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Certain partitions on a set and their applications to different classes of graded algebras

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Abstract. Let $(\mathfrak{A}, \epsilon_u)$ and $(\mathfrak{B}, \epsilon_b)$ be two pointed sets. Given a family of three maps $\mathcal{F} = \{f_1 \colon \mathfrak{A} \to \mathfrak{A}; f_2 \colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}; f_3 \colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B}\}$, this family provides an adequate decomposition of $\mathfrak{A} \setminus \{\epsilon_u\}$ as the orthogonal disjoint union of well-described $\mathcal{F}$-invariant subsets. This decomposition is applied to the structure theory of graded involutive algebras, graded quadratic algebras and graded weak H^*-algebras.

1 Introduction and previous definitions

Let $(\mathfrak{A}, \epsilon_u)$ and $(\mathfrak{B}, \epsilon_b)$ be two *pointed sets* (that is, ϵ_u and ϵ_b are distinguish elements of $\mathfrak{A}$ and $\mathfrak{B}$ respectively), and consider a family of three maps

$$\mathcal{F} = \{f_1 \colon \mathfrak{A} \to \mathfrak{A}; f_2 \colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}; f_3 \colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B}\}.$$

In Section 2 we will introduce connections techniques on the pair $(\mathfrak{A}, \mathcal{F})$ to get a decomposition of $\mathfrak{A} \setminus \{\epsilon_u\}$ as the disjoint union of well-described subsets

$$\mathfrak{A} \setminus \{\epsilon_u\} = \dot{\cup}_{i \in I} \mathfrak{A}_i .$$

This decomposition will be $\mathcal{F}$*-invariant* in the sense that

$$f_1(\mathfrak{A}_i) \subset \mathfrak{A}_i \cup \{\epsilon_u\} \text{ and } f_2(\mathfrak{A}_i, \mathfrak{A}) \cup f_2(\mathfrak{A}, \mathfrak{A}_i) \subset \mathfrak{A}_i \cup \{\epsilon_u\}$$

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for any $i \in I$, and *orthogonal* in the sense that

$$f_3(\mathfrak{A}_i, \mathfrak{A}_j) = \{\epsilon_b\}$$

for any $i, j \in I$ with $i \neq j$.

We recall that an *algebra* A is a linear space, over a base field $\mathbb{K}$, endowed with a bilinear map (the product)

$$\begin{aligned} A \times A &\to A \\ (x, y) &\mapsto xy\,. \end{aligned}$$

An *ideal* I of A is a linear subspace satisfying $AI + IA \subset I$.

An *involution* on an algebra A is a linear map

$$\begin{aligned} *\colon A &\to A \\ x &\mapsto x^* \end{aligned}$$

such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for any $x, y \in A$.

Definition 1. An algebra A endowed with an involution $*\colon A \to A$ is called an *involutive algebra*. An *ideal* of an involutive algebra A is a linear subspace I satisfying $AI + IA \subset I$ and $I^* \subset I$.

Remark 1. We note that given an algebra A over a base field $(\mathbb{K}, -)$, where $\lambda \mapsto \bar{\lambda}$ is an involution on $\mathbb{K}$, endowed with a mapping $*\colon A \to A$ satisfying $(x+y)^* = x^*+y^*$, $(\lambda x)^* = \bar{\lambda}x^*$, $(x^*)^* = x$ and $(xy)^* = y^*x^*$, for any $x, y \in A$ and $\lambda \in \mathbb{K}$, is called an *algebra with an involution of second type*. We note that the results in the present paper also hold for an algebra with an involution of second type (following the same arguments).

In Lie algebras theory we find several concepts dealing with a Lie algebra endowed with a bilinear form. For instance, a *quadratic Lie algebra* is a Lie algebra $(L, [\cdot,\cdot])$ with a non-degenerate skew-symmetric bilinear form $\omega\colon L \times L \to \mathbb{K}$ satisfying the equality

$$\omega([x, y], z) + \omega(y, [x, z]) = 0$$

for any $x, y, z \in L$ (see for instance [2], [3]).

A *weak Lie H^*-algebra* is a Lie algebra $(L, [\cdot,\cdot])$ endowed with a non-degenerate symmetric bilinear form $\omega\colon L \times L \to \mathbb{K}$ and an involution $*\colon L \to L$ in such a way that the following identity holds

$$\omega([x, y], z) = \omega(x, [z, y^*])$$

for any $x, y, z \in L$ (see for instance [1], [4], [5]).

The above concepts can be extended to arbitrary algebras in a natural way.

Definition 2. A *quadratic algebra* is an algebra A endowed with a non-degenerate skew-symmetric bilinear form $\omega\colon A \times A \to \mathbb{K}$ such that

$$\omega(xy, z) = -\omega(y, xz) = -\omega(x, zy)$$

for any $x, y, z \in A$.

Definition 3. A *weak H^*-algebra* is an algebra A endowed with a non-degenerate symmetric bilinear form $\omega\colon A\times A\to\mathbb{K}$ and an involution $*\colon A\to A$ in such a way that the identity

$$\omega(xy,z)=\omega(x,zy^*)=\omega(y,x^*z)$$

holds for any $x,y,z\in A$. An *ideal* of a weak H^*-algebra A is a linear subspace I satisfying $AI+IA\subset I$ and $I^*\subset I$.

Two ideals I and J of a quadratic algebra or a weak H^*-algebra are said to be *orthogonal* if $\omega(I,J)=0$.

Finally, we recall that an algebra A is said to be *graded* by means of a nonempty set $\mathfrak{A}$ if it decomposes as the direct sum of linear subspaces

$$A=\bigoplus_{i\in\mathfrak{A}}A_i$$

in such a way that for any $i,j\in\mathfrak{A}$ there exists $k\in\mathfrak{A}$ such that

$$A_iA_j\subset A_k\,.$$

As references of graded algebras we can mention ([6], [7], [8]).

In case A is furthermore a quadratic algebra we will say that A is a *graded quadratic algebra*.

In case a graded algebra $A=\bigoplus_{i\in\mathfrak{A}}A_i$ is also an involutive algebra or a weak H^*-algebra we will say that A is a *graded involutive algebra* (resp. *graded weak H^*-algebra*) if for any $i\in\mathfrak{A}$ there exists $j\in\mathfrak{A}$ such that $(A_i)^*\subset A_j$.

In Sections 3, 4 and 5 we will apply the main result in Section 2 to the grading set $\mathfrak{A}$ of a graded involutive algebra, a graded quadratic algebra and a graded weak H^*-algebra $A=\bigoplus_{i\in\mathfrak{A}}A_i$ respectively, so as to provide a decomposition of A as the orthogonal direct sum of graded ideals.

2 Connections in $(\mathfrak{A},\mathcal{F})$ techniques

Let $(\mathfrak{A},\epsilon_u)$ and $(\mathfrak{B},\epsilon_b)$ be two pointed sets. Let us consider a family of three maps

$$\mathcal{F}=\{f_1\colon\mathfrak{A}\to\mathfrak{A};f_2\colon\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A};f_3\colon\mathfrak{A}\times\mathfrak{A}\to\mathfrak{B}\}\,.$$

We will denote by $\mathcal{P}(\mathfrak{A})$ the power set of $\mathfrak{A}$.

For each $i\in\mathfrak{A}$, a new variable $\bar{i}\notin\mathfrak{A}$ is introduced and we denote by

$$\overline{\mathfrak{A}}:=\{\bar{i}:i\in\mathfrak{A}\}$$

the set of all these new symbols. Given any $\bar{i}\in\overline{\mathfrak{A}}$ we denote $\overline{(\bar{i})}:=i$.

Now we introduce the next two operations:

$$*_2\colon\mathfrak{A}\times(\mathfrak{A}\cup\bar{\mathfrak{A}})\to\mathcal{P}(\mathfrak{A})$$

given by

$$*_2(i,j):=\{f_2(i,j),f_2(j,i)\}\setminus\{\epsilon_u\}$$

$$*_2(i,\bar{j}) := \{k \in \mathfrak{A} \setminus \{\epsilon_u\} : f_2(k,j) = i\} \cup \{k \in \mathfrak{A} \setminus \{\epsilon_u\} : f_2(j,k) = i\}$$

for any $i, j \in I$.

$$*_3 \colon \mathfrak{A} \times (\mathfrak{A} \cup \bar{\mathfrak{A}}) \to \mathcal{P}(\mathfrak{A})$$

given by

$$*_3(i,j) := \begin{cases} \{i,j\}, & \text{if } f_3(i,j) \neq \epsilon_b \text{ or } f_3(j,i) \neq \epsilon_b; \\ \emptyset, & \text{if } f_3(i,j) = f_3(j,i) = \epsilon_b. \end{cases}$$

$$*_3(i,\bar{j}) := \begin{cases} \{k \in \mathfrak{A} \setminus \{\epsilon_u\} : f_3(i,k) \neq \epsilon_b\} \\ \qquad\qquad \cup \{k \in \mathfrak{A} \setminus \{\epsilon_u\} : f_3(k,i) \neq \epsilon_b\}, & \text{if } i = j \ ; \\ *_3(i,j), & \text{if } i \neq j. \end{cases}$$

for any $i, j \in I$.

We fix now an external element $\nu \notin \mathfrak{A}$ such that $\bar{\nu} \notin \mathfrak{A}$ and denote $\overline{(\bar{\nu})} := \nu$.

Then, we consider the mapping

$$\circ \colon \mathcal{P}(\mathfrak{A}) \times (\mathfrak{A} \cup \bar{\mathfrak{A}} \cup \{\nu\} \cup \{\bar{\nu}\}) \to \mathcal{P}(\mathfrak{A}),$$

defined as

$$\circ(\emptyset, \mathfrak{A} \cup \bar{\mathfrak{A}} \cup \{\nu\} \cup \{\bar{\nu}\}) := \emptyset,$$

$$\circ(J,k) := \bigcup_{j \in J} (*_2(j,k) \cup *_3(j,k))$$

$$\circ(J,\nu) := f_1(J)$$

and

$$\circ(J,\bar{\nu}) := f_1^{-1}(J)$$

for any $\emptyset \neq J \in P(\mathfrak{A})$ and $k \in \mathfrak{A} \cup \bar{\mathfrak{A}}$.

Lemma 1. *The following assertions hold:*

(i) Let $i, j \in \mathfrak{A}$ and $k \in \mathfrak{A} \cup \overline{\mathfrak{A}}$. Then we have that

$$i \in *_2(j,k) \text{ if and only if } j \in *_2(i,\bar{k}).$$

(ii) Let $i, j \in \mathfrak{A}$ and $k \in \mathfrak{A} \cup \overline{\mathfrak{A}}$. Then we have that

$$i \in *_3(j,k) \text{ if and only if } j \in *_3(i,\bar{k}).$$

(iii) Let $J \in \mathcal{P}(\mathfrak{A})$ and $k \in \mathfrak{A} \cup \overline{\mathfrak{A}}$. Then

$$j \in \circ(J,k) \text{ if and only if } \circ(\{j\},\bar{k}) \cap J \neq \emptyset.$$

Proof. (i) Let $i, j \in I$ and $k \in \mathfrak{A} \cup \overline{\mathfrak{A}}$ such that

$$i \in *_2(j,k).$$

We have two cases to consider. In the first one $k \in \mathfrak{A}$. Then

$$i \in \{f_2(j,k), f_2(k,j)\}.$$

Hence $j \in *_2(i,\bar{k})$. In the second case $k \in \overline{\mathfrak{A}}$. Then $j \in \{f_2(i,\bar{k}), f_2(\bar{k},i)\}$ and we get $j \in *_2(i,\bar{k})$. We have already proved both directions, just replace i and j.

(ii) Take $i,j \in I$ and $k \in \mathfrak{A} \cup \overline{\mathfrak{A}}$ such that

$$i \in *_3(j,k).$$

We will also distinguish two cases. First $k \in \mathfrak{A}$. Then necessarily $f_3(j,k) \neq \epsilon_b$ or $f_3(k,j) \neq \epsilon_b$, being $i \in \{j,k\}$. Form here, either $i = j$ and so $j \in *_3(i,\bar{i}) = *_3(i,\bar{k})$ when $k = i$ and $j \in *_3(i,\bar{k})$ when $k \neq i$; or $i = k$, being then $j \in *_3(i,\bar{i}) = *_3(i,\bar{k})$. Second, $k \in \overline{\mathfrak{A}}$. In this case we will consider two possibilities. In the first one $\bar{k} = j$. Then $f_3(i,j) \neq \epsilon_b$ or $f_3(j,i) \neq \epsilon_b$ what implies $j \in *_3(i,j) = *_3(i,\bar{k})$. In the second one $\bar{k} \neq j$. Then the fact $i \in *_3(j,k)$ means $i \in *_3(j,\bar{k}) = \{j,\bar{k}\}$. Now observe that $j \in *_3(j,\bar{k}) = *_3(i,\bar{k})$ when $i = j$ and $j \in *_3(\bar{k},k) = *_3(i,\bar{k})$ when $i = \bar{k}$. We have proved that the fact $i \in *_3(j,k)$ implies $j \in *_3(i,\bar{k})$ in any case. The converse is just replace i and j.

(iii) We have $j \in \circ(J,k)$ if and only if there exists $t \in J$ and $p \in \{2,3\}$ such that

$$j \in *_p(t,k).$$

Items (i) and (ii) give us that this fact is equivalent to $t \in *_p(j,\bar{k})$ and so equivalent to $t \in \circ(\{j\},\bar{k}) \cap J$. □

Definition 4. Let i and j be a couple of elements in $\mathfrak{A} \setminus \{\epsilon_u\}$. We say that i is *connected* to j if either $i = j$ or there exists a sequence

$$a_1, a_2, \ldots, a_n \in (\mathfrak{A} \setminus \{\epsilon_u\}) \,\dot\cup\, \overline{(\mathfrak{A} \setminus \{\epsilon_u\})} \,\dot\cup\, \{\nu\} \,\dot\cup\, \{\bar{\nu}\},$$

such that the following conditions hold:

$$\circ(\{i\},a_1) \neq \emptyset.$$

$$\circ(\circ(\{i\},a_1),a_2) \neq \emptyset.$$

$$\circ(\circ(\circ(\{i\},a_1),a_2),a_3) \neq \emptyset$$

$$\vdots$$

$$j \in \circ(\cdots\circ(\circ(\circ(\{i\},a_1),a_2),a_3)\cdots,a_n).$$

In this case we say that $(a_1,a_2,\ldots,a_n)$ is a *connection* from i to j.

Proposition 1. *The relation $\sim$ in $\mathfrak{A} \setminus \{\epsilon_u\}$, defined by $i \sim j$ if and only if i is connected to j, is an equivalence relation.*

Proof. By Definition 4 the relation $\sim$ is reflexive.

To show the transitivity, consider $i, j, k \in I$ such that $i \sim j$ and $j \sim k$. If either $i = j$ or $j = k$ it is clear that $i \sim k$. Suppose that $i \neq j$ and $j \neq k$. Then, we can find connections $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_m)$ from i to j and from j to k respectively. We can easily verify that

$$(i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_m)$$

is a connection from i to k.

Finally, to verify the symmetric character of $\sim$, consider a connection

$$(i_1, i_2, \ldots, i_{n-1}, i_n) \tag{1}$$

from i to j; $i, j \in \mathfrak{A} \setminus \{\epsilon_u\}$, and let us show that

$$(\bar{i}_n, \bar{i}_{n-1}, ..., \bar{i}_2, \bar{i}_1)$$

is then a connection from j to i. We will prove it by induction on n:

If $n = 1$, we have three possibilities for the connection (1). Either it is $(i_1) = (\nu)$ or $(i_1) = (\bar{\nu})$ or (i_1) with $i_1 \in (\mathfrak{A} \setminus \{\epsilon_u\}) \dot{\cup} (\overline{\mathfrak{A} \setminus \{\epsilon_u\}})$. In the first case $j \in \circ(\{i\}, \nu)$, so $f_1(i) = j$ and then $i = f_1^{-1}(j) = \circ(\{j\}, \bar{\nu})$. Hence, $(\bar{\nu}) = (\bar{i_1})$ is a connection from j to i. In the second case we argue similarly. Finally, in the third case we have that $j \in \circ(\{i\}, i_1)$, then $j \in *_p(i, i_1)$ for some $p \in \{2, 3\}$ and by Lemma 1.(i)–(ii) we get $i \in *_p(j, \bar{i}_1)$. From here, $(\bar{i}_1)$ is a connection from j to i.

Suppose now that the assertion holds for the value r in Equation (1) and let us show that this assertion also holds for $r + 1$. So consider a connection

$$(i_1, i_2, \ldots, i_r, i_{r+1})$$

from i to j and let us show that

$$(\bar{i}_{r+1}, \bar{i}_r, \ldots, \bar{i}_2, \bar{i}_1)$$

is a connection from j to i:

Since

$$j \in \circ(\cdots \circ (\circ(\circ(\{i\}, i_1), i_2), i_3) \cdots, i_{r+1}),$$

if we denote

$$J := \circ(\cdots \circ (\circ(\circ(\{i\}, i_1), i_2), i_3) \cdots, i_r), \tag{2}$$

then $j \in \circ(J, i_{r+1})$ and so Lemma 1.(iii) gives us

$$\circ(\{j\}, \bar{i}_{r+1}) \cap J \neq \emptyset$$

which allows us to take an element

$$k \in \circ(\{j\}, \bar{i}_{r+1}) \cap J\,.$$

Therefore, the sequence

$$(\bar{i}_{r+1}) \tag{3}$$

is a connection from j to k.

Now, by induction hypothesis (take into account the fact that $k \in J$ and Equation (2)), we can take a connection

$$(\bar{i}_r, \bar{i}_{r-1}, \ldots, \bar{i}_2, \bar{i}_1)$$

from k to i. From here, taking into account Equation (3), we can conclude that

$$(\bar{i}_{r+1}, \bar{i}_r, ..., \bar{i}_1)$$

is a connection from j to i, which proves that $\sim$ is symmetric and consequently an equivalence relation. □

By the above Proposition we can introduce the quotient set

$$(\mathfrak{A} \setminus \{\epsilon_u\})/\sim = \{[i] : i \in \mathfrak{A} \setminus \{\epsilon_u\}\},$$

where $[i]$ is the set of elements in $\mathfrak{A} \setminus \{\epsilon_u\}$ which are connected to i, and write so

$$\mathfrak{A} \setminus \{\epsilon_u\} = \dot{\bigcup_{[i] \in (\mathfrak{A} \setminus \{\epsilon_u\})/\sim}} [i]\,. \tag{4}$$

Lemma 2. *Let $[i], [j] \in (\mathfrak{A} \setminus \{\epsilon_u\})/\sim$ be. Then the following assertions hold.*

(i) $f_1([i]) \subset [i] \cup \{\epsilon_u\}$.

(ii) $f_2([i], [i]) \subset [i] \cup \{\epsilon_u\}$.

(iii) If $f_2([i], [j]) \neq \{\epsilon_u\}$ *then* $[i] = [j]$.

(iv) If $f_3([i], [j]) \neq \{\epsilon_b\}$ *then* $[i] = [j]$.

Proof. (i) Let $\epsilon_u \neq j \in f_1([i])$. Then there exists $k \in [i]$ such that $f_1(k) = j$. From here $\circ(\{k\}, \nu) = \{j\}$. So (ν) is a connection from k to j and we get $j \in [k] = [i]$.

(ii) Suppose $f_2(i_1, i_2) = k \neq \epsilon_u$ for some $i_1, i_2 \in [i]$. Then $k \in *_2(i_1, i_2)$ and so the sequence (i_2) is a connection from i_1 to k. Hence $k \in [i_1] = [i]$.

(iii) Suppose $f_2(i_1, j_1) = k \neq \epsilon_u$ for some $i_1 \in [i]$ and $j_1 \in [j]$. Then $k \in *_2(i_1, j_1)$ and $j_1 \in *_2(k, \bar{i_1})$. From here the sequence $(j_1, \bar{i_1})$ is a connection from i_1 to j_1 and we get $[i] = [i_1] = [j_1] = [j]$.

(iv) Suppose $f_3(i_1, j_1) = b \neq \epsilon_b$ for some $i_1 \in [i]$ and $j_1 \in [j]$. Then $*_3(i_1, j_1) = \{i_1, j_1\}$, so $j_1 \in \circ(\{i_1\}, j_1\}$ and then the sequence (j_1) is a connection from i_1 to j_1. Hence $[i] = [i_1] = [j_1] = [j]$. □

Theorem 1. *Let $(\mathfrak{A}, \epsilon_u)$, $(\mathfrak{B}, \epsilon_b)$ be two pointed sets and $\mathcal{F}$ a family of three maps*

$$\mathcal{F} = \{f_1 \colon \mathfrak{A} \to \mathfrak{A}; f_2 \colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}; f_3 \colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B}\}\,.$$

Then

$$\mathfrak{A} \setminus \{\epsilon_u\} = \dot{\bigcup_{[i] \in (\mathfrak{A} \setminus \{\epsilon_u\})/\sim}} [i]$$

is the orthogonal disjoint union of the $\mathcal{F}$-invariant family of subsets

$$\{[i] : [i] \in (\mathfrak{A} \setminus \{\epsilon_u\})/\sim\}\,.$$

Proof. Consequence of Equation (4) and Lemma 2. □

3 Application to graded involutive algebras

Let $(A,*)$ be an involutive algebra, over a base field $\mathbb{K}$, which is also graded by means of a nonempty set $\mathfrak{A}$ as

$$A=\bigoplus_{i\in\mathfrak{A}} A_i\,. \tag{5}$$

That is, for any $i,j\in\mathfrak{A}$ there exists $k\in\mathfrak{A}$ such that $A_iA_j\subset A_k$, and for any $i\in\mathfrak{A}$ there is $j\in\mathfrak{A}$ with $(A_i)^*\subset A_j$.

Let us fix an external element τ to $\mathfrak{A}$ (that is $\tau\notin\mathfrak{A}$). We can define on the set $\mathfrak{A}\cup\tau$ the following maps:

$$f_1\colon(\mathfrak{A}\cup\tau)\to(\mathfrak{A}\cup\tau)$$

as $f_1(i)=j$ where $(A_i)^*\subset A_j$ for any $i\in\mathfrak{A}$; and $f_1(\tau)=\tau$.

$$f_2\colon(\mathfrak{A}\cup\tau)\times(\mathfrak{A}\cup\tau)\to(\mathfrak{A}\cup\tau)$$

as

$$f_2(i,j)=\begin{cases}k, & \text{if } 0\neq A_iA_j\subset A_k;\\ \tau, & \text{if } A_iA_j=0.\end{cases}$$

for any $i,j\in\mathfrak{A}$; and $f_2(\tau,\mathfrak{A}\cup\tau)=f_2(\mathfrak{A},\tau)=\tau$.

$$f_3\colon(\mathfrak{A}\cup\tau)\times(\mathfrak{A}\cup\tau)\to\mathbb{K}$$

as $f_3(\mathfrak{A}\cup\tau,\mathfrak{A}\cup\tau)=0$.

Consider now the pointed sets $(\mathfrak{A}\cup\tau,\tau)$ and $(\mathbb{K},0)$; and the family of the above maps $\mathcal{F}=\{f_1,f_2,f_3\}$. By applying Theorem 1 and taking into account Lemma 1 we have that

$$\mathfrak{A}=\dot{\bigcup_{[i]\in(\mathfrak{A}/\sim)}}[i] \tag{6}$$

in such a way that

$$f_1([i])\subset[i]\cup\{\tau\},f_2([i],\mathfrak{A})\cup f_2(\mathfrak{A},[i])\subset[i]\cup\{\tau\} \tag{7}$$

and

$$f_3([i],[j])=0$$

when $[i]\neq[j]$.

For any $[i]\in\mathfrak{A}/\sim$, denote by $I_{[i]}$ to the graded linear subspace of A given by

$$I_{[i]}:=\bigoplus_{k\in[i]}A_k\,.$$

From Equations (5) and (6) we have that A decomposes as

$$A=\bigoplus_{[i]\in(\mathfrak{A}/\sim)}I_{[i]}\,.$$

Equation (7) shows that $(I_{[i]})^*\subset I_{[i]}$ and that $I_{[i]}A+AI_{[i]}\subset I_{[i]}$ for any $[i]\in(\mathfrak{A}/\sim)$. That is, each $I_{[i]}$ is a graded ideal of A. Hence we have proved the next result.

Theorem 2. *Let* $(A = \bigoplus_{i \in \mathfrak{A}} A_i, *)$ *be a graded involutive algebra. Then A decomposes as the next direct sum of (involutive) graded ideals*

$$A = \bigoplus_{[i] \in (\mathfrak{A}/\sim)} I_{[i]} .$$

4 Application to graded quadratic algebras

Let (A, ω) be an orthogonal algebra, over a base field $\mathbb{K}$, which is also graded by means of a nonempty set $\mathfrak{A}$ as

$$A = \bigoplus_{i \in \mathfrak{A}} A_i . \tag{8}$$

Let us fix an external element τ to $\mathfrak{A}$ (that is $\tau \notin \mathfrak{A}$). We can define on $\mathfrak{A} \cup \tau$ the following maps:

$$f_1 \colon (\mathfrak{A} \cup \tau) \to (\mathfrak{A} \cup \tau)$$

as $f_1(k) = k$ for any $k \in \mathfrak{A} \cup \tau$.

$$f_2 \colon (\mathfrak{A} \cup \tau) \times (\mathfrak{A} \cup \tau) \to (\mathfrak{A} \cup \tau)$$

as

$$f_2(i, j) = \begin{cases} k, & \text{if } 0 \neq A_i A_j \subset A_k; \\ \tau, & \text{if } A_i A_j = 0. \end{cases}$$

for any $i, j \in \mathfrak{A}$; and $f_2(\tau, \mathfrak{A} \cup \tau) = f_2(\mathfrak{A}, \tau) = \tau$.

$$f_3 \colon (\mathfrak{A} \cup \tau) \times (\mathfrak{A} \cup \tau) \to \mathbb{K}$$

as

$$f_3(i, j) = \begin{cases} 1, & \text{if } \omega(A_i, A_j) \neq 0; \\ 0, & \text{if } \omega(A_i, A_j) = 0. \end{cases}$$

for any $i, j \in \mathfrak{A}$; and $f_3(\tau, \mathfrak{A} \cup \tau) = f_3(\mathfrak{A}, \tau) = 0$.

Consider now the pointed sets $(\mathfrak{A} \cup \tau, \tau)$ and $(\mathbb{K}, 0)$; and the family of maps $\mathcal{F} = \{f_1, f_2, f_3\}$. By applying Theorem 1 in this setup we have that

$$\mathfrak{A} = \dot{\bigcup_{[i] \in (\mathfrak{A}/\sim)}} [i] \tag{9}$$

in such a way that

$$f_1([i]) \subset [i] \cup \tau, f_2([i], \mathfrak{A}) \cup f_2(\mathfrak{A}, [i]) \subset [i] \cup \tau \tag{10}$$

and

$$f_3([i], [j]) = 0 \tag{11}$$

when $[i] \neq [j]$.

For any $[i] \in \mathfrak{A}/\sim$ denote by $I_{[i]}$ to the graded linear space

$$I_{[i]} := \bigoplus_{k \in [i]} A_k .$$

From Equations (8) and (9) we have that A decomposes as

$$A = \bigoplus_{[i]\in(\mathfrak{A}/\sim)} I_{[i]} .$$

Equation (10) shows that any $I_{[i]}$ is a graded ideal of A and finally Equation (11) gives us that $\omega(I_{[i]}, I_{[j]}) = 0$ when $[i] \neq [j]$. Hence we have proved the next result.

Theorem 3. *Let $(A = \bigoplus_{i\in\mathfrak{A}} A_i, \omega)$ be a graded orthogonal algebra. Then A decomposes as the next orthogonal direct sum of graded ideals*

$$A = \bigoplus_{[i]\in(\mathfrak{A}/\sim)} I_{[i]} .$$

5 Application to graded weak H^*-algebras

Let $(A, *, \omega)$ be a weak H^*-algebra, over a base field $\mathbb{K}$, which is also graded by means of a nonempty set $\mathfrak{A}$ as

$$A = \bigoplus_{i\in\mathfrak{A}} A_i . \tag{12}$$

Let us fix an external element τ to $\mathfrak{A}$ (that is $\tau \notin \mathfrak{A}$). We can define on $\mathfrak{A} \cup \tau$ the following maps:

$$f_1 \colon (\mathfrak{A} \cup \tau) \to (\mathfrak{A} \cup \tau)$$

as $f_1(i) = j$ where $(A_i)^* \subset A_j$ for any $i \in \mathfrak{A}$; and $f_1(\tau) = \tau$.

$$f_2 \colon (\mathfrak{A} \cup \tau) \times (\mathfrak{A} \cup \tau) \to (\mathfrak{A} \cup \tau)$$

as

$$f_2(i,j) = \begin{cases} k, & \text{if } 0 \neq A_iA_j \subset A_k; \\ \tau, & \text{if } A_iA_j = 0. \end{cases}$$

for any $i, j \in \mathfrak{A}$; and $f_2(\tau, \mathfrak{A} \cup \tau) = f_2(\mathfrak{A}, \tau) = \tau$.

$$f_3 \colon (\mathfrak{A} \cup \tau) \times (\mathfrak{A} \cup \tau) \to \mathbb{K}$$

as

$$f_3(i,j) = \begin{cases} 1, & \text{if } \omega(A_i, A_j) \neq 0; \\ 0, & \text{if } \omega(A_i, A_j) = 0. \end{cases}$$

for any $i, j \in \mathfrak{A}$; and $f_3(\tau, \mathfrak{A} \cup \tau) = f_3(\mathfrak{A}, \tau) = 0$.

Consider now the pointed sets $(\mathfrak{A} \cup \tau, \tau)$ and $(\mathbb{K}, 0)$; and the family of maps $\mathcal{F} = \{f_1, f_2, f_3\}$. By applying Theorem 1 in this setup we have that

$$\mathfrak{A} = \dot{\bigcup_{[i]\in(\mathfrak{A}/\sim)}} [i] \tag{13}$$

in such a way that

$$f_1([i]) \subset [i] \cup \{\tau\}, f_2([i], \mathfrak{A}) \cup f_2(\mathfrak{A}, [i]) \subset [i] \cup \{\tau\} \tag{14}$$

and

$$f_3([i], [j]) = 0 \tag{15}$$

when $[i] \neq [j]$.

For any $[i] \in \mathfrak{A}/\sim$ denote by $I_{[i]}$ to the graded linear space

$$I_{[i]} := \bigoplus_{k \in [i]} A_k \,.$$

From Equations (12) and (13) we have that A decomposes as

$$A = \bigoplus_{[i] \in (\mathfrak{A}/\sim)} I_{[i]} \,.$$

Equation (14) shows that any $I_{[i]}$ is an (involutive) graded ideal of A and finally Equation (15) gives us that $\omega(I_{[i]}, I_{[j]}) = 0$ when $[i] \neq [j]$. Hence we have proved the next result.

Theorem 4. *Let* $(A = \bigoplus_{i \in \mathfrak{A}} A_i, *, \omega)$ *be a graded weak* H^**-algebra. Then* A *decomposes as the next orthogonal direct sum of (involutive) graded ideals*

$$A = \bigoplus_{[i] \in (\mathfrak{A}/\sim)} I_{[i]} \,.$$

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