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A generalisation of Amitsur's A-polynomials

Adam Owen, Susanne Pumplün

Abstract. We find examples of polynomials $f \in D[t;\sigma,\delta]$ whose eigenring $\mathcal{E}(f)$ is a central simple algebra over the field $F = C \cap \mathrm{Fix}(\sigma) \cap \mathrm{Const}(\delta)$.

Introduction

Let K be a field of characteristic 0 and $R = K[t;\delta]$ be the ring of differential polynomials with coefficients in K. In order to derive results on the structure of the left R-modules R/Rf, Amitsur studied spaces of linear differential operators via differential transformations [2], [3], [4]. He observed that every central simple algebra B over a field F of characteristic 0 that is split by an algebraically closed field extension K of F, is isomorphic to the eigenspace of some polynomial $f \in K[t;\delta]$, for a suitable derivation δ of K. This identification of a central simple algebra B with a suitable differential polynomial $f \in K[t;\delta]$ he called *A-polynomial* also holds when K has prime characteristic p [2, Section 10], [18].

Let D be a central division algebra of degree d over C, σ an endomorphism of D and δ a left σ-derivation of D. Our aim is to provide a partial answer to the following generalisation of Amitsur's investigation:

"For which polynomials f in a skew polynomial ring $D[t;\sigma,\delta]$ is the eigenring $\mathcal{E}(f)$ a central simple algebra over its subfield $F = C \cap \mathrm{Fix}(\sigma) \cap \mathrm{Const}(\delta)$?"

After the preliminaries in Section 1, we investigate two different setups, always assuming that f has degree $m \geq 1$ and that the minimal left divisor of f is square-free. We look at generalised A-polynomials in $D[t;\sigma]$ in Section 2, where σ is an automorphism of D with $\sigma^n = \iota_u$ for some $u \in D^\times$. Then f is a generalised A--polynomial in R if and only if f right divides $u^{-1}t^n - a$ for some $a \in F$ (Theorem 2). If n is prime and not equal to d, then f is a generalised A-polynomial in R if

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Affiliation:
School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom
E-mail: adam.owen@nottingham.ac.uk; susanne.pumpluen@nottingham.ac.uk

and only if one of the following holds: (i) There exists some $a \in F^\times$ such that $ua \neq \prod_{j=1}^{n} \sigma^{n-j}(b)$ for every $b \in D$, and $f(t) = t^n - ua$. In this case f is an irreducible polynomial in R. (ii) $m \leq n$ and there exist $c_1, c_2, \ldots, c_{m-1}, b \in D^\times$, such that $u^{-1} \prod_{j=0}^{n-1} \sigma^{n-j}(b) \in F^\times$, and $f(t) = \prod_{i=1}^{m-1} (t - \Omega_{c_i}(b))(t - \Omega_1(b))$. (Theorem 3). In particular, f is a generalised A-polynomial in $R = K[t;\sigma]$, K a field, if and only if f right divides $t^n - a$ in R (Theorem 4). If moreover n is prime then f is a generalised A-polynomial in $R = K[t;\sigma]$, if and only if one of the following holds: (i) There exists some $a \in F^\times$ such that $a \neq N_{K/F}(b)$ for any $b \in K$, and $f(t) = t^n - a$. In this case f is irreducible. (ii) $m \leq n$ and there exist some constants $c_1, c_2, \ldots, c_{m-1}, b \in K^\times$, such that $f(t) = \prod_{i=1}^{m-1} (t - \Omega_{c_i}(b))(t - \Omega_1(b))$ (Corollary 1).

In Section 3, we study generalised A-polynomials in $D[t;\delta]$, where C has prime characteristic p and δ is an algebraic derivation of D with minimum polynomial $g(t) \in F[t]$ of degree p^e such that $g(\delta) = \delta_c$ for some nonzero $c \in D$. Then f is a generalised A-polynomial in $D[t;\delta]$ if and only if f right divides $g(t) - (b + c)$ for some $b \in F$. In particular, $\deg(f) \leq p^e$ (Theorem 6). In the special case that $g(t) = t^p - at$, f is a generalised A-polynomial in R if and only if one of the following holds: (i) $f(t) = h(t) = t^p - at - (b+c)$, and $V_p(\alpha) - a\alpha - (b+c) \neq 0$ for all $\alpha \in D$. In this case f is irreducible in R. (ii) $h(t) = t^p - at - (b + c)$ for some $a, b \in F$, $m \leq p$ and $f(t) = \prod_{i=1}^{m-1} (t - \Omega_{c_i}(\alpha))(t - \Omega_1(\alpha))$ for some $c_1, c_2, \ldots, c_{m-1}\alpha \in D^\times$, such that $V_p(\alpha) - a\alpha - (b + c) = 0$ (Theorem 7).

The results are part of the first author's PhD thesis written under the supervision of the second author.

1 Preliminaries

1.1 Skew Polynomial Rings ([12], [13], [15], [16])

Let D be a unital associative division algebra over its center C, σ an endomorphism of D, and δ a left σ-derivation of D, i.e. δ is an additive map on D satisfying $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$ for all $x, y \in D$. For $u \in D^\times$, $\iota_u(a) = uau^{-1}$ is called an *inner automorphism* of D. If there exists $n \in \mathbb{Z}^+$ such that $\sigma^n = \iota_u$ for some $u \in D^\times$, and σ^i is a not an inner derivation for $1 \leq i < n$, then σ is said to have *finite inner order* n. For $c \in D$, the derivation $\delta_c(a) = [c, a] = ca - ac$ for all $a \in D$ is called an *inner derivation*. The *skew polynomial ring* $R = D[t;\sigma,\delta]$ is the set of skew polynomials $a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ with $a_i \in D$, endowed with term-wise addition and multiplication defined by $ta = \sigma(a)t + \delta(a)$ for all $a \in D$. R is a unital associative ring. If $\delta = 0$, we write $R = D[t;\sigma]$. If $\sigma = \mathrm{id}_D$, we write $R = D[t;\delta]$.

For $f(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ with $a_m \neq 0$, the *degree* of f, denoted by $\deg(f)$, is m, and by convention $\deg(0) = -\infty$. If $a_m = 1$, we call f *monic*. We have $\deg(fg) = \deg(f) + \deg(g)$ and $\deg(f + g) \leq \max(\deg(f), \deg(g))$ for all $f, g \in R$. A polynomial $f \in R$ is called *reducible* if $f = gh$ for some $g, h \in R$ such that $\deg(g), \deg(h) < \deg(f)$, otherwise we call f *irreducible*. A polynomial

$f \in R$ is called *right (resp. left) invariant* if $fR \subseteq Rf$ (resp. $Rf \subseteq fR$), i.e. Rf (resp. fR) is a two-sided ideal of R. We call f *invariant* if it is both right and left invariant. Two skew polynomials $f, g \in R$ are *similar*, written $f \sim g$, if $R/Rf \cong R/Rg$.

R is a left principal ideal domain. The *left idealiser* $\mathcal{I}(f) = \{g \in R : fg \in Rf\}$ of $f \in R$ is the largest subring of R within which Rf is a two-sided ideal. We define the *eigenring* of f as $\mathcal{E}(f) = \mathcal{I}(f)/Rf = \{g \in R : \deg(g) < m \text{ and } fg \in Rf\}$. A nonzero $f \in R$ is said to be *bounded* if there exists another nonzero skew polynomial $f^\star \in R$, called a *bound* of f, such that $Rf^\star$ is the unique largest two-sided ideal of R contained in the left ideal Rf. Equivalently, a nonzero polynomial in $f \in R$ is said to be bounded if there exists a right invariant polynomial $f^\star \in R$, which is called a bound of f, such that $Rf^\star = \mathrm{Ann}_R(R/Rf) \neq \{0\}$. The annihilator $\mathrm{Ann}_R(R/Rf)$ of the left R-module R/Rf is a two-sided ideal of R. When f is bounded and of positive degree, the nontrivial zero divisors in the eigenspace of f are in one-to-one correspondence with proper right factors of f in R: If f is bounded and $\sigma \in \mathrm{Aut}(D)$, then f is irreducible if and only if $\mathcal{E}(f)$ has no non-trivial zero divisors. Each non-trivial zero divisor q of f in $\mathcal{E}(f)$ gives a proper factor $\mathrm{gcrd}(q, f)$ of f [10, Lemma 3, Proposition 4].

If D has finite dimension as an algebra over its center C, then $R = D[t;\sigma,\delta]$ is either a twisted polynomial ring or a differential polynomial ring [13, Theorem 1.1.21].

1.2 Generalized A-polynomials

Unless stated otherwise, from now on let D be a unital associative division ring with center C, $\sigma \in \mathrm{End}(D)$, δ a left σ-derivation of D, and let $F = C \cap \mathrm{Fix}(\sigma) \cap \mathrm{Const}(\delta)$. We are interested in the question:

"For $f \in R = D[t;\sigma,\delta]$ when is $\mathcal{E}(f)$ a central simple algebra over the field F?"

We call $f \in R$ a *generalised A-polynomial* if $\mathcal{E}(f)$ is a central simple algebra over F. For each $v \in D^\times$, we define a map $\Omega_v : D \longrightarrow D$ by $\Omega_v(\alpha) = \sigma(v)\alpha v^{-1} + \delta(v)v^{-1}$.

Lemma 1. *[2, Lemma 2 for $\sigma = \mathrm{id}$] Let $\alpha, \beta \in D$. Then $(t-\alpha) \sim (t-\beta)$ in $D[t;\sigma,\delta]$ if and only if $\Omega_v(\alpha) = \beta$ for some $v \in D^\times$.*

Proof. $(t-\alpha) \sim (t-\beta)$ is equivalent to the existence of $v, w \in D^\times$ such that $w(t-\alpha) = (t-\beta)v$ [14, pg. 33], i.e. there exists $v, w \in D^\times$ such that $w(t-\alpha) = \sigma(v)t + \delta(v) - \beta v$. This is the case if and only if $w = \sigma(v)$ and $w\alpha = \sigma(v)\alpha = \beta v - \delta(v)$. The result follows immediately. □

2 Generalised A-polynomials in $D[t;\sigma]$

Let D be a central division algebra over C of degree d and σ an automorphism of D of finite inner order n, with $\sigma^n = \iota_u$ for some $u \in D^\times$. Let $R = D[t;\sigma]$. Then R has center $F[u^{-1}t^n] \cong F[x]$. We define the *minimal central left multiple of f in R* to be the unique polynomial of minimal degree $h \in C(R) = F[u^{-1}t^n]$ such that $h = gf$ for some $g \in R$, and such that $h(t) = \hat{h}(u^{-1}t^n)$ for some monic $\hat{h}(x) \in F[x]$.

If the greatest common right divisor $(f,t)_r$ of f and t is one, then $f^\star \in C(R)$ [10, Lemma 2.11]), and the minimal central left multiple of f equals $f^\star$ up to a scalar multiple in $D^\times$. For the remainder of this section we therefore assume that $f \in R$ is a monic polynomial of degree $m \geq 1$ such that $(f,t)_r = 1$. Then $f^\star \in C(R)$. Define $E_{\hat{h}} = F[x]/(\hat{h}(x))$. $E_{\hat{h}} = F[x]/(\hat{h}(x))$ is a field if and only if $\hat{h}(x) \in F[x]$ is irreducible.

Since $F[x]$ is a unique factorisation domain, we have

$$\hat{h}(x) = \hat{\pi}_1^{e_1}(x)\hat{\pi}_2(x)^{e_2}\cdots\hat{\pi}_z(x)^{e_z}$$

for some irreducible polynomials $\hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_z \in F[x]$ such that $\hat{\pi}_i \neq \hat{\pi}_j$ for $i \neq j$, and some exponents $e_1, e_2, \dots, e_z \in \mathbb{N}$. Henceforth we assume that $e_1 = e_2 = \cdots = e_z = 1$, i.e. that $\hat{h}$ is square-free. By the Chinese Remainder Theorem for commutative rings [9, §5] $E_{\hat{h}} \cong E_{\hat{\pi}_1} \oplus E_{\hat{\pi}_2} \oplus \cdots \oplus E_{\hat{\pi}_z}$, where $E_{\hat{\pi}_i} = F[x]/(\hat{\pi}_i(x))$ for each i. $\mathcal{E}(f)$ is a semisimple algebra over its center $E_{\hat{h}}$ [17]. Thus $\mathcal{E}(f)$ has center F if and only if $z = 1$ and $E_{\hat{\pi}_1} = F$, i.e. if and only if $\hat{h}$ is a degree 1 polynomial in $F[x]$. Hence under the global assumption that $\hat{h}$ is square-free, we see that for f to be a generalised A-polynomial it is necessary that $\hat{h}$ be irreducible. So assume that $\hat{h}$ is irreducible. Then the eigenspace of f is a central simple algebra over the field $E_{\hat{h}}$:

Theorem 1. *[17] Suppose that $\hat{h}(x)$ is irreducible in $F[x]$. Then $f = f_1 f_2 \cdots f_l$ where $f_1, f_2, \dots, f_l$ are irreducible polynomials in R such that $f_i \sim f_j$ for all i, j. Moreover,*

$$\mathcal{E}(f) \cong M_\ell(\mathcal{E}(f_i))$$

is a central simple algebra of degree $s = \frac{\ell d n}{k}$ over the field $E_{\hat{h}}$ where k is the number of irreducible factors of $h(t) \in R$. In particular, $deg(\hat{h}) = \deg(h)/n = \frac{dm}{s}$ and $[\mathcal{E}(f) : F] = mds$.

Theorem 2. *Suppose that $\hat{h}(x)$ is irreducible in $F[x]$. Then f is a generalised A-polynomial in R if and only if $\hat{h}(x) = x - a$ for some $a \in F$ if and only if f right divides $u^{-1}t^n - a$ for some $a \in F$. In particular, if f is a generalised A-polynomial, then $m \leq n$.*

Proof. Suppose that f is a generalised A-polynomial in R. By the paragraph preceding Theorem 1, for f to be a generalised A-polynomial it is necessary that $\hat{h}(x) = x - a$ for some $a \in F$. Conversely if $\hat{h}(x) = x - a \in F[x]$, then $E_{\hat{h}} = F[x]/(x-a) = F$. Hence $\mathcal{E}(f)$ is a central simple algebra over F by Theorem 1, i.e. f is a generalised A-polynomial. It is easy to see that $\hat{h}(x) = x - a$ is equivalent to f being a right divisor of $u^{-1}t^n - a$ by definition of the minimal central left multiple. Moreover, if f right divides $u^{-1}t^n - a$, then $\deg(f) \leq n$. □

For n prime we are able to provide a more concrete description of f:

Theorem 3. *Suppose that $\hat{h}(x)$ is irreducible in $F[x]$. Suppose that n is prime and not equal to d. Then f is a generalised A-polynomial in R if and only if one of the following holds:*

1. *There exists some* $a \in F^\times$ *such that* $ua \neq \prod_{j=1}^{n} \sigma^{n-j}(b)$ *for every* $b \in D$, *and* $f(t) = t^n - ua$. *In this case* f *is an irreducible polynomial in* R.
2. $m \leq n$ *and there exist* $c_1, c_2, \ldots, c_{m-1}, b \in D^\times$, *such that*
$$u^{-1} \prod_{j=0}^{n-1} \sigma^{n-j}(b) \in F^\times\,, \quad \text{and} \quad f(t) = \prod_{i=1}^{m-1} (t - \Omega_{c_i}(b))(t - \Omega_1(b))\,.$$

Proof. By Theorem 2, f is a generalised A-polynomial in R if and only if f right divides $u^{-1}t^n - a$ for some $a \in F^\times$. So suppose that f is a generalised A-polynomial in R, then there exists some $a \in F^\times$ and some nonzero $g \in R$ such that
$$u^{-1}t^n - a = gf. \tag{1}$$
In the notation of Theorem 1, $\ell dn = ks$ and since f is a generalised A-polynomial $\deg(\hat{h}) = \frac{dm}{s} = 1$, i.e. $dm = s$. Combining these yields $\frac{n}{k} = \frac{m}{\ell} \in \mathbb{N}$. That is k must divide n, and so we must have that $k = 1$ or $k = n$ as n is prime. We analyse the cases $k = 1$ and $k = n$ separately.

First suppose that $k = 1$, then $h(t)$ is irreducible. Therefore Equation (1) becomes $u^{-1}t^n - a = gf(t)$ for some $a \in F^\times$ and some $g \in D^\times$. This yields $g = u^{-1}$ and $f(t) = t^n - ua$ for some $a \in F^\times$. Suppose that f were reducible, then f would be the product of n linear factors as n is prime, hence f is irreducible if and only if $ua \neq \prod_{j=1}^{n} \sigma^{n-j}(b)$ for any $b \in D$, by [7, Corollary 3.4].

On the other hand, if $k = n$, then $h(t)$ is equal to a product of n linear factors in R, all of which are similar. Also, since $\frac{n}{k} = \frac{m}{\ell}$ and $n = k$, we have $m = \ell \leq n$. Hence f is the product of $m \leq n$ linear factors in R, all of which are similar to each other.

So there exist constants $b_1, b_2, \ldots, b_m \in D^\times$ such that $(t - b_i) \sim (t - b_j)$ for all $i, j \in \{1, 2, \ldots, m\}$, and $f(t) = \prod_{i=1}^{m} (t - b_i)$. In particular $(t - b_i) \sim (t - b_m)$ for all $i \neq m$, which is true if and only if there exist constants $c_1, c_2, \ldots, c_{m-1}, c_m \in D^\times$ such that $b_i = \Omega_{c_i}(b_m)$ for all i by Lemma 1. Hence setting $b = b_m$ and $c_m = 1$ yields $f(t) = \prod_{i=1}^{m} (t - \Omega_{c_i}(b))$. Finally, we note that $(t-b)|_r(t^n - ua)$ for some $a \in F^\times$ if and only if $u^{-1} \prod_{j=0}^{n-1} \sigma^{n-j}(b) = a \in F^\times$, by [7, Corollary 3.4]. □

If $e_i > 1$ for at least one i, then it is not clear to the authors when $\mathcal{E}(f)$ is a central simple algebra over the field F.

2.1 Generalised A-polynomials in $K[t;\sigma]$

Throughout this section we suppose that $R = K[t;\sigma]$ with K a field, and that σ is an automorphism of K of finite order n with fixed field F. Now the center of R is $F[t^n] \cong F[x]$. Let $f \in R$ be of degree $m \geq 1$ and satisfy $(f, t)_r = 1$, and suppose that f has minimal central left multiple $h(t) = \hat{h}(t^n)$, $\hat{h} \in F[x]$ an irreducible monic polynomial. Again, we consider only those $f \in R$ where $\hat{h} \in F[x]$ is square-free.

Theorem 4. *f is a generalised A-polynomial in R if and only if $\hat{h}(x) = x - a$ for some $a \in F[x]$ if and only if f right divides $t^n - a$ in R.*

This follows from Theorem 2. If n is prime, then the following is an immediate corollary to both Theorem 2 and Theorem 3:

Corollary 1. *Let n be prime. Then f is a generalised A-polynomial in R if and only if one of the following holds:*

1. *There exists some $a \in F^\times$ such that $a \neq N_{K/F}(b)$ for any $b \in K$, and $f(t) = t^n - a$. In this case f is an irreducible polynomial in R.*
2. *$m \leq n$ and there exist some constants $c_1, c_2, \dots, c_{m-1}, b \in K^\times$, such that $f(t) = \prod_{i=1}^{m-1} (t - \Omega_{c_i}(b))(t - \Omega_1(b))$.*

Proof. The proof is identical to the proof of Theorem 3 with $d = u = 1$. The condition that $\prod_{j=0}^{n-1} \sigma^j(b)$ lies in $F^\times$ is always satisfied, since $\prod_{j=0}^{n-1} \sigma^j(b) = N_{K/F}(b) \in F^\times$ for all $b \neq 0$. $\square$

In particular, let $K = \mathbb{F}_{q^n}$, where $q = p^e$ for some prime p and exponent $e \geq 1$, and where $\sigma : K \longrightarrow K$, $a \mapsto a^q$ is the Frobenius automorphism of order n, with fixed field $F = \mathbb{F}_q$. Here the only central division algebra over $\mathbb{F}_q$ is $\mathbb{F}_q$ itself. The following result is therefore an easy consequence of Theorems 1 and 2:

Corollary 2. *Suppose that $f \in \mathbb{F}_{q^n}[t, \sigma]$ satisfies $(f, t)_r = 1$, and has minimal central left multiple $h(t) = \hat{h}(t^n)$ for some irreducible polynomial $\hat{h} \in \mathbb{F}_q[x]$. Then f is an A-polynomial if and only if $m \leq n$ and there exist some constants $c_1, c_2, \dots, c_{m-1}, b \in \mathbb{F}_{q^n}^\times$, such that $f(t) = \prod_{i=1}^{m-1} (t - \Omega_{c_i}(b))(t - \Omega_1(b))$. In particular, f is a reducible polynomial in $\mathbb{F}_{q^n}[t, \sigma]$, unless $m = 1$.*

The result follows identically to the $n = k$ case in the proof of Theorem 3.

3 Generalised A-polynomials in $D[t; \delta]$

From now on let $R = D[t; \delta]$ where D is a central division algebra of degree d over C. Assume that C has prime characteristic p, and that δ is an algebraic derivation of D with minimum polynomial $g(t) = t^{p^e} + \gamma_1 t^{p^{e-1}} + \dots + \gamma_e t \in F[t]$, such that $g(\delta)(a) = [c, a] = ca - ac$ for some nonzero $c \in D$ and for all $a \in D$. Here, $F = C \cap \mathrm{Const}(\delta)$ ($D = K$ is a field is included here as special case). Then R has center $F[g(t) - c] \cong F[x]$. For every $f \in R$, the *minimal central left multiple of f in R* is the unique polynomial of minimal degree $h \in C(R) = F[x]$ such that $h = gf$ for some $g \in R$, and such that $h(t) = \hat{h}(g(t) - c)$ for some monic $\hat{h}(x) \in F[x]$. All $f \in R = D[t; \delta]$ have a unique minimal central left multiple, which is a bound of f.

Again we can restrict our investigation to the case $\hat{h}$ is square-free in $F[x]$, and note that it is necessary that $\hat{h}$ be irreducible in $F[x]$ for f to be a generalised A-polynomial in R.

Theorem 5. *[17] Suppose that $\hat{h}(x)$ is irreducible in $F[x]$. Then $f = f_1 f_2 \cdots f_l$ where $f_1, f_2, \ldots, f_l$ are irreducible polynomials in R such that $f_i \sim f_j$ for all i, j. Moreover,*

$$\mathcal{E}(f) \cong M_\ell(\mathcal{E}(f_i))$$

is a central simple algebra of degree $s = \frac{\ell d p^e}{k}$ over the field $E_{\hat{h}}$ where k is the number of irreducible factors of $h \in R$. In particular $\deg(\hat{h}) = \deg(h)/p^e = \frac{dm}{s}$ and $[\mathcal{E}(f) : F] = mds$.

We obtain the following:

Theorem 6. *Suppose that $\hat{h}(x)$ is irreducible in $F[x]$. Then f is a generalised A-polynomial in R if and only if f right divides $g(t) - (b + c)$ for some $b \in F$. In particular, $\deg(f) \leq p^e$.*

Proof. Suppose that f is a generalised A-polynomial in R. For f to be a generalised A-polynomial it is necessary that $\hat{h}(x) = x - b$ for some $b \in F$. Conversely if $\hat{h}(x) = x - b \in F[x]$, then $E_{\hat{h}} = F[x]/(x - b) = F$. Hence $\mathcal{E}(f)$ is a central simple algebra over F by Theorem 5, i.e. f is a generalised A-polynomial. It is easy to see that $\hat{h}(x) = x - b$ is equivalent to f being a right divisor of $g(t) - (b + c)$ by definition of the minimal central left multiple. Moreover, if f right divides $g(t) - (b + c)$, then $\deg(f) \leq \deg(g(t) - (b + c)) = p^e$. $\square$

In $D[t; \delta]$, we have $(t - b)^p = t^p - V_p(b)$, $V_p(b) = b^p + \delta^{p-1}(b) + \nabla_b$ for all $b \in D$, where ∇_b is a sum of commutators of $b, \delta(b), \delta^2(b), \ldots, \delta^{p-2}(b)$ [13, pg. 17–18]. In particular, if D is commutative, then $\nabla_b = 0$ and $V_p(b) = b^p + \delta^{p-1}(b)$ for all $b \in D$. Using the identities $t^p = (t - b)^p + V_p(b)$ and $t = (t - b) + b$ for all $b \in D$, we arrive at:

Lemma 2. *[13, Proposition 1.3.25 (for $e = 1$)] Let $f(t) = t^p - a_1 t - a_0 \in D[t; \delta]$ and $b \in D$. Then $(t - b)|_r f(t)$ if and only if $V_p(b) - a_1 b - a_0 = 0$.*

If $e = 1$ (i.e. δ is an algebraic derivation of D of degree p), we can determine necessary and sufficient conditions for f to be an A-polynomial in R:

Theorem 7. *Let δ be an algebraic derivation of D of degree p with minimum polynomial $g(t) = t^p - at$ such that $g(\delta) = \delta_c$ for some $c \in D$. Suppose that $\hat{h}(x)$ is irreducible in $F[x]$. Then f is a generalised A-polynomial in R if and only if one of the following holds:*

1. *$f(t) = h(t) = t^p - at - (b + c)$, and $V_p(\alpha) - a\alpha - (b + c) \neq 0$ for all $\alpha \in D$. In this case f is irreducible in R.*
2. *$h(t) = t^p - at - (b + c)$ for some $a, b \in F$, $m \leq p$ and*

$$f(t) = \prod_{i=1}^{m-1} (t - \Omega_{c_i}(\alpha))(t - \Omega_1(\alpha))$$

for some $c_1, c_2, \ldots, c_{m-1}\alpha \in D^\times$, such that $V_p(\alpha) - a\alpha - (b + c) = 0$.

Proof. By Theorem 6, f is a generalised A-polynomial in R if and only if f right divides $t^p - at - (b+c)$ for some $b \in F$. So suppose that f is a generalised A--polynomial in R, then there exists some $b \in F$ and some nonzero $f' \in R$ such that

$$t^p - at - (b+c) = f'f \tag{2}$$

In the notation of Theorem 5, $\ell dp = ks$ and since f is a generalised A-polynomial, $\deg(\hat{h}) = \frac{dm}{s} = 1$, i.e. $dm = s$. Combining these yields $\frac{p}{k} = \frac{m}{\ell} \in \mathbb{N}$. That is k must divide p, and so we must have that $k = 1$ or $k = p$ as p is prime.

First suppose that $k = 1$, then $h(t)$ is irreducible in R. Therefore Equation (2) becomes $t^p - at - (b+c) = f'f$ for some $b \in F^\times$ and some $f' \in D^\times$. This yields $f' = 1$ and $f(t) = t^p - at - (b+c)$. Suppose that f were reducible, then f would be the product of p linear factors as p is prime, hence f is irreducible if and only if $V_p(\alpha) - a\alpha - (b+c) \neq 0$ for any $\alpha \in D$, by Lemma 2.

On the other hand, if $k = p$, then $h(t)$ is equal to a product of p linear factors in R, all of which are similar to one another. Also, since $\frac{p}{k} = \frac{m}{\ell}$ and $p = k$, we have $m = \ell \leq p$. Hence f is the product of $m \leq p$ linear factors in R, all of which are mutually similar to each other.

So there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_m \in D^\times$ such that $f(t) = \prod_{i=1}^{m} (t - \alpha_i)$, and $(t-\alpha_i) \sim (t-\alpha_j)$ for all $i, j \in \{1, 2, \ldots, m\}$. In particular $(t-\alpha_i) \sim (t-\alpha_m)$ for all $i \neq m$, which is true if and only if there exist constants $c_1, c_2, \ldots, c_{m-1}, c_m \in D^\times$ such that $\alpha_i = \Omega_{c_i}(\alpha_m)$ for all i by Lemma 1. Hence setting $\alpha = \alpha_m$ and $c_m = 1$ yields $f(t) = \prod_{i=1}^{m} (t - \Omega_{c_i}(\alpha))$. Finally, we note that $(t-\alpha)$ right divides $t^p - at - (b+c)$ if and only if $V_p(\alpha) - a\alpha - (b+c) = 0$ by Lemma 2. $\square$

Remark 1. Suppose on the other hand that C has characteristic 0 and δ is the inner derivation δ_c. Then R has center $C[t-c] \cong C[x]$. i.e. $F = C$. In this case the A-polynomials are trivial: if $\hat{h}(x)$ is irreducible in $C[x]$ then f is a generalised A-polynomial in R if and only if $f(t) = (t-c) + a$ for some $a \in C$. In this case, $\mathcal{E}(f) = D$.

References

[1] A.A. Albert: On nonassociative division algebras. *Transactions of the American Mathematical Society* 72 (2) (1952) 296–309.

[2] A.S. Amitsur: Differential polynomials and division algebras. *Annals of Mathematics* (1954) 245–278.

[3] A.S. Amitsur: Non-commutative cyclic fields. *Duke Mathematical Journal* 21 (1) (1954) 87–105.

[4] A.S. Amitsur: Generic splitting fields of central simple algebras. *Annals of Mathematics* 62 (2) (1955) 8–43.

[5] N. Bourbaki: *Elements of mathematics*. Springer (2003).

[6] C. Brown, S. Pumplün: How a nonassociative algebra reflects the properties of a skew polynomial. *Glasgow Mathematical Journal* 63 (1) (2021) 6–26.

[7] C. Brown: Petit algebras and their automorphisms. PhD Thesis, University of Nottingham. Online at arXiv:1806.00822 (2018)

[8] J. Carcanague: Quelques résultats sur les anneaux de Ore. *CR Acad. Sci. Paris Sr. AB* 269 (1969) A749–A752.

[9] P.M. Cohn: Noncommutative unique factorization domains. *Transactions of the American Mathematical Society* 109 (2) (1963) 313–331.

[10] J. Gòmez-Torrecillas, F. J. Lobillo, G. Navarro: Computing the bound of an Ore polynomial. Applications to factorization. *Journal of Symbolic Computation* 92 (2019) 269–297.

[11] J. Gòmez-Torrecillas: Basic module theory over non-commutative rings with computational aspects of operator algebras. With an appendix by V. Levandovskyy.. In: M. Barkatou, T. Cluzeau, G. Regensburger, M. Rosenkranz: *Algebraic and Algorithmic Aspects of Differential and Integral Operators. AADIOS 2012. Lecture Notes in Computer Science*. Springer (2014) 23–82.

[12] K. R. Goodearl, J. W. Bruce, R. B. Warfield: *An introduction to noncommutative Noetherian rings*. Cambridge University Press (2004).

[13] N. Jacobson: *Finite-dimensional division algebras over fields*. Springer (1996).

[14] N. Jacobson: *The theory of rings*. American Mathematical Society (1943).

[15] J.C. McConnell, C.J. Robson, L.W. Small: *Noncommutative noetherian rings*. American Mathematical Soc. (2001).

[16] O. Ore: Theory of non-commutative polynomials. *Annals of Mathematics* (1933) 480–508.

[17] A. Owen: On the right nucleus of Petit algebras. PhD Thesis, University of Nottingham, in preparation.

[18] S. Pumplün: Algebras whose right nucleus is a central simple algebra. *Journal of Pure and Applied Algebra* 222 (9) (2018) 2773–2783.

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