The image of multilinear polynomials evaluated on 3×3 upper triangular matrices

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Abstract. We describe the images of multilinear polynomials of arbitrary degree evaluated on the 3×3 upper triangular matrix algebra over an infinite field.

1 Introduction

A famous open problem known as Lvov-Kaplansky's conjecture asserts: the image of a multilinear polynomial in noncommutative variables on the matrix algebra $M_n(K)$ over a field K is a vector space [5].

A major breakthrough in this direction was made by Kanel-Belov, Malev and Rowen [6], [9], with the solution of the conjecture for $n = 2$. Also for 3×3 matrices the authors obtained significant results [7], but the complete problem for matrices of order ≥ 3 is still open. For the state-of-art of the Lvov-Kaplansky conjecture see [8].

This conjecture motivated many different studies related to images of polynomials. For instance, papers on images of Lie, and Jordan polynomials, and also for other algebras have been published since then. The particular case of $n \times n$ upper triangular matrices over a field K, $UT_n(K)$ (or simply UT_n), was studied by the author and Fagundes in [4], where the Lvov-Kaplansky's conjecture for $UT_n(K)$ was proved to be equivalent to the following

Conjecture 1. The image of a multilinear polynomial on $UT_n(K)$ is either $\{0\}$, UT_n or J^k for some integer $k \geq 1$, where $J = J(UT_n)$ is the Jacobson radical of UT_n, i.e., the set of strictly upper triangular matrices.

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We remark that in the papers [3], [4] J^k was denoted by $UT_n^{(k-1)}$. This is the set of matrices whose entries (i,j) are zero if $i \geq j + k$, for $k \geq 1$.

A general result in the direction was given in [3], where the author proves the conjecture for polynomials of arbitrary degree evaluated on strictly upper triangular matrices.

In [4], the authors prove the conjecture for polynomials of degree up to 4 over fields of zero characteristic.

For polynomials of arbitrary degree, the problem was solved in the case $n = 2$ by Fagundes in his master's degree dissertation [2], but the text was written in Portuguese and the result was not published elsewhere. Some time latter, in [10] (see also [11] for a correction of the paper), Wang gives another proof of this result.

In this short note, we present Fagundes' proof of the above result, since it is extremely simpler then Wang's proof, and we prove the above conjecture for $n = 3$.

The techniques here used are based on the paper [4] and on the theory of algebras with polynomial identities (PI-algebras). We assume the reader has the basic knowledge of this subject. For more information and notation, see the book [1]. We believe the technique used in this paper may be adapted to solve the general problem for UT_n. This will be considered in a future project.

2   Preliminary Results

In this section we recall some results on the theory of PI-algebras and results on images of multilinear polynomials given in [4]. We denote by $K\langle X \rangle$ the free associative algebra freely generated by the set X, i.e., the algebra of noncommutative polynomials in the variables of X, and by P_n the set of multilinear polynomials of degree n in $K\langle X \rangle$. If $S \subseteq K\langle X \rangle$, we denote by $\langle S \rangle^T$, the T-ideal generated by S.

Theorem 1 (Theorem 5.2.1 of [1]). *Let K be an infinite field and let $UT_n(K)$ be the algebra of $n \times n$ upper triangular matrices over K.*

1. *The polynomial identity*

$$[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] = 0$$

   forms a basis of the polynomial identities of $UT_n(K)$.

2. *The relatively free algebra $F(UT_n(K))$ has a basis consisting of all products*

$$x_1^{a_1} \ldots x_m^{a_m} [x_{i_{11}}, x_{i_{21}}, \ldots, x_{i_{p_1 1}}] \ldots [x_{i_{1r}}, x_{i_{2r}}, \ldots, x_{i_{p_r r}}],$$

   where the number r of participating commutators is $\leq n - 1$ and the indexes in each commutator $[x_{i_{1s}}, x_{i_{2s}}, \ldots, x_{i_{p_s s}}]$ satisfy $i_{1s} > i_{2s} \leq \cdots \leq i_{p_s s}$.

The following is an improvement of [4, Proposition 8] and is easy to prove.

Lemma 1. *Let $f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$. Then*

1. *If $\sum_{\sigma \in S_n} \alpha_\sigma \neq 0$, then the image of f on a unitary algebra A is A.*

2. *$f \in \langle [x_1, x_2] \rangle^T$ if and only if $\sum_{\sigma \in S_n} \alpha_\sigma = 0$.*

Lemma 2 (Lemma 10 of [4]). *Let K be a field with at least n elements and let $d_1, \ldots, d_n \in K$ be distinct elements. Then for $D = \mathrm{diag}(d_1, \ldots, d_n)$ and $k \geq 0$, we have*

$$[J^k, D] = J^k \text{ and } [UT_n, D] = J.$$

3   The case n=2

In this section we give Fagundes' proof [2, Proposition 4.7] of the case $n = 2$ of the above conjecture. This is a simple application of Lemma 1.

Proposition 1. *Let K be a field. The image of a multilinear polynomial on UT_2 is UT_2, J, or $\{0\}$.*

Proof. Let $f(x_1, \ldots, x_m) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be a multilinear polynomial. If $\sum_{\sigma \in S_n} \alpha_\sigma \neq 0$, then the image of f on UT_2 is UT_2. Suppose $\sum_{\sigma \in S_n} \alpha_\sigma = 0$. Then $Im(f) \subseteq J$. Now if f is a polynomial identity, the image of f is $\{0\}$. Otherwise, since J is a one-dimensional subspace of UT_2, and the image of f is closed under scalar multiplication, the image of f is J.   □

4   The case n=3

Theorem 2. *Let K be an infinite field and $f \in P_n$. The image of f in UT_3 is either UT_3, J, J^2 or $\{0\}$.*

Proof. Write $f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$. If f is a polynomial identity for UT_3, then the image of f is $\{0\}$. So from now on we suppose f is not an identity for UT_3.

If $\sum_{\sigma \in S_n} a_\sigma \neq 0$, then item 1 of the Lemma 1 asserts that the image of f is UT_3.

So let us suppose that $\sum_{\sigma \in S_n} a_\sigma = 0$. Item 2 of Lemma 1 yields $f \in \langle [x_1, x_2] \rangle^T$. As a consequence, the image of f is a subset of J.

By Theorem 1, up to an identity of UT_3, f can be written as a linear combination of the following two families of polynomials:

1. $x_{i_1} \cdots x_{i_k}[x_{i_{k+1}}, \ldots, x_{i_{k+t}}]$, with $k + t = n$, $i_1 < i_2 < \cdots < i_k$ and $i_{k+1} > i_{k+2} < i_{k+3} < \cdots < i_{k+t}$

2. $x_{i_1} \cdots x_{i_k}[x_{i_{k+1}}, \ldots, x_{i_{k+t}}][x_{i_{k+t+1}}, \ldots, x_{i_{k+t+s}}]$, with $k + t + s = n$, $i_1 < i_2 < \cdots < i_k$, $i_{k+1} > i_{k+2} < i_{k+3} < \cdots < i_{k+t}$ and $i_{k+t+1} > i_{k+t+2} < i_{k+t+3} < \cdots < i_{k+t+s}$.

Let us first assume that $f \notin \langle [x_1, x_2][x_3, x_4] \rangle^T$. So there exists at least one polynomial of the first family of polynomials above with nonzero coefficient in the decomposition of f as a linear combination of such polynomials.

Let us consider one such polynomial with t minimal. Now perform the substitution $x_{i_j} \mapsto I$, for $j = 1, \ldots, k$. Here I stands for the 3×3 identity matrix. Then, after this substitution, the polynomial f is a nonzero linear combination of polynomials $[x_{j_1}, \ldots, x_{j_t}]$ with $j_1 > j_2 < j_3 < \cdots < j_t$ and some polynomials in $\langle [x_1, x_2][x_3, x_4] \rangle^T$.

Now let d_1, d_2, d_3 be distinct elements in K and $D = \text{diag}(d_1, d_2, d_3)$. By Lemma 2 $[UT_3, D] = J$ and $[J, D] = J$. As a consequence, $[UT_n, D, \ldots, D] = J$. After substituting all variables but one by D, the polynomials in $\langle [x_1, x_2][x_3, x_4] \rangle^T$ vanish. All the other polynomials but one also vanish, and we obtain that the image of f is J.

To finish the proof, we assume $f \in \langle [x_1, x_2][x_3, x_4] \rangle^T$. Of course, the image of f is a subset of J^2, which is 1-dimensional. Since f is not a polynomial identity for UT_3, its image is J^2 and the theorem is proved.

$\square$

References

[1] V. Drensky: *Free Algebras and PI-Algebras. Graduate course in algebra.* Springer-Verlag (2000).

[2] P. S. Fagundes: Imagens De Polinômios Multilineares Sobre Algumas Subálgebras de Matrizes. Dissertação de Mestrado em Matemática Aplicada, Universidade Federal de São Paulo (2019).

[3] P. S. Fagundes: The images of multilinear polynomials on strictly upper triangular matrices. *Linear Algebra Appl.* 563 (2019) 287–301.

[4] P.S. Fagundes, T.C. de Mello: Images of multilinear polynomials of degree up to four on upper triangular matrices. *Oper. Matrices* 13 (1) (2019) 283–292.

[5] V.T. Filippov, V.K. Kharchenko, I.P. Shestakov: Dniester notebook: unsolved problems in the theory of rings and modules. In: L. Sabinin, L. Sbitneva, I. Shestakov: *Non-associative algebra and its applications, Lect. Notes Pure Appl. Math. 246.* Chapman & Hall/CRC (2006) 461–516. Translated from the 1993 Russian edition by Murray R. Bremner and Mikhail V. Kochetov

[6] A. Kanel-Belov, S. Malev, L. Rowen: The images of non-commutative polynomials evaluated on 2×2 matrices. *Proc. Amer. Math. Soc.* 140 (2012) 465–478.

[7] A. Kanel-Belov, S. Malev, L. Rowen: The images of non-commutative polynomials evaluated on 3×3 matrices. *Proc. Amer. Math. Soc.* 144 (2016) 7–19.

[8] A. Kanel-Belov, S. Malev, L. Rowen: Evaluations of Noncommutative Polynomials on Finite Dimensional Algebras. L'vov-Kaplansky Conjecture. *SIGMA* 16 (2020) 61 pp.

[9] A. Malev: The image of non-commutative polynomials evaluated on 2×2 matrices over an arbitrary field. *J. Algebra Appl.* 13 (6) (2014) 12 pp.

[10] Y. Wang: The images of multilinear polynomials on 2×2 upper triangular matrix algebras. *Linear and Multilinear Algebra* 67 (11) (2019) 2366–2372.

[11] Y. Wang, P. Liu, J. Bai: The images of multilinear polynomials on 2×2 upper triangular matrix algebras (Correction). *Linear and Multilinear Algebra* 67 (11) (2019) i–vi.