

## Actions of the additive group $G_a$ on certain noncommutative deformations of the plane

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**Abstract.** We connect the theorems of Rentschler [18] and Dixmier [10] on locally nilpotent derivations and automorphisms of the polynomial ring  $A_0$  and of the Weyl algebra  $A_1$ , both over a field of characteristic zero, by establishing the same type of results for the family of algebras

$$A_h = \langle x, y \mid yx - xy = h(x) \rangle,$$

where  $h$  is an arbitrary polynomial in  $x$ . In the second part of the paper we consider a field  $\mathbb{F}$  of prime characteristic and study  $\mathbb{F}[t]$ -comodule algebra structures on  $A_h$ . We also compute the Makar-Limanov invariant of absolute constants of  $A_h$  over a field of arbitrary characteristic and show how this subalgebra determines the automorphism group of  $A_h$ .

### 1 Introduction

The purpose of this note is to connect two groundbreaking papers which appeared in 1968: in [18], Rentschler classified the actions of the additive group  $G_a$  on the 2-dimensional plane and in [10] Dixmier determined the automorphism group of the Weyl algebra  $A_1 = \langle x, y \mid yx - xy = 1 \rangle$ , the algebra of differential operators with polynomial coefficients in one variable, both over a field of characteristic 0. What they have in common is the use of locally nilpotent derivations as a

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fundamental tool to obtain their respective main results, each related to a corresponding automorphism group. Indeed, a consequence of Rentschler's Theorem is a description of the automorphism group of the polynomial ring in two variables  $A_0 = \langle x, y \mid yx = xy \rangle$ .

Although polynomial rings and Weyl algebras can seem to be on opposite ends of the spectrum when it comes to certain algebraic properties (e.g., one is commutative, has plenty of prime ideals and can be made into a Hopf algebra in a natural way, while the other is noncommutative and simple, with no Hopf structure), it should not be surprising that they are quite strongly related. A striking connection is the fact that the Jacobian conjecture is equivalent to the (weak) Dixmier conjecture (see [3], [21] and [22] for definitions and details).

One way of explicitly connecting  $A_0$  and  $A_1$  is through a family of algebras  $A_h$ , parametrized by a polynomial  $h(x)$  in  $x$ , which was introduced and studied in [4], [5] and [6]. The algebra  $A_h$  can be defined as the unital associative algebra with generators  $x, y$  satisfying the commutation relation  $yx - xy = h(x)$ . When  $h = 0, 1$  we retrieve the polynomial algebra  $A_0$  and the Weyl algebra  $A_1$ , respectively. Other choices of  $h$  give algebras like the enveloping algebra of the two-dimensional non-abelian Lie algebra, as  $A_x$ , the Jordan plane, as  $A_{x^2}$ , and many others. In characteristic 0, one can think of all of these algebras as deformations of the coordinate ring of the 2-dimensional plane, the polynomial ring  $A_0$ . This can be made explicit by means of the so-called *Groenewold-Moyal product*. Consider the derivations  $\phi = \frac{d}{dy}$  and  $\psi = h(x)\frac{d}{dx}$  of  $A_0$ . Then the infinitesimal  $\phi \wedge \psi$  defines an associative star product on  $A_0[[\hbar]]$ , with

$$a \star b = \sum_{n \geq 0} \frac{\phi^n(a)\psi^n(b)}{n!} \hbar^n, \quad \text{for } a, b \in A_0.$$

It is easy to verify that

$$x \star x = x^2, \quad y \star y = y^2, \quad y \star x = yx + h(x)\hbar, \quad x \star y = xy,$$

so  $y \star x - x \star y = h(x)\hbar$ . Since  $\phi$  is locally nilpotent, we can specialize at  $\hbar = 1$ , hence retrieving  $A_h$  as a deformation of the commutative polynomial algebra  $A_0$ .

We show in Section 2 that, over a field of characteristic 0, the descriptions given in Dixmier and Rentschler's aforementioned papers still hold in general for  $A_h$ , for any  $h$ , although in a more rigid form, in case  $h$  is not a constant polynomial. After describing explicitly the locally nilpotent derivations of  $A_h$ , we determine the so-called Makar-Limanov invariant of absolute constants,  $\text{ML}(A_h)$  and use it to give an alternative proof of [1, Prop. 3.6], that the automorphism group of  $A_h$  is tame (generated by affine and triangular automorphisms). See [16] for the corresponding results for the free Poisson algebra, [14] for the free generic Poisson algebra and the recent papers [9] and [12] for related results on the free algebra (all cases mentioned are in rank two).

In Section 3, we consider the case of fields of positive characteristic  $p$ . In this case, locally nilpotent derivations lose some of their properties, and they do not capture enough information, as often (although not always) the  $p$ -th power of a

locally nilpotent derivation will be trivial. The natural analogue in prime characteristic comes from the notion of an action of the additive group  $G_a$  of the field. In algebraic terms, this corresponds to a comodule algebra structure or, equivalently, to a locally nilpotent iterative higher derivation. This point of view fits in naturally with viewing  $A_h$  as a deformation of the polynomial ring  $A_0$ , allowing for a generalization of the geometric notion of an action on a space, which in this case could be thought of as a noncommutative space. See [19] and [20] for results in this direction in the case of the Weyl algebra  $A_1$ . Thus, we define the prime characteristic analogue of the Makar-Limanov invariant, as in [8], and compute it for  $A_h$  for any non-constant polynomial  $h$ . This again gives sufficient information for computing  $\text{Aut}(A_h)$  over a field of prime characteristic.

## 2 The locally nilpotent derivations of $A_h$ in characteristic 0

Throughout this section,  $\mathbb{F}$  denotes an arbitrary field of characteristic 0. For a unital associative  $\mathbb{F}$ -algebra  $A$ , we denote by  $\text{LND}(A)$  the set of all locally nilpotent derivations of  $A$ . In detail,  $\text{LND}(A)$  is the set of all linear maps  $\partial: A \rightarrow A$  satisfying the Leibniz identity  $\partial(ab) = a\partial(b) + \partial(a)b$  and such that the set

$$N(\partial, a) = \{n \geq 0 \mid \partial^n(a) \neq 0\}$$

is finite, for all  $a, b \in A$ . We set  $A^\partial = \ker \partial$ , a subalgebra of  $A$ . It is well known that every  $\partial \in \text{LND}(A)$  induces a degree function on  $A$  by setting:

$$\text{deg}_\partial(0) = -\infty, \quad \text{deg}_\partial(a) = \max N(\partial, a), \quad \text{for } 0 \neq a \in A. \tag{1}$$

This degree function has especially nice properties in case  $A$  is a domain and  $\text{char}(\mathbb{F}) = 0$ .

**Proposition 1 ([15]).** *Assume that  $A$  is a domain and  $\mathbb{F}$  is a field of characteristic zero. For any  $\partial \in \text{LND}(A)$  and  $a, b \in A$ , we have:*

- (a)  $\text{deg}_\partial(ab) = \text{deg}_\partial(a) + \text{deg}_\partial(b)$ ;
- (b)  $\text{deg}_\partial(a + b) \leq \max \{\text{deg}_\partial(a), \text{deg}_\partial(b)\}$ , with equality if  $\text{deg}_\partial(a) \neq \text{deg}_\partial(b)$ ;
- (c)  $\text{deg}_\partial(\partial(a)) = \text{deg}_\partial(a) - 1$  if  $\text{deg}_\partial(a) \neq 0$ .

*It follows from (a) above that  $A^\partial$  is factorially closed: if  $a, b \in A \setminus \{0\}$  and  $ab \in A^\partial$ , then  $a, b \in A^\partial$ .*

**Remark 1.** The hypotheses on  $A$  and  $\mathbb{F}$  in Proposition 1 are needed only for part (a); the remaining parts hold in general.

There is a strong connection between locally nilpotent derivations and algebra automorphisms of  $A$ . Given  $\partial \in \text{LND}(A)$ , there is a well-defined map  $\exp(\partial): A \rightarrow A$  with

$$\exp(\partial)(a) = \sum_{k \geq 0} \frac{\partial^k(a)}{k!}$$

and it is easy to see that  $\exp(\partial)$  is an algebra automorphism of  $A$ . Although the set  $\text{LND}(A)$  is not in general closed under sums or commutators, the automorphism group  $\text{Aut}(A)$  acts on  $\text{LND}(A)$  by conjugation, and it follows that  $\{\exp(\partial) \mid \partial \in \text{LND}(A)\}$  generates a normal subgroup of  $\text{Aut}(A)$ .

Another connection with automorphisms of  $A$  is via the so-called Makar-Limanov invariant of absolute constants,  $\text{ML}(A)$ , introduced in [15]. By definition,

$$\text{ML}(A) = \bigcap_{\partial \in \text{LND}(A)} A^\partial \tag{2}$$

and clearly the subalgebra  $\text{ML}(A)$  is invariant under automorphisms of  $A$ .

**Example 1.** For  $\alpha \in \mathbb{F}$ , let  $A_\alpha$  be the unital associative  $\mathbb{F}$ -algebra generated by elements  $x, y$ , subject to the relation  $[y, x] = \alpha$ , where  $[a, b] = ab - ba$  is the commutator.

- (a) If  $\alpha = 0$ , then  $A_0 = \mathbb{F}[x, y]$  is the usual commutative polynomial algebra of rank 2. Then the partial derivatives  $\partial_x = \frac{d}{dx}$  and  $\partial_y = \frac{d}{dy}$  are locally nilpotent and it is easy to see that  $A_0^{\partial_x} \cap A_0^{\partial_y} = \mathbb{F}$ . Hence,  $\text{ML}(A_0) = \mathbb{F}$ .
- (b) If  $\alpha \neq 0$ , then  $A_\alpha$  is isomorphic to  $A_1$ , the first Weyl algebra (the algebra of differential operators on  $\mathbb{F}[x]$  with polynomial coefficients), with defining relation  $yx - xy = 1$ . It is well known that all derivations of  $A_1$  are inner (see e.g. [11, 4.6.8]) and thus of the form  $\text{ad}_a$ , for some  $a \in A_1$ , where  $\text{ad}_a(b) = [a, b]$ . Let  $\partial_x = \text{ad}_x$  and  $\partial_y = \text{ad}_y$ . It is easy to see that  $\partial_x, \partial_y \in \text{LND}(A_1)$  and  $A_1^{\partial_x} \cap A_1^{\partial_y} = \mathbb{F}$ . Hence,  $\text{ML}(A_1) = \mathbb{F}$ .

Although it was easy to compute  $\text{ML}(A_\alpha)$  without explicitly determining  $\text{LND}(A_\alpha)$ , in these two cases the invariant in itself is of no use for computing  $\text{Aut}(A_\alpha)$ . However, in [10] and [18] the authors describe the automorphism groups of the polynomial algebra  $A_0$  and of the first Weyl algebra  $A_1$ , respectively, using a characterization of the locally nilpotent derivations of the corresponding algebra. Specifically, given  $\alpha \in \mathbb{F}$  (up to isomorphism, it can be assumed that either  $\alpha = 0$  or  $\alpha = 1$ ), let  $G_\alpha$  be the subgroup of  $\text{Aut}(A_\alpha)$  generated by the affine automorphisms (those which leave the 3-dimensional subspace  $\mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}1$  invariant) and the triangular automorphisms (those of the form  $x \mapsto x, y \mapsto y + p(x)$ , with  $p(x) \in \mathbb{F}[x]$ ).

**Theorem 1 ([10, 8.9] and [18, Théoreme]).** *Assume that  $\text{char}(\mathbb{F}) = 0$  and let  $\alpha \in \mathbb{F}$ . Then, for any  $\partial \in \text{LND}(A_\alpha)$ , there exists  $\Gamma \in G_\alpha$  such that  $\Gamma \circ \partial \circ \Gamma^{-1}(x) = 0$  and  $\Gamma \circ \partial \circ \Gamma^{-1}(y) = p(x)$ , for some  $p(x) \in \mathbb{F}[x]$ .*

**Remark 2.** Using the notation of Theorem 1, in case  $\alpha = 0$  we have

$$\Gamma \circ \partial \circ \Gamma^{-1} = p(x) \frac{d}{dy}$$

and in case  $\alpha \neq 0$  we have  $\Gamma \circ \partial \circ \Gamma^{-1} = \text{ad}_{f(x)}$ , where  $p(x) = -f'(x)$ .

From Theorem 1 it is easy to deduce that  $\text{Aut}(A_\alpha) = G_\alpha$ . For example, in the case of the Weyl algebra  $A_1$  we can argue as follows (compare [10, 8.10]). Let  $\phi \in \text{Aut}(A_1)$  and set  $(u, v) = (\phi(x), \phi(y))$ . Then  $\text{ad}_u = \phi \circ \text{ad}_x \circ \phi^{-1} \in \text{LND}(A_1)$ . By Theorem 1, there exists  $\Gamma \in G_1$  such that  $\Gamma \circ \text{ad}_u \circ \Gamma^{-1} = \text{ad}_{f(x)}$ , for some  $f(x) \in \mathbb{F}[x]$ . Thus,  $\text{ad}_{f(x)} = \text{ad}_{\Gamma(u)}$  and since  $A_1$  has trivial center when  $\text{char}(\mathbb{F}) = 0$ , we deduce that  $\Gamma \circ \phi(x) = g(x)$ , where  $g(x)$  differs from  $f(x)$  by a constant. Moreover, since  $C_{A_1}(x) = \mathbb{F}[x]$ , where  $C_{A_1}$  stands for the centralizer in  $A_1$ , we have that  $C_{A_1}(g(x)) = \mathbb{F}[g(x)]$ . As  $g \notin \mathbb{F}$ , it is easy to see that  $C_{A_1}(g(x)) = \mathbb{F}[x]$ , so  $g(x) = ax + b$  for some  $a, b \in \mathbb{F}$  with  $a \neq 0$ . Now, applying  $\Gamma \circ \phi$  to the defining relation  $[y, x] = 1$ , one concludes that  $\Gamma \circ \phi(y) = a^{-1}y + p(x)$ , for some  $p(x) \in \mathbb{F}[x]$ , which shows that  $\Gamma \circ \phi \in G_1$  and  $\text{Aut}(A_1) = G_1$ . The proof for the polynomial algebra  $A_0$  follows similar reasoning, with a few adaptations.

Our goal in this note is to point out that these ideas apply more generally to a family  $A_h$  of algebras parametrized by arbitrary polynomials  $h \in \mathbb{F}[x]$ . This family was introduced in [6], where the automorphism groups  $\text{Aut}(A_h)$  were studied using different methods.

**Definition 1.** Let  $h \in \mathbb{F}[x]$ . The algebra  $A_h$  is the unital associative algebra over  $\mathbb{F}$  with generators  $x, y$  and defining relation  $[y, x] = h$ , where  $[y, x] = yx - xy$ .

The algebras  $A_h$  include the polynomial algebra as  $A_0$ , the Weyl algebra as  $A_1$ , the enveloping algebra of the two-dimensional non-abelian Lie algebra as  $A_x$ , the Jordan plane as  $A_{x^2}$ , and many others (see [6], [4], [5] for more details on these algebras).

For a general  $h \in \mathbb{F}[x]$ , there are derivations of  $A_h$  that are analogues of the derivations  $p(x) \frac{d}{dy}$  of  $A_0$  and  $\text{ad}_{f(x)}$  of  $A_1$ . Given  $p(x) \in \mathbb{F}[x]$ , the derivation  $D_{p(x)}$  is determined by

$$D_{p(x)}: A_h \rightarrow A_h, \quad D_{p(x)}(x) = 0, \quad D_{p(x)}(y) = p(x). \tag{3}$$

It is easy to see that  $D_{p(x)} \in \text{LND}(A_h)$ . Next, we generalize Theorem 1 to the algebras  $A_h$ . Notice that the result implies that these algebras are more rigid (in the sense of [7]) when  $h \notin \mathbb{F}$ .

**Proposition 2.** Assume that  $\text{char}(\mathbb{F}) = 0$  and let  $h \in \mathbb{F}[x] \setminus \mathbb{F}$ . Then,

$$\text{LND}(A_h) = \{D_{p(x)} \mid p(x) \in \mathbb{F}[x]\}$$

and  $\text{ML}(A_h) = \mathbb{F}[x]$ .

*Proof.* Let  $\partial \in \text{LND}(A_h)$ . Then,

$$\partial(h) = [\partial(y), x] + [y, \partial(x)]. \tag{4}$$

In particular,  $\partial(h) \in [A_h, A_h]$  and by [6, Lem. 6.1],  $[A_h, A_h] \subseteq hA_h$ , so  $\partial(h) = h\theta$ , for some  $\theta \in A_h$ . If  $\partial(h) \neq 0$ , then  $\deg_\partial(h) - 1 = \deg_\partial(\partial(h)) = \deg_\partial(h) + \deg_\partial(\theta)$ , which is a contradiction as  $\deg_\partial$  does not take on the value  $-1$ . Thus,  $\partial(h) = 0$ .

Let  $n$  be the degree of  $h$  as a polynomial in  $x$ . By hypothesis  $n \geq 1$ , and by Proposition 1,  $0 = \deg_\partial(h) = n \deg_\partial(x)$ . Therefore,  $\partial(x) = 0$ . Now, using (4),

we conclude that  $\partial(y) \in C_{A_h}(x) = \mathbb{F}[x]$ , where the last equality comes from [6, Lem. 6.3]. We thus conclude that  $\partial = D_{p(x)}$ , where  $p(x) = \partial(y)$ . The final statement follows from the fact that  $A_h^{D_{p(x)}} = \mathbb{F}[x]$  for all  $0 \neq p(x) \in \mathbb{F}[x]$ .  $\square$

Each  $D_{p(x)} \in \text{LND}(A_h)$  determines a triangular automorphism

$$\phi_{p(x)} = \exp(D_{p(x)})$$

with  $\phi_{p(x)}(x) = x$  and  $\phi_{p(x)}(y) = y + p(x)$ . There are also affine automorphisms  $\tau_{(\alpha,\beta)}$  such that  $\tau_{(\alpha,\beta)}(x) = \alpha x + \beta$  and  $\tau_{(\alpha,\beta)}(y) = \alpha^{n-1}y$ , for every  $\alpha, \beta \in \mathbb{F}$  with  $\alpha \neq 0$  and  $h(\alpha x + \beta) = \alpha^n h(x)$ , where  $n$  is the degree of  $h$  as a polynomial in  $x$  (see [6, Sec. 8] for more details). Let  $G_h$  be the subgroup of  $\text{Aut}(A_h)$  generated by the triangular and the affine automorphisms defined above. As a corollary of Proposition 2 we get the analogue of Jung’s Theorem [13] for the polynomial ring  $A_0$  and of Dixmier’s Theorem [10] for the Weyl algebra  $A_1$ . This result was obtained in [6] using different methods but here we wish to underline the common features and properties of the locally nilpotent derivations of the algebras  $A_h$  as a whole, showing how they fit into the approach used by Dixmier and Rentschler in [10] and [18], respectively, and how their structure under the action of the group  $G_h$  allows for the description of their automorphism groups. Another example of this phenomenon occurs in [2], where the authors study the isomorphisms and automorphisms of a family of generalized Weyl algebras over a polynomial algebra of rank one.

**Corollary 1.** *Assume that  $\text{char}(\mathbb{F}) = 0$  and let  $h \in \mathbb{F}[x] \setminus \mathbb{F}$ . Then  $\text{Aut}(A_h) = G_h$ , i.e.  $\text{Aut}(A_h)$  is generated by the triangular automorphisms  $\phi_{p(x)} = \exp(D_{p(x)})$  and the affine automorphisms  $\tau_{(\alpha,\beta)}$ .*

*Proof.* Let  $\phi \in \text{Aut}(A_h)$  with  $h \in \mathbb{F}[x]$  of degree  $n$ . By Proposition 2,  $\text{ML}(A_h) = \mathbb{F}[x]$ , so there are  $\alpha, \beta \in \mathbb{F}$  with  $\alpha \neq 0$  such that  $\phi(x) = \alpha x + \beta$ . Applying  $\phi$  to the defining relation of  $A_h$  we obtain  $[\phi(y), x] = \alpha^{-1}h(\alpha x + \beta)$ . Let  $\partial = \text{ad}_{-x} = D_h$ . Then the relation obtained implies that  $\text{deg}_{\partial}(\phi(y)) = 1$ . It is not hard to see that the set of  $\theta \in A_h$  with  $\text{deg}_{\partial}(\theta) = 1$  is  $(\mathbb{F}[x]y + \mathbb{F}[x]) \setminus \mathbb{F}[x]$ , so there are  $f, g \in \mathbb{F}[x]$  with  $f \neq 0$  such that  $\phi(y) = fy + g$ . Substituting into  $[\phi(y), x] = \alpha^{-1}h(\alpha x + \beta)$ , we deduce that  $\alpha fh = h(\alpha x + \beta)$ . Hence, comparing the coefficients of  $x^n$  on both sides, we get  $f = \alpha^{n-1} \in \mathbb{F}^*$  and  $\alpha^n h = h(\alpha x + \beta)$ . Finally, notice that  $\phi = \phi_{\alpha^{1-n}g} \circ \tau_{(\alpha,\beta)} \in G_h$ .  $\square$

### 3 Higher derivations of $A_h$

Unless stated otherwise, throughout this section  $\mathbb{F}$  denotes a field of arbitrary characteristic. As remarked after Proposition 1, the fundamental properties of  $\text{deg}_{\partial}$  hold over fields of arbitrary characteristic, except for the multiplicative property.

**Example 2.** Assume that  $\text{char}(\mathbb{F}) = p > 0$ . Then the Weyl algebra has non-inner derivations. One such is  $E_x$ , defined by  $E_x(x) = y^{p-1}$  and  $E_x(y) = 0$ . This derivation is locally nilpotent and  $\text{deg}_{E_x}(x) = 1$ ,  $\text{deg}_{E_x}(y) = 0$ . Since  $A_1$  is a domain, we can still deduce that  $\text{deg}_{E_x}(x^p) \leq p$ , but in fact we have  $E_x(x^p) = -1$ , so  $\text{deg}_{E_x}(x^p) = 1$  (see [5] for more details).

One way of circumventing this problem is to follow along the generalization introduced in [8], motivated by the more geometric notion of an action of the additive group  $G_a$  on a variety  $V$ . From the algebraic point of view, the affine group scheme  $G_a$  is represented by the Hopf algebra  $\mathbb{F}[t]$ , with comultiplication  $\Delta: t \mapsto t \otimes 1 + 1 \otimes t$ , counit  $\epsilon: t \mapsto 0$  and antipode  $S: t \mapsto -t$ . The action of  $G_a$  on  $V$  then corresponds to a  $\mathbb{F}[t]$ -comodule algebra structure on the coordinate ring of  $V$ . This is the setting of Rentschler's Theorem in [18], where his result is phrased in terms of actions of the additive group  $G_a$  on the affine plane, represented by the polynomial ring  $A_0$ .

Let us very briefly explain the connection between this algebraic setting and derivations. Let  $A$  be a unital associative  $\mathbb{F}$ -algebra (not necessarily commutative). Then a (right)  $\mathbb{F}[t]$ -comodule algebra structure on  $A$  is a map  $\delta: A \rightarrow A \otimes \mathbb{F}[t]$  satisfying the following axioms (dualizing the axioms for an action):

- (i)  $\delta$  is an algebra homomorphism;
- (ii)  $(\text{Id}_A \otimes \Delta) \circ \delta = (\delta \otimes \text{Id}_{\mathbb{F}[t]}) \circ \delta$ ;
- (iii)  $(\text{Id}_A \otimes \epsilon) \circ \delta = \mu$ ;

where  $\mu: A \rightarrow A \otimes \mathbb{F}$  is the canonical isomorphism. Given such a map  $\delta$ , write

$$\delta(a) = \sum_{k \geq 0} \partial_k(a) \otimes t^k, \tag{5}$$

where, for each  $a \in A$ , the sum is finite and  $\partial_k: A \rightarrow A$ . Then the above axioms are equivalent to the following properties for all  $a, b \in A$  and  $k \geq 0$ :

- (i)  $\partial_k$  is a linear map;
- (ii)  $\partial_0 = \text{Id}_A$ ;
- (iii)  $\partial_k(ab) = \sum_{i=0}^k \partial_i(a)\partial_{k-i}(b)$ ;
- (iv)  $\partial_k \circ \partial_j = \binom{k+j}{k} \partial_{k+j}$ ;
- (v)  $\{k \geq 0 \mid \partial_k(a) \neq 0\}$  is finite.

A sequence  $\partial = \{\partial_k\}_{k \geq 0}$  satisfying properties (i)–(iii) above is called a higher derivation of  $A$ . If in addition  $\partial$  satisfies (iv), we say that it is iterative, and if it satisfies (v) then we say that it is locally nilpotent. We have thus encoded  $G_a$  group actions and, more generally,  $\mathbb{F}[t]$ -comodule algebra structures, using locally nilpotent iterative higher derivations.

Let  $\partial = \{\partial_k\}_{k \geq 0}$  be a locally nilpotent iterative higher derivation of  $A$ . Notice that, in particular,  $\partial_1$  is a derivation of  $A$ . In case the base field  $\mathbb{F}$  has characteristic 0, it is easy to show that  $\partial_k = \frac{\partial_1^k}{k!}$ , for all  $k \geq 0$ , and it follows that  $\partial_1$  is locally nilpotent as a derivation of  $A$ . Conversely (still in characteristic 0), a locally nilpotent derivation  $\partial_1$  of  $A$  determines the locally nilpotent iterative higher derivation  $\left\{ \frac{\partial_1^k}{k!} \right\}_{k \geq 0}$ . So, over fields of characteristic 0 these two concepts coincide,

but it is not longer so in characteristic  $p > 0$ , as the maps  $\partial_{p^k}$ , for  $k \geq 0$ , are in a sense independent. Specifically, it can be proved that, writing  $k = \sum_{i=0}^m k_i p^i$ , with  $0 \leq k_i < p$ , then

$$\partial_k = \frac{\partial_{p^0}^{k_0} \circ \partial_{p^1}^{k_1} \circ \dots \circ \partial_{p^m}^{k_m}}{k_0! k_1! \dots k_m!}.$$

Therefore, the natural generalization of the Makar-Limanov invariant over fields of positive characteristic uses locally nilpotent iterative higher derivations. We denote the set of such higher derivations of an algebra  $A$  by  $\text{LNIHD}(A)$  and set

$$\text{ML}(A) = \bigcap_{\partial \in \text{LNIHD}(A)} A^\partial, \tag{6}$$

where  $A^\partial = \{a \in A \mid \partial_k(a) = 0, \text{ for all } k \geq 1\}$ . We pursue this further by setting  $N(\partial, a) = \{k \geq 0 \mid \partial_k(a) \neq 0\}$  and

$$\text{deg}_\partial(0) = -\infty, \quad \text{deg}_\partial(a) = \max N(\partial, a), \quad \text{for } 0 \neq a \in A. \tag{7}$$

Since  $\text{Aut}(A)$  acts on  $\text{LNIHD}(A)$  by conjugation, the subalgebra  $\text{ML}(A)$  is invariant under automorphisms.

With these definitions, the new notions coincide with the existing ones when the base field has characteristic 0. It is easy to see that this extended notion of  $\text{deg}_\partial$ , for  $\partial \in \text{LNIHD}(A)$ , still satisfies the additive property from Proposition 1 (b) and, as long as  $A$  is a domain, it satisfies the multiplicative property (a) as well. See [8] for more details, especially in the case that  $A$  is commutative.

Now we return to the algebras  $A_h$  and compute the invariant  $\text{ML}(A_h)$  over a field of arbitrary characteristic.

**Lemma 1.** *Let  $h \in \mathbb{F}[x] \setminus \mathbb{F}$ . Then, for any  $\partial \in \text{LNIHD}(A_h)$ ,  $x \in A_h^\partial$  and  $\partial_k(y)$  commutes with  $x$ , for every  $k \geq 1$ .*

*Proof.* Fix  $\partial = \{\partial_k\}_{k \geq 0} \in \text{LNIHD}(A_h)$  and recall that  $\partial$  defines an algebra homomorphism  $\delta: A_h \rightarrow A_h \otimes \mathbb{F}[t]$  as in (5). Then, applying  $\delta$  to the relation  $h = [y, x]$  we see that  $\delta(h) \in [A_h \otimes \mathbb{F}[t], A_h \otimes \mathbb{F}[t]] \subseteq [A_h, A_h] \otimes \mathbb{F}[t] \subseteq hA_h \otimes \mathbb{F}[t]$ . Thus, for every  $k \geq 0$ ,  $\partial_k(h) = h\theta_k$ , for some  $\theta_k \in A_h$ .

Let  $k \geq 1$  and  $\ell = \text{deg}_\partial(h)$ . For any  $j > \ell - k$ , we have

$$\partial_j(\partial_k(h)) = \binom{k+j}{k} \partial_{k+j}(h) = 0.$$

Thus,

$$\ell + \text{deg}_\partial(\theta_k) = \text{deg}_\partial(h\theta_k) = \text{deg}_\partial(\partial_k(h)) \leq \ell - k.$$

It follows that  $\text{deg}_\partial(\theta_k) = -\infty$ , so  $\theta_k = 0 = \partial_k(h)$ . This shows that  $\text{deg}_\partial(h) = 0$ . Now, as in the proof of Proposition 2, the latter implies that  $\text{deg}_\partial(x) = 0$ , so  $x \in A_h^\partial$ .

Applying  $\delta$  once again to the defining relation of  $A_h$  gives:

$$h \otimes 1 = \delta(h) = \sum_{k \geq 0} [\partial_k(y), x] \otimes t^k,$$

whence the final claim. □



Guided by the above result, below we construct locally nilpotent iterative higher derivations of  $A_h$  which in positive characteristic take the role of the locally nilpotent derivations  $p(x)\frac{d}{dy}$  of  $A_0$ ,  $\text{ad}_{f(x)}$  of  $A_1$ , and  $D_{p(x)}$  of  $A_h$ , as defined in (3). For related results on the polynomial algebra  $A_0$  and on the Weyl algebra  $A_1$  see [17] and [20], respectively.

**Lemma 2.** *Assume that  $\text{char}(\mathbb{F}) = p > 0$  and fix  $h \in \mathbb{F}[x]$ . Let  $P = \{P_i(x)\}_{i \geq 0}$  be a family of elements of  $\mathbb{F}[x]$  such that  $P_i = 0$  for  $i \gg 0$ . Then there is a locally nilpotent iterative higher derivation of  $A_h$ , denoted by  $\partial_P = \{(\partial_P)_k\}_{k \geq 0}$ , such that:*

- (a)  $(\partial_P)_0 = \text{Id}_{A_h}$ ,
- (b)  $(\partial_P)_k(x) = 0$ , for all  $k \geq 1$ ,
- (c)  $(\partial_P)_k(y) = P_i(x)$ , if  $k = p^i$  for some  $i \geq 0$ ,
- (d)  $(\partial_P)_k(y) = 0$  for all other  $k \geq 1$ .

*Proof.* As seen in (5), the claim is tantamount to the statement that

$$\delta = \sum_{k \geq 0} (\partial_P)_k \otimes t^k$$

defines a  $\mathbb{F}[t]$ -comodule algebra structure on  $A_h$ , which is what will be proved.

Let  $c_k = (\partial_P)_k(y)$ , so that  $c_0 = y$ ,  $c_{p^i} = P_i(x)$  and  $c_k = 0$  for all other values of  $k$ . Then, as

$$\left[ \sum_{k \geq 0} c_k \otimes t^k, x \otimes 1 \right] = \sum_{k \geq 0} [c_k, x] \otimes t^k = [y, x] \otimes 1 = h \otimes 1,$$

it follows that there is a unique algebra homomorphism  $\delta: A_h \rightarrow A_h \otimes \mathbb{F}[t]$  such that  $\delta(x) = x \otimes 1$  and  $\delta(y) = \sum_{k \geq 0} c_k \otimes t^k$ . This already shows that  $\partial_P$  is a higher derivation of  $A_h$ , and it is locally nilpotent by the finiteness assumption on  $P = \{P_i(x)\}_{i \geq 0}$ . Hence it remains to prove the iterative property, which is equivalent to the following equality:

$$\delta(c_k) = \sum_{j \geq 0} \binom{k+j}{k} c_{k+j} \otimes t^j, \quad \text{for all } k \geq 0. \tag{8}$$

If  $k = 0$ , then (8) reduces to  $\delta(y) = \delta(c_0)$ , which clearly holds. Otherwise, there is a unique  $i \geq 0$  such that  $p^i \leq k < p^{i+1}$ . Then either  $c_k = P_i(x)$  or  $c_k = 0$ ; regardless,  $\delta(c_k) = c_k \otimes 1$ . Now consider the right-hand side of (8). If  $j = 0$ , the corresponding summand is  $c_k \otimes 1$ , thus we need to show that for  $j > 0$ ,  $\binom{k+j}{k} c_{k+j} = 0$ . Suppose that  $c_{k+j} \neq 0$ . Then, as  $k, j > 0$ , there is  $\ell > i$  such that  $k + j = p^\ell$  and by Lucas' Theorem,

$$\binom{k+j}{k} = \binom{p^\ell}{k} \equiv \prod_{m=0}^{\ell} \binom{a_m}{b_m} \pmod{p},$$

where  $p^\ell = \sum_m a_m p^m$  and  $k = \sum_m b_m p^m$  are the  $p$ -adic expansions. Since  $a_m = 0$  for all  $m < \ell$  and  $p^i \leq k < p^{i+1}$ , it follows that  $a_i = 0$  and  $b_i > 0$ , so  $\binom{a_i}{b_i} = 0$  and  $\binom{k+j}{k} \equiv 0 \pmod{p}$ . This proves that  $\binom{k+j}{k} c_{k+j} = 0$  for all  $j > 0$ , so  $\partial_P$  is iterative.  $\square$

Now we are ready to compute the Makar-Limanov invariant  $\text{ML}(A_h)$  in case  $\text{char}(\mathbb{F}) = p > 0$  and see that this information is enough to describe the automorphism group of  $A_h$ , as in Corollary 1.

**Corollary 2.** *Let  $h \in \mathbb{F}[x] \setminus \mathbb{F}$ . Then,  $\text{ML}(A_h) = \mathbb{F}[x]$  and  $\text{Aut}(A_h) = G_h$ .*

*Proof.* In view of Proposition 2 and Corollary 1, we can assume that  $\text{char}(\mathbb{F}) = p > 0$ . Then, by Lemma 1,  $\mathbb{F}[x] \subseteq \text{ML}(A_h)$ . Now, by Lemma 2, for every  $\ell \geq 0$ , there is  $\partial_P \in \text{LNIHD}(A_h)$  such that  $\deg_{\partial_P}(y) = p^\ell \geq 1$ . For any such higher derivation,  $A_h^{\partial_P} = \mathbb{F}[x]$ , proving equality. Now, as in Corollary 1, this implies that any automorphism of  $A_h$  sends  $x$  to  $\alpha x + \beta$  for some  $\alpha, \beta \in \mathbb{F}$  with  $\alpha \neq 0$ . This result and its analogue for the inverse automorphism imply that  $y$  is sent to  $\alpha^{n-1}y + g$ , for some  $g \in \mathbb{F}[x]$ , as in the proof of Corollary 1.  $\square$

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