs $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$ such that $i > \beta > \alpha > j$,

$$D \cap (\mathcal{R}_\alpha \cup \mathcal{C}_\alpha) = D \cap (\mathcal{R}_\beta \cup \mathcal{C}_\beta) = \emptyset,$$

$D \cap (\mathcal{R}_k \cup \mathcal{C}_k) \neq \emptyset$ for all $\beta > k > \alpha$, and if $(p, q) \in D$, $(i, j) > (p, q)$, $(\alpha, j) \not> (p, q)$ then $(i, \beta) > (p, q)$. For each pair $(i, j) \in C^{\mathcal{I}}_{(i,j)}(D)$, we put

$$D^{\alpha,\beta,\mathcal{I}}_{(i,j)} = (D \setminus \{(i, j)\}) \cup \{(i, \beta), (\alpha, j)\}.$$

Example 6. Let $n = 8$, $D = \{(4,1), (8,2), (7,6)\}$, then $(3,5) \in C^{\mathcal{I}}_{(8,2)}(D)$, so

$D =$  , $D^{3,5,\mathcal{I}}_{(8,2)} =$  .

Finally, we denote

$$N^{+}_{\mathcal{I}}(D) = \bigcup_{(i,j)\in D} \left\{ D^{\alpha,\beta,\mathcal{I}}_{(i,j)}, (\alpha,\beta) \in C^{\mathcal{I}}_{(i,j)}(D) \right\},$$
$$N_{\mathcal{I}}(D) = N^{-}_{\mathcal{R}}(D) \cup N^{0}_{\mathcal{I}}(D) \cup N^{+}_{\mathcal{I}}(D).$$

Immediate predecessors in $\mathcal{I}(n)$ are described by the following F. Incitti's theorem (see also [6, Subsection 2.4]).

Theorem 2 ([12, Theorem 5.1]). *Let $D \in \mathcal{I}(n)$. Then $L_{\mathcal{I}}(D) = N_{\mathcal{I}}(D)$.*

3 Kerov map and the main result

In this section, we introduce our main technical tool, Kerov orthogonal rook placements, and, using them, prove that $\mathcal{R}(n)$ is graded.

Definition 3. Let $n \geq 3$, and D be a rook placement from $\mathcal{R}(n)$. A *Kerov rook placement* corresponding to D is, by definition, the orthogonal rook placement $K(D) \in \mathcal{I}(2n-2)$ constructed by the following rule: if

$$D = \{(i_1, j_1), \ldots, (i_s, j_s)\},$$

then

$$K(D) = (2i_1 - 2, 2j_1 - 1) \ldots (2i_s - 2, 2j_s - 1).$$

(Kerov rook placements were introduced in the paper [13]). We call the map $K\colon \mathcal{R}(n) \to \mathcal{I}(2n-2)$ given by the rule $D \mapsto K(D)$ the *Kerov map*.

Example 7. If $n = 8$ and $D = \{(3,1), (6,2), (7,3), (5,4), (8,6)\} \in \mathcal{R}(8)$, then

$$\begin{aligned} K(D) &= (4,1)\cdot(10,3)\cdot(12,5)\cdot(8,7)\cdot(14,11) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 4 & 2 & 10 & 1 & 12 & 6 & 8 & 7 & 9 & 3 & 14 & 5 & 13 & 11 \end{pmatrix} \in \mathcal{I}(14). \end{aligned}$$

The following proposition is evident.

Proposition 1. *Let D, $T \in \mathcal{R}(n)$. Then the conditions $T \leq D$ and $K(T) \leq K(D)$ are equivalent.*

The following theorem plays the crucial role in the proof of the main result.

Theorem 3. *Let D, $T \in \mathcal{R}(n)$ be rook placements. Then the conditions $T \in L_{\mathcal{R}}(D)$ and $K(T) \in L_{\mathcal{I}}(K(D))$ are equivalent.*

Proof. Clearly, $K(T) \in L_{\mathcal{I}}(D)$ implies $T \in L_{\mathcal{R}}(D)$. Indeed, since there are no orthogonal involutions from $\mathcal{I}(2n-2)$ between $K(T)$ and $K(D)$, we conclude that, in particular, there are no Kerov involutions between them. It remains to prove that the converse is also true.

Assume that $T \in L_{\mathcal{R}}(D)$. By Theorem 1, this is equivalent to

$$T \in N_{\mathcal{R}}(D) = N^{-}(D) \cup N^{0}_{\mathcal{R}}(D) \cup N^{+}_{\mathcal{R}}(D)\,.$$

We will consider these variants case-by-case.

First, suppose that $T \in N^{-}_{\mathcal{R}}(D)$. This means that $T = D^{-}_{(i,j)}$ for a certain root $(i,j) \in M(D)$. Automatically, $K(T) = K(D) \setminus \{(2i-2, 2j-1)\}$. It follows immediately from $(i,j) \in \widetilde{M}(D)$ that $(2i-2, 2j-1) \in \widetilde{M}(K(D))$. Since $(i,j) \in M(D)$, we see that $D \cap \mathcal{R}_k$ and $D \cap \mathcal{C}_k$ are nonempty if $i < k < j$. This shows that $K(D \cap \mathcal{R}_{2k-2})$ and $K(D) \cap \mathcal{C}_{2k-1}$ are nonempty for all such k. Thus,

$$(2i-2, 2j-1) \in M(K(D))\,,$$

i.e., $K(T) \in N^{-}_{\mathcal{I}}(K(D))$. By Theorem 2, $K(T) \in L_{\mathcal{I}}(K(D))$.

Next, assume that $T \in N^{0}_{\mathcal{R}}(D)$. If $T = D^{(\alpha,\beta),\mathcal{R}}_{(i,j)}$ for some $(i,j) \in D$, $(\alpha,\beta) \in \mathcal{B}^{\mathcal{R}}_{(i,j)}(D)$, then it is easy to see that

$$(2\alpha-2, 2\beta-1) \in \mathcal{B}^{\mathcal{R}}_{(2i-2,2j-1)}(K(D))$$

and

$$K(T) = K(D)^{(2\alpha-2,2\beta-1),\mathcal{R}}_{(2i-2,2j-1)} \in N^{0}_{\mathcal{R}}(K(D))\,,$$

hence

$$K(T) \in N^{0}_{\mathcal{I}}(D) \subset L_{\mathcal{I}}(K(D))\,.$$

Now consider the case when $T = D^{\to,\mathcal{R}}_{(i,j)}$ for some $(i,j) \in A^{\mathcal{R}}_{\to}$. (The case $T = D^{\uparrow,\mathcal{R}}_{(i,j)}$, $(i,j) \in A^{\mathcal{R}}_{\uparrow}$ can be considered similarly.) Let $T = (D \setminus \{(i,j)\}) \cup \{(i,m)\}$, then

$$K(T) = (K(D) \setminus \{(2i-2, 2j-1)\}) \cup \{(2i-2, 2m-1)\}\,.$$

Since there are no root in D which is less than (i,j) but not less than (i,m), we have a similar condition for $K(D)$. Since $D \cap \mathcal{C}_k \neq \emptyset$ forP. Heymans: Pfaffians and skew-symmetric matrices $j < k < m$, one has $K(D) \cap \mathcal{C}_{2k-1} \neq \emptyset$ for such k. On the other hand, $D \cap \mathcal{R}_k$ is nonempty for $j < k \leq m$, so $K(D) \cap \mathcal{R}_{2k-2}$ is also nonempty for such k. Thus, $K(D) \cap (\mathcal{R}_k \cup \mathcal{C}_k) \neq \emptyset$ for $2j-1 < k < 2m-1$, which means that $(2i-2, 2j-1) \in A^{\mathcal{I}}_{\to}$ and $K(T) = K(D)^{\to,\mathcal{I}}_{(2i-2,2j-1)}$. Hence, by Theorem 2, $K(T) \in L_{\mathcal{I}}(K(D))$, as required.

Finally, suppose that $T \in N^+_{\mathcal{R}}(D)$, i.e., $T = D^{\alpha,\beta,\mathcal{R}}_{(i,j)}$ for certain $(i,j) \in D$ and $(\alpha,\beta) \in C^{\mathcal{R}}_{(i,j)}(D)$. Since $i > \beta \geq \alpha > j$, we have

$$2i - 2 > 2\beta - 1 > 2\alpha - 2 > 2j - 1\,.$$

It follows from $D \cap \mathcal{R}_\alpha = D \cap \mathcal{C}_\beta = \emptyset$ that

$$K(D) \cap \mathcal{R}_{2\alpha-2} = K(D) \cap \mathcal{C}_{2\beta-1} = \emptyset\,.$$

Since $K(D)$ is a Kerov rook placement, the condition

$$K(D) \cap \mathcal{C}_{2\alpha-2} = K(D) \cap \mathcal{R}_{2\beta-1} = \emptyset$$

is satisfied automatically. If $\alpha = \beta$ then there is nothing to prove. If $\beta > \alpha$ then $D \cap \mathcal{R}_k \neq \emptyset$ and $D \cap \mathcal{C}_k \neq \emptyset$ for all k from $\alpha+1$ to $\beta-1$, hence $K(D) \cap \mathcal{R}_{2k-2} \neq \emptyset$ and $K(D) \cap \mathcal{C}_{2k-1} \neq \emptyset$ for all such k. Furthermore, $D \cap \mathcal{R}_\beta$ and $D \cap \mathcal{C}_\alpha$ are nonempty, which implies that $K(D) \cap \mathcal{R}_{2\beta-2}$ and $D \cap \mathcal{C}_{2\alpha-1}$ are also nonempty. Thus, we obtain $K(D) \cap (\mathcal{R}_k \cap \mathcal{C}_k) \neq \emptyset$ for all k from $2\alpha - 1$ to $2\beta - 2$, sa required. We conclude that $(2\alpha - 2, 2\beta - 1) \in C^{\mathcal{I}}_{(2i-2,2j-1)}(D)$ and $K(T) = K(D)^{2\alpha-2,2\beta-1,\mathcal{I}}_{(2i-2,2j-1)}$. Theorem 2 guarantees that $K(T) \in L_{\mathcal{I}}(K(D))$. The proof is complete.

□

Corollary 1. *For each $n \geqslant 2$ the poset $\mathcal{R}(n)$ is graded with the rank function*

$$\rho(D) = \frac{l(w_{K(D)}) + |D|}{2},$$

where $l(w)$ is the length of a permutation w in the corresponding symmetric group.

Proof. As we mentioned in the introduction, F. Incitti showed that the set $\mathcal{I}(2n-2)$ of orthogonal rook placements is graded. Precisely [11, Theorem 5.3.2], the rank function on this poset has the form

$$\rho(D) = \frac{l(w_D) + |D|}{2}.$$

Applying Theorem 3, we see that the restriction of this rank function to $K(\mathcal{R}(n))$ in fact provided the rank function of the required form on $\mathcal{R}(n)$. This concludes the proof. □

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