Conservative algebras of 2-dimensional algebras, III

Farhodjon Arzikulov, Nodirbek Umriaqov

Abstract. In the present paper we prove that every local and 2-local derivation on conservative algebras of 2-dimensional algebras are derivations. Also, we prove that every local and 2-local automorphism on conservative algebras of 2-dimensional algebras are automorphisms.

1 Introduction

The present paper is devoted to the study of conservative algebras. In 1972 Kantor [12] introduced conservative algebras as a generalization of Jordan algebras (also, see a good written survey about the study of conservative algebras [25]).

In 1990 Kantor [14] defined the multiplication \( \cdot \) on the set of all algebras (i.e. all multiplications) on the \( n \)-dimensional vector space \( V_n \) over a field \( \mathbb{F} \) of characteristic zero as follows: \( A \cdot B = [L^A_e, B] \), where \( A \) and \( B \) are multiplications and \( e \in V_n \) is some fixed vector. If \( n > 1 \), then the algebra \( W(n) \) does not belong to any well-known class of algebras (such as associative, Lie, Jordan, or Leibniz algebras). The algebra \( W(n) \) is a conservative algebra [12].

In [12] Kantor classified all conservative 2-dimensional algebras and defined the class of terminal algebras as algebras satisfying some certain identity. He proved that every terminal algebra is a conservative algebra and classified all simple finite-dimensional terminal algebras with left quasi-unit over an algebraically closed field of characteristic zero [13]. Terminal algebras were also studied in [18], [19].

In 2017 Kaygorodov and Volkov [16] described automorphisms, one-sided ideals, and idempotents of \( W(2) \). Also a similar problem is solved for the algebra \( W_2 \) of all commutative algebras on the 2-dimensional vector space and for the algebra

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$S_2$ of all commutative algebras with zero multiplication trace on the 2-dimensional vector space. The papers [15], [17] are also devoted to the study of conservative algebras and superalgebras.

Let $A$ be an algebra. A linear operator $\nabla$ on $A$ is called a local derivation if for every $x \in A$ there exists a derivation $\phi_x$ of $A$, depending on $x$, such that $\nabla(x) = \phi_x(x)$. The history of local derivations had begun from the paper of Kadison [11]. Kadison introduced the concept of local derivation and proved that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation.

A similar notion, which characterizes nonlinear generalizations of derivations, was introduced by Šemrl as 2-local derivations. In his paper [26] was proved that a 2-local derivation of the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space $H$ is a derivation. After his works, appear numerous new results related to the description of local and 2-local derivations of associative algebras (see, for example, [1], [3], [4], [20], [21], [23]).

The study of local and 2-local derivations of non-associative algebras was initiated in some papers of Ayupov and Kudaybergenov (for the case of Lie algebras, see [5], [6]). In particular, they proved that there are no non-trivial local and 2-local derivations on semisimple finite-dimensional Lie algebras. In [8] examples of 2-local derivations on nilpotent Lie algebras which are not derivations, were also given. Later, the study of local and 2-local derivations was continued for Leibniz algebras [7], Malcev algebras and Jordan algebras [2]. Local automorphisms and 2-local automorphisms, also were studied in many cases, for example, they were studied on Lie algebras [5], [10].

Now, a linear operator $\nabla$ on $A$ is called a local automorphism if for every $x \in A$ there exists an automorphism $\phi_x$ of $A$, depending on $x$, such that $\nabla(x) = \phi_x(x)$. The concept of local automorphism was introduced by Larson and Sourour [22] in 1990. They proved that, invertible local automorphisms of the algebra of all bounded linear operators on an infinite-dimensional Banach space $X$ are automorphisms.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by Šemrl in [26] as 2-local automorphisms. Namely, a map $\Delta: A \to A$ (not necessarily linear) is called a 2-local automorphism, if for every $x, y \in A$ there exists an automorphism $\phi_{x,y}: A \to A$ such that $\Delta(x) = \phi_{x,y}(x)$ and $\Delta(y) = \phi_{x,y}(y)$. After the work of Šemrl, it appeared numerous new results related to the description of local and 2-local automorphisms of algebras (see, for example, [5], [7], [9], [10], [21]).

In the present paper, we continue the study of derivations, local and 2-local derivations of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local derivation of the conservative algebras of 2-dimensional algebras are derivations. In the present paper, we continue the study of automorphisms, local and 2-local automorphisms in the case of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local automorphism of the conservative algebras of 2-dimensional algebras are automorphisms.
2 Preliminaries

Throughout this paper $\mathbb{F}$ is some fixed field of characteristic zero. A multiplication on 2-dimensional vector space is defined by a $2 \times 2 \times 2$ matrix. Their classification was given in many papers (see, for example, [24]). Let consider the space $W(2)$ of all multiplications on the 2-dimensional space $V_2$ with a basis $v_1, v_2$. The definition of the multiplication · on the algebra $W(2)$ is defined as follows: we fix the vector $v_1 \in V_2$ and define

$$(A \cdot B)(x, y) = A(v_1, B(x, y)) - B(A(v_1, x), y) - B(x, A(v_1, y))$$

for $x, y \in V_2$ and $A, B \in W(2)$. The algebra $W(2)$ is conservative [14]. Let consider the multiplications $\alpha^k_{i, j}$ $(i, j, k = 1, 2)$ on $V_2$ defined by the formula $\alpha^k_{i, j}(v_t, v_l) = \delta_{il}\delta_{jl}v_k$ for all $t, l \in \{1, 2\}$. It is easy to see that $\{\alpha^k_{i, j}|i, j, k = 1, 2\}$ is a basis of the algebra $W(2)$. The multiplication table of $W(2)$ in this basis is given in [15]. In this work we use another basis for the algebra $W(2)$ (from [16]). Let introduce the notation

$$e_1 = \alpha_{11}^1 - \alpha_{12}^2 - \alpha_{21}^2, \quad e_2 = \alpha_{12}^1, \quad e_3 = \alpha_{22}^2 - \alpha_{12}^1 - \alpha_{21}^1, \quad e_4 = \alpha_{22}^1, \quad e_5 = 2\alpha_{11}^1 + \alpha_{12}^2 + \alpha_{21}^2,$$

$$e_6 = 2\alpha_{22}^2 + \alpha_{12}^1 + \alpha_{21}^1, \quad e_7 = \alpha_{12}^1 - \alpha_{21}^1, \quad e_8 = \alpha_{12}^2 - \alpha_{21}^2.$$

It is easy to see that the multiplication table of $W(2)$ in the basis $e_1, \ldots, e_8$ is the following.

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The subalgebra generated by the elements $e_1, \ldots, e_6$ is the conservative (and, moreover, terminal) algebra $W_2$ of commutative 2-dimensional algebras. The subalgebra generated by the elements $e_1, \ldots, e_4$ is the conservative (and, moreover, terminal) algebra $S_2$ of all commutative 2-dimensional algebras with zero multiplication trace [15].

Let $A$ be an algebra. A linear map $D: A \to A$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for any two elements $x, y \in A$.

Our main tool for study of local and 2-local derivations of the algebras $S_2, W_2$ and $W(2)$ is the following lemma [15, Theorem 6], where the matrix of a derivation is calculated in the new basis $e_1, \ldots, e_8$. 


Lemma 1. A linear map $D : W(2) \to W(2)$ is a derivation if and only if the matrix of $D$ has the following matrix form:

$$
\begin{pmatrix}
0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
2\alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3\alpha & 2\beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

(1)

where $\alpha, \beta$ are elements in $\mathbb{F}$.

Now, we give a characterization of automorphisms on conservative algebras of 2-dimensional algebras.

Let $A$ be an algebra. A bijective linear map $\phi : A \to A$ is called an automorphism, if $\phi(xy) = \phi(x)\phi(y)$ for any elements $x, y \in A$.

Our principal tool for study of local and 2-local automorphisms of the algebras $S_2$, $W_2$ and $W(2)$ is the following lemma, which was proved in [16, Theorem 11].

Lemma 2. A linear map $\phi : W(2) \to W(2)$ is an automorphism if and only if the matrix of $\phi$ has the following matrix form:

$$
\begin{pmatrix}
1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\
2ab & a^2b & b & 0 & 0 & 0 & 0 & 0 \\
3a^2b^2 & a^3b^2 & 3ab^2 & b^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -ab & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b & ab \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

(2)

where $a, b$ are elements in $\mathbb{F}$ and $b \neq 0$.

3 Local derivations of conservative algebras of 2-dimensional algebras

In this section we give a characterization of derivations on conservative algebras of 2-dimensional algebras.

Let $A$ be an algebra. A linear map $\nabla : A \to A$ is called a local derivation, if for any element $x \in A$ there exists a derivation $D_x : A \to A$ such that $\nabla(x) = D_x(x)$.

Theorem 1. Every local derivation of the algebra $W(2)$ is a derivation.

Proof. Let $\nabla$ be an arbitrary local derivation of $W(2)$ and write

$$
\nabla(x) = B\bar{x}, x \in W(2),
$$
where \( B = (b_{i,j})_{j=1}^8, \bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \) is the vector corresponding to \( x \). Then for every \( x \in W(2) \) there exist elements \( a_x, b_x \) in \( \mathbb{F} \) such that

\[
B \bar{x} = \begin{pmatrix}
0 & a_x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -b_x & 0 & 0 & 0 & 0 & 0 & 0 \\
2a_x & 0 & b_x & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3a_x & 2b_x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_x & b_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_x & a_x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8
\end{pmatrix}.
\]

In other words

\[
\begin{align*}
b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 + b_{1,4}x_4 + b_{1,5}x_5 + b_{1,6}x_6 + b_{1,7}x_7 + b_{1,8}x_8 &= a_xx_2; \\
b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 + b_{2,4}x_4 + b_{2,5}x_5 + b_{2,6}x_6 + b_{2,7}x_7 + b_{2,8}x_8 &= -b_xx_2; \\
b_{3,1}x_1 + b_{3,2}x_2 + b_{3,3}x_3 + b_{3,4}x_4 + b_{3,5}x_5 + b_{3,6}x_6 + b_{3,7}x_7 + b_{3,8}x_8 &= 2a_xx_1 + b_xx_3; \\
b_{4,1}x_1 + b_{4,2}x_2 + b_{4,3}x_3 + b_{4,4}x_4 + b_{4,5}x_5 + b_{4,6}x_6 + b_{4,7}x_7 + b_{4,8}x_8 &= 3a_xx_3 + 2b_xx_4; \\
b_{5,1}x_1 + b_{5,2}x_2 + b_{5,3}x_3 + b_{5,4}x_4 + b_{5,5}x_5 + b_{5,6}x_6 + b_{5,7}x_7 + b_{5,8}x_8 &= 0; \\
b_{6,1}x_1 + b_{6,2}x_2 + b_{6,3}x_3 + b_{6,4}x_4 + b_{6,5}x_5 + b_{6,6}x_6 + b_{6,7}x_7 + b_{6,8}x_8 &= -a_xx_5 + b_xx_6; \\
b_{7,1}x_1 + b_{7,2}x_2 + b_{7,3}x_3 + b_{7,4}x_4 + b_{7,5}x_5 + b_{7,6}x_6 + b_{7,7}x_7 + b_{7,8}x_8 &= b_xx_7 + a_xx_8; \\
b_{8,1}x_1 + b_{8,2}x_2 + b_{8,3}x_3 + b_{8,4}x_4 + b_{8,5}x_5 + b_{8,6}x_6 + b_{8,7}x_7 + b_{8,8}x_8 &= 0.
\end{align*}
\]

Taking \( x = (1, 0, 0, 0, 0, 0, 0, 0) \), \( x = (0, 0, 1, 0, 0, 0, 0, 0) \), etc, from this it follows that

\[
\begin{align*}
b_{1,1} &= b_{1,3} = b_{1,4} = b_{1,5} = b_{1,6} = b_{1,7} = b_{1,8} = 1; \\
b_{2,1} &= b_{2,3} = b_{2,4} = b_{2,5} = b_{2,6} = b_{2,7} = b_{2,8} = 0; \\
b_{3,1} &= b_{3,2} = b_{3,4} = b_{3,5} = b_{3,6} = b_{3,7} = b_{3,8} = 0; \\
b_{4,1} &= b_{4,2} = b_{4,3} = b_{4,5} = b_{4,6} = b_{4,7} = b_{4,8} = 0; \\
b_{5,1} &= b_{5,2} = b_{5,3} = b_{5,4} = b_{5,5} = b_{5,6} = b_{5,7} = b_{5,8} = 0; \\
b_{6,1} &= b_{6,2} = b_{6,3} = b_{6,4} = b_{6,5} = b_{6,6} = b_{6,7} = b_{6,8} = 0; \\
b_{7,1} &= b_{7,2} = b_{7,3} = b_{7,4} = b_{7,5} = b_{7,6} = 0; \\
b_{8,1} &= b_{8,2} = b_{8,3} = b_{8,4} = b_{8,5} = b_{8,6} = b_{8,7} = b_{8,8} = 0.
\end{align*}
\]

Then for every \( x \in W(2) \) there exist elements \( a_x, b_x \) in \( \mathbb{F} \) such that

\[
\begin{align*}
b_{1,2}x_2 &= a_xx_2; \\
b_{2,2}x_2 &= -b_xx_2; \\
b_{3,1}x_1 + b_{3,3}x_3 &= 2a_xx_1 + b_xx_3; \\
b_{4,3}x_3 + b_{4,4}x_4 &= 3a_xx_3 + 2b_xx_4; \\
b_{6,5}x_5 + b_{6,6}x_6 &= -a_xx_5 + b_xx_6; \\
b_{7,7}x_7 + b_{7,8}x_8 &= b_xx_7 + a_xx_8.
\end{align*}
\]
Using 1-th and 3-th equalities of system (3) we get
\[
\begin{align*}
2b_{1,2}x_1x_2 &= 2a_xx_1x_2; \\
b_{3,1}x_1x_2 + b_{3,3}x_2x_3 &= 2a_xx_1x_2 + b_xx_2x_3.
\end{align*}
\]

and
\[
(b_{3,1} - 2b_{1,2})x_1x_2 + b_{3,3}x_2x_3 = b_xx_2x_3.
\]

Hence, \(b_{3,1} = 2b_{1,2}\). Similarly, using equalities of (3) we get
\[
b_{4,3} = 3b_{1,2}, b_{2,2} = -b_{3,3}, b_{4,4} = -2b_{2,2}.
\]

Using 1-th and 5-th equalities of system (3) we get
\[
\begin{align*}
b_{1,2}x_2x_5 &= a_xx_2x_5; \\
b_{6,5}x_5x_2 + b_{6,6}x_6x_2 &= -a_xx_5x_2 + b_xx_6x_2.
\end{align*}
\]

and
\[
(b_{6,5} + b_{1,2})x_2x_5 + b_{6,6}x_6x_2 = b_xx_6x_2.
\]

Hence, \(b_{6,5} = -b_{1,2}\).

Using 2-th and 5-th equalities of system (3) we get
\[
\begin{align*}
b_{2,2}x_2x_6 &= -b_xx_2x_6; \\
b_{6,5}x_5x_2 + b_{6,6}x_6x_2 &= -a_xx_5x_2 + b_xx_6x_2.
\end{align*}
\]

and
\[
b_{6,5}x_5x_2 + (b_{6,6} + b_{2,2})x_6x_2 = -a_xx_5x_2.
\]

Hence, \(b_{6,6} = -b_{2,2}\).

Using 1-th and 6-th equalities of system (3) we get
\[
\begin{align*}
b_{1,2}x_2x_8 &= a_xx_2x_8; \\
b_{7,7}x_7x_2 + b_{7,8}x_8x_2 &= b_xx_7x_2 + a_xx_8x_2.
\end{align*}
\]

and
\[
b_{7,7}x_7x_2 + (b_{7,8} - b_{1,2})x_8x_2 = b_xx_7x_2.
\]

Hence, \(b_{7,8} = b_{1,2}\).

Using 2-th and 6-th equalities of system (3) we get
\[
\begin{align*}
b_{2,2}x_2x_7 &= -b_xx_2x_7; \\
b_{7,7}x_7x_2 + b_{7,8}x_8x_2 &= b_xx_7x_2 + a_xx_8x_2.
\end{align*}
\]

and
\[
(b_{7,7} + b_{2,2})x_7x_2 + b_{7,8}x_8x_2 = a_xx_8x_2.
\]

Hence, \(b_{7,7} = -b_{2,2}\).

These equalities show that the matrix of the linear map \(\nabla\) is of the form (1). Therefore, by lemma 1 \(\nabla\) is a derivation. This completes the proof.

Since a derivation on \(W(2)\) is invariant on the subalgebras \(S_2\) and \(W_2\), we have the following corollary.

**Corollary 1.** Every local derivation of the algebras \(S_2\) and \(W_2\) is a derivation.
4 2-Local derivations of conservative algebras of 2-dimensional algebras

In this section we give another characterization of derivations on conservative algebras of 2-dimensional algebras.

A (not necessary linear) map $\Delta: A \rightarrow A$ is called a 2-local derivation, if for any elements $x, y \in A$ there exists a derivation $D_{x,y}: A \rightarrow A$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

**Theorem 2.** Every 2-local derivation of the algebras $S_2$, $W_2$ and $W(2)$ is a derivation.

**Proof.** We will prove that every 2-local derivation of $W(2)$ is a derivation.

Let $\Delta$ be an arbitrary 2-local derivation of $W(2)$. Then, by the definition, for every element $a \in W(2)$, there exists a derivation $D_{a,e_2}$ of $W(2)$ such that

$$\Delta(a) = D_{a,e_2}(a), \quad \Delta(e_2) = D_{a,e_2}(e_2).$$

By lemma 1, the matrix $A_{a,e_2}$ of the derivation $D_{a,e_2}$ has the following matrix form:

$$A_{a,e_2} = \begin{pmatrix}
0 & \alpha_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
2\alpha_{a,e_2} & 0 & \beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3\alpha_{a,e_2} & 2\beta_{a,e_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$

Let $v$ be an arbitrary element in $W(2)$. Then there exists a derivation $D_{v,e_2}$ of $W(2)$ such that

$$\Delta(v) = D_{v,e_2}(v), \quad \Delta(e_2) = D_{v,e_2}(e_2).$$

By lemma 1, the matrix $A_{v,e_2}$ of the derivation $D_{v,e_2}$ has the following matrix form:

$$A_{v,e_2} = \begin{pmatrix}
0 & \alpha_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
2\alpha_{v,e_2} & 0 & \beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3\alpha_{v,e_2} & 2\beta_{v,e_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$

Since $\Delta(e_2) = D_{a,e_2}(e_2) = D_{v,e_2}(e_2)$, we have

$$\alpha_{a,e_2} = \alpha_{v,e_2}, \quad \beta_{a,e_2} = \beta_{v,e_2},$$
that it
\[ D_{v,e_2} = D_{a,e_2}. \]
Therefore, for any element \( a \) of the algebra \( W(2) \)
\[ \Delta(a) = D_{v,e_2}(a), \]
that it \( D_{v,e_2} \) does not depend on \( a \). Hence, \( \Delta \) is a derivation by lemma 1.

The cases of the algebras \( S_2 \) and \( W_2 \) are also similarly proved. This ends the proof. \( \square \)

5 2-Local automorphisms of conservative algebras of 2-dimensional algebras

A (not necessary linear) map \( \Delta : \mathcal{A} \to \mathcal{A} \) is called a 2-local automorphism, if for any elements \( x, y \in \mathcal{A} \) there exists an automorphism \( \phi_{x,y} : \mathcal{A} \to \mathcal{A} \) such that
\[ \Delta(x) = \phi_{x,y}(x), \Delta(y) = \phi_{x,y}(y). \]

**Theorem 3.** Every 2-local automorphism of the algebras \( S_2, W_2 \) and \( W(2) \) is an automorphism.

**Proof.** We prove that every 2-local automorphism of \( W(2) \) is an automorphism.

Let \( \Delta \) be an arbitrary 2-local automorphism of \( W(2) \). Then, by the definition, for every element \( x \in W(2) \),
\[ x = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 + x_8 e_8, \]
there exist elements \( a_{x,e_2}, b_{x,e_2} \) such that
\[
\begin{pmatrix}
1 & a_{x,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2a_{x,e_2} b_{x,e_2} & a_{x,e_2}^2 b_{x,e_2} & b_{x,e_2} & 0 & 0 & 0 & 0 & 0 \\
3a_{x,e_2}^2 b_{x,e_2}^2 & a_{x,e_2}^3 b_{x,e_2}^2 & 3a_{x,e_2} b_{x,e_2}^2 & b_{x,e_2}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{x,e_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{x,e_2} b_{x,e_2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[ \Delta(x) = A_{x,e_2} \bar{x}, \]
where \( \bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \) is the vector corresponding to \( x \), and
\[ \Delta(e_2) = A_{x,e_2} e_2 = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a_{x,e_2}^2 b_{x,e_2}, a_{x,e_2}^3 b_{x,e_2}^2, 0, 0, 0, 0). \]

Since the element \( x \) was chosen arbitrarily, we have
\[ \Delta(e_2) = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a_{x,e_2}^2 b_{x,e_2}, a_{x,e_2}^3 b_{x,e_2}^2, 0, 0, 0, 0) \]
\begin{align*}
= (a_{y,e_2}, \frac{1}{b_{y,e_2}}, a_{y,e_2}^2 b_{y,e_2}, a_{y,e_2}^3 b_{y,e_2}^2, 0, 0, 0, 0),
\end{align*}
for each pair \( x, y \) of elements in \( W(2) \). Hence, \( a_{x,e_2} = a_{y,e_2}, b_{x,e_2} = b_{y,e_2} \). Therefore
\[ \Delta(x) = A_{y,e_2} x \]
for any \( x \in W(2) \) and the matrix \( A_{y,e_2} \) does not depend on \( x \). Thus, by lemma 2 \( \Delta \) is an automorphism.

The cases of the algebras \( S_2 \) and \( W_2 \) are also similarly proved. The proof is complete. \( \square \)

6 Local automorphisms of conservative algebras of 2-dimensional algebras

Let \( A \) be an algebra. A linear map \( \nabla: A \to A \) is called a local automorphism, if for any element \( x \in A \) there exists an automorphism \( \phi_x: A \to A \) such that \( \nabla(x) = \phi_x(x) \).

**Theorem 4.** Every local automorphism of the algebras \( S_2, W_2 \) and \( W(2) \) is an automorphism.

**Proof.** We prove that every local automorphism of \( W(2) \) is an automorphism.

Let \( \nabla \) be an arbitrary local automorphism of \( W(2) \) and \( B \) be its matrix, i.e.,
\[ \nabla(x) = B \bar{x}, x \in W(2), \]
where \( \bar{x} \) is the vector corresponding to \( x \). Then, by the definition, for every element \( x \in W(2) \),
\[ x = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 + x_8 e_8, \]
there exist elements \( a_x, b_x \) such that
\[
A_x = \begin{pmatrix}
1 & a_x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & b_x & 0 & 0 & 0 & 0 & 0 \\
2a_x b_x & a_x^2 b_x & 0 & 0 & 0 & 0 & 0 & 0 \\
3a_x^2 b_x^2 & a_x^3 b_x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
and
\[ \nabla(x) = B \bar{x} = A_x \bar{x}. \]
Using these equalities and by choosing subsequently \( x = e_1, x = e_2, \ldots, x = e_8 \) we get

\[
B = \begin{pmatrix}
1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{b_{e_2}} & a_{e_2}b_{e_2} & b_{e_3} & 0 & 0 & 0 & 0 \\
2a_{e_1}b_{e_1} & a_{e_2}^2b_{e_2} & b_{e_3} & 0 & 0 & 0 & 0 & 0 \\
3a_{e_1}^2b_{e_1}^2 & a_{e_2}^3b_{e_2}^2 & 3a_{e_3}b_{e_3}^2 & b_{e_4}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{e_7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Since \( \nabla (e_6 + e_7) = \nabla (e_6) + \nabla (e_7) \), we have

\[
b_{e_6 + e_7} = b_{e_6}, b_{e_6 + e_7} = b_{e_7}.
\]

Hence,

\[
b_{e_6} = b_{e_7}.
\]

Similarly to this equality we get \( b_{e_3} = b_{e_6} \) and \( b_{e_6} = b_{e_2} \neq 0 \). Hence,

\[
b_{e_2} = b_{e_3} = b_{e_6} = b_{e_7}.
\] (4)

Since \( \nabla (e_5 + e_8) = \nabla (e_5) + \nabla (e_8) \), we have

\[
a_{e_5 + e_8}b_{e_5 + e_8} = a_{e_5}b_{e_5}, a_{e_5 + e_8}b_{e_5 + e_8} = a_{e_8}b_{e_8}.
\]

From this it follows that

\[
a_{e_5}b_{e_5} = a_{e_8}b_{e_8}.
\]

Similarly to this equality we get \( a_{e_1}b_{e_1} = a_{e_8}b_{e_8} \). Hence,

\[
a_{e_1}b_{e_1} = a_{e_5}b_{e_5} = a_{e_8}b_{e_8}.
\] (5)

Since \( \nabla (e_4 + e_6) = \nabla (e_4) + \nabla (e_6) \), we have

\[
b_{e_4 + e_6}^2 = b_{e_4}^2, b_{e_4 + e_6}^2 = b_{e_6}^2.
\]

From this it follows that

\[
b_{e_4}^2 = b_{e_6}^2.
\]

Hence, by (4), we get

\[
b_{e_4}^2 = b_{e_2}^2.
\] (6)

Since \( \nabla (e_2 + e_8) = \nabla (e_2) + \nabla (e_8) \), we have

\[
a_{e_2} = a_{e_2 + e_8}, a_{e_2 + e_8}^2b_{e_2 + e_8} = a_{e_2}^2b_{e_2}, a_{e_2 + e_8}b_{e_2 + e_8} = a_{e_8}b_{e_8}.
\]

Hence,

\[
b_{e_2 + e_8} = b_{e_2}, a_{e_2 + e_8}b_{e_2 + e_8} = a_{e_2}b_{e_2}.
\]
and, therefore,
\[ a_{e_2} b_{e_2} = a_{e_8} b_{e_8}. \] (7)

Similarly, since \( \nabla(e_2 + e_3) = \nabla(e_2) + \nabla(e_3) \), we have
\[
a_{e_2} = a_{e_2+e_3}, \quad b_{e_2}^{-1} = b_{e_2+e_3}^{-1}, \quad a_{e_2+e_3}^3 b_{e_2+e_3}^2 + 3a_{e_2+e_3} b_{e_2+e_3}^2 = a_{e_2}^3 b_{e_2}^2 + 3a_{e_3} b_{e_3}^2.
\]

Hence,
\[ b_{e_2} = b_{e_2+e_3} \]
and by (4) and \( a_{e_2} = a_{e_2+e_3} \) we get
\[ a_{e_2}^3 + 3a_{e_2} = a_{e_2}^3 + 3a_{e_3}.
\]
Therefore, \( a_{e_2} = a_{e_3} \) and
\[ a_{e_2} b_{e_2}^2 = a_{e_3} b_{e_3}^2. \] (8)

Finally, since \( \nabla(e_1 + e_8) = \nabla(e_1) + \nabla(e_8) \), we have
\[
a_{e_1+e_8} b_{e_1+e_8} = a_{e_1} b_{e_1}, \quad a_{e_1+e_8} b_{e_1+e_8} = a_{e_8} b_{e_8}.
\]
Hence,
\[ a_{e_1} b_{e_1} = a_{e_8} b_{e_8}. \]

By (7), from the last equalities it follows that
\[ a_{e_1} b_{e_1} = a_{e_2} b_{e_2}, \quad a_{e_1}^2 b_{e_1}^2 = (a_{e_1} b_{e_1})^2 = (a_{e_2} b_{e_2})^2 = a_{e_2} b_{e_2}^2. \] (9)

By (4), (5), (6), (7), (8), (9) the matrix \( B \) has the following matrix form
\[
B = \begin{pmatrix}
1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & a_{e_2} & b_{e_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & a_{e_2}^2 b_{e_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_{e_2} b_{e_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & a_{e_2} b_{e_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{e_2} b_{e_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Hence, by lemma 2, the local automorphism \( \nabla \) is an automorphism.

The cases of the algebras \( S_2 \) and \( W_2 \) are also similarly proved. This ends the proof. \[ \square \]

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References


[2] Sh. Ayupov, F. Arzikulov: 2-Local derivations on associative and Jordan matrix rings  


[7] Sh. Ayupov, K. Kudaybergenov, B. Omirov: Local and 2-local derivations and  
automorphisms on simple Leibniz algebras. Bulletin of the Malaysian Mathematical  


117–121.


[18] I. Kaygorodov, A. Khudoyberdiyev, A. Sattarov: One-generated nilpotent terminal  

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