Communications in Mathematics 29 (2021) 395–405
DOI: 10.2478/cm-2021-0027

G-tridiagonal majorization on $\mathbf{M}_{n,m}$

Ahmad Mohammadhasani, Yamin Sayyari, Mahdi Sabzvari

Abstract. For $X, Y \in \mathbf{M}_{n,m}$, it is said that X is *g-tridiagonal* majorized by Y (and it is denoted by $X \prec_{gt} Y$) if there exists a tridiagonal g-doubly stochastic matrix A such that $X = AY$. In this paper, the linear preservers and strong linear preservers of $\prec_{gt}$ are characterized on $\mathbf{M}_{n,m}$.

1 Introduction

One of the most interesting problems in linear algebra is called a preserver problem. With the development of majorization problem, preserving majorization have attracted much attention of mathematicians as an active subject of research in linear algebra. For more information we refer the reader to [3], and [5]. For complete references on majorization, we refer the reader to books by Bahatia [4] and Marshall, Olkin, and Arnold [9].

In this work, we study some kind of majorization and we try to find its (strong) linear preservers on matrices. A tridiagonal matrix is a band matrix that has nonzero elements only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal.

An n-by-n real matrix (not necessarily nonnegative) A is *g-doubly stochastic* (generalized doubly stochastic) if all its row and column sums are one. Let $X, Y \in \mathbf{M}_{n,m}$. The matrix X is said to be gt-majorized by Y and it is denoted by $X \prec_{gt} Y$, if there exists an n-by-n tridiagonal g-doubly stochastic matrix A such that $X = AY$.

2020 MSC: 15A04, 15A21
Key words: G-doubly stochastic matrix, gt-majorization, (strong) linear preserver, tridiagonal matrices.
Affiliation:
Ahmad Mohammadhasani – Department of Mathematics, Sirjan University Of Technology, Sirjan, Iran
E-mail: a.mohammadhasani53@gmail.com
Yamin Sayyari – Department of Mathematics, Sirjan University Of Technology, Sirjan, Iran
E-mail: y.sayyari@sirjantech.ac.ir
Mahdi Sabzvari – Department of Mathematics, Hormozgan University, Hormozgan, Iran
E-mail: sabzevari@hormozgan.ac.ir

Some of our notations and symbols are explained as the following.
$\mathbf{M}_{n,m}$: the set of all n-by-m real matrices.
$\mathbf{M}_n$: the abbreviation of $\mathbf{M}_{n,n}$.
$\mathbb{R}^n$: the set of all n-by-1 real column vectors.
$\{e_1, \dots, e_n\}$: the standard basis of $\mathbb{R}^n$.
E_{ij}: the n-by-n matrix whose (i,j) entry is one and all other entries are zero.
$[X_1 \mid \dots \mid X_m]$: the n-by-m matrix with columns $X_1, \dots, X_m \in \mathbb{R}^n$.
$tr(x)$: the summation of all components of a vector x in $\mathbb{R}^n$.
$\mathbb{N}_k$: the set $\{1, \dots, k\} \subset \mathbb{N}$.
A^t: the transpose of a given matrix A.
$[T]$: the matrix representation of a linear operator
$T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ with respect to the standard basis.
J: the matrix with all entries equal to one.
e: the vector with all entries equal to one.
P: the backward identity matrix.
$(A)_i$: the i^{th} column of the matrix A.
Ω_n^t: the set of all n-by-n tridiagonal g-doubly stochastic matrices.

$$A_\mu = \begin{pmatrix} 1-\mu_1 & \mu_1 & & & 0 \\ \mu_1 & 1-\mu_1-\mu_2 & \mu_2 & & \\ & & \ddots & & \\ & & & & \mu_{n-1} \\ 0 & & & \mu_{n-1} & 1-\mu_{n-1} \end{pmatrix},$$

where $\mu = (\mu_1, \dots, \mu_{n-1})^t \in \mathbb{R}^{n-1}$. It is easy to show that $\Omega_n^t = \{A_\mu | \mu \in \mathbb{R}^{n-1}\}$. A linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserves a relation $\prec$ in $\mathbf{M}_{n,m}$, if $TX \prec TY$ whenever $X \prec Y$. Also, T is said to strongly preserve $\prec$ if for all X, $Y \in \mathbf{M}_{n,m}$

$$X \prec Y \Leftrightarrow TX \prec TY.$$

For $x, y \in \mathbb{R}^n$, it is said that x is *g-tridiagonal* majorized by y (denoted by $x \prec_{gt} y$) if there exists some $A \in \Omega_n^t$ such that $x = Ay$.

In [1] and [2], the authors found the strong linear preservers of $\prec_{gt}$ on $\mathbb{R}^n$ and linear preservers of $\prec_{gt}$ on $\mathbb{R}^n$, respectively, as follows.

Lemma 1. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then T strongly preserves $\prec_{gt}$ if and only if there exist $a, b \in \mathbb{R}$ such that $(a-b)(a+(n-1)b) \neq 0$ and $[T]$ is one of the following matrices*

$$\begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b & b & b & \cdots & a \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} b & b & \cdots & b & a \\ b & b & \cdots & a & b \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a & b & \cdots & b & b \end{pmatrix}.$$

In other words T strongly preserves $\prec_{gt}$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha(\alpha + n\beta) \neq 0$ and $[T] = \alpha I + \beta \mathbf{J}$ or $[T] = \alpha P + \beta \mathbf{J}$.

Theorem 1. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then T preserves $\prec_{gt}$ if and only if there exist α, $\beta \in \mathbb{R}$ and $a \in \mathbb{R}^n$ such that one of the following holds.*

(i) $Tx = tr(x)a$, $\forall x \in \mathbb{R}^n$.

(ii) $Tx = \alpha x + \beta Jx$, $\forall x \in \mathbb{R}^n$.

(iii) $Tx = \alpha Px + \beta Jx$, $\forall x \in \mathbb{R}^n$.

In this paper, we characterize all of (strong) linear preservers of $\prec_{gt}$ on $\mathbf{M}_{n,m}$, as follows.

Theorem 2. *A linear operator $T\colon \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ preserves $\prec_{gt}$ if and only if T satisfies one of the following conditions.*

(I) *There exist $A_1, A_2, \dots, A_m \in \boldsymbol{M}_{n,m}$ such that*

$$TX = \sum_{j=1}^{m}(\sum_{i=1}^{n} x_{ij})A_j, \quad \forall X = [x_{ij}] \in \boldsymbol{M}_{n,m}.$$

(II) *There exist $R, S \in \boldsymbol{M}_m$ such that*

$$TX = XR + JXS, \quad \forall X \in \boldsymbol{M}_{n,m}.$$

(III) *There exist $R, S \in \boldsymbol{M}_m$ such that*

$$TX = PXR + JXS, \quad \forall X \in \boldsymbol{M}_{n,m}.$$

Theorem 3. *Let $T\colon \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. The following assertions are equivalent.*

(a) *T is invertible and preserves $\prec_{gt}$.*

(b) *There exist $R, S \in \boldsymbol{M}_m$ such that $R(R + nS)$ is invertible and T has one of the forms*

$$TX = XR + JXS, \quad \text{or} \quad \mathrm{TX = PXR + JXS}, \quad \forall \mathrm{X} \in \boldsymbol{M}_{\mathrm{n,m}}.$$

(c) *T strongly preserves $\prec_{gt}$.*

The next section of this paper studies some facts of this concept that are necessary for studying the (strong) linear preservers of $\prec_{gt}$ on $\mathbf{M}_{n,m}$. Also, the (strong) linear preservers of $\prec_{gt}$ on $\mathbf{M}_{n,m}$ are obtained.

2 Gt-majorization on $\mathbf{M}_{n,m}$ and its (strong) linear preservers

First, we review some sticking point of $\prec_{gt}$ on $\mathbb{R}^n$, and then, we bring some properties to prove the main theorems. Also, we characterize all linear operators $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserving (resp. strongly preserving) $\prec_{gt}$.

Lemma 2. *[[1], Theorem 2.3] Let x, y be two distinct vectors in $\mathbb{R}^n$. Assume that $i_1 < i_2 < \cdots < i_k$ and $\{i_1, i_2, \ldots, i_k\} = \{j : j \in \mathbb{N}_{n-1}, y_j = y_{j+1}\}$. Then $x \prec_{gt} y$ if and only if $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$, for every l ($l \in \mathbb{N}_{k+1}$), where $i_{k+1} = n$ and $i_0 = 0$.*

The principle significance of the following lemma is in the assertion of the next theorems.

Lemma 3. *Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{gt}$. If T satisfies two forms of Theorem 1, then T has the third form of this theorem.*

Proof. We consider three cases. Case 1. If T satisfies forms of (i) and (ii) of Theorem 1, then there exist some $\alpha, \beta \in \mathbb{R}$ and $a \in \mathbb{R}^n$ such that $Tx = tr(x)a$ and $Tx = \alpha x + \beta Jx$, $\forall x \in \mathbb{R}^n$. So $\alpha = 0$ and $a = \beta e$. It implies that $Tx = \beta tr(x)e = \beta Jx = \alpha Px + \beta Jx$, $\forall x \in \mathbb{R}^n$, and hence T has the form (iii). Case 2. If T satisfies forms of (i) and (iii) of Theorem 1, then there exist some $\alpha, \beta \in \mathbb{R}$ and $a \in \mathbb{R}^n$ such that $Tx = tr(x)a$ and $Tx = \alpha Px + \beta Jx$, $\forall x \in \mathbb{R}^n$. Hence $\alpha = 0$ and $a = \beta e$, and then T has the form (ii). Case 3. If T satisfies forms of (ii) and (iii) of Theorem 1, then there exist some $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$ such that $Tx = \alpha x + \beta Jx$ and $Tx = \alpha' Px + \beta' Jx$, $\forall x \in \mathbb{R}^n$. We conclude that $\alpha = \alpha' = 0$ and $\beta = \beta'$.

Then there exist some $\alpha, \beta \in \mathbb{R}$ such that $Tx = \alpha x + \beta Jx$ and $Tx = \alpha Px + \beta Jx$, $\forall x \in \mathbb{R}^n$. We conclude that $\alpha = 0$. So $Tx = \beta Jx = \beta tr(x)e = tr(x)(\beta e)$. We see that T has the form (i). ☐

Remark 1. If T satisfies only in (i), then $a \notin Span\{e\}$. Also, if T satisfies just in (ii) or only in (iii), then $\alpha \neq 0$.

Remark 2. In the case $n = 2$, T satisfies the form of (ii) if and only if T satisfies the form of (iii). Because $Tx = \alpha x + \beta Jx = (-\alpha)Px + (\alpha + \beta)Jx$, $\forall \alpha, \beta \in \mathbb{R}$, and $\forall x \in \mathbb{R}^2$.

The following idea is useful for finding the structure of linear preservers of gt-majorization.

Suppose that $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{R}^m$. For each $i, j \in \mathbb{N}_m$, consider the embedding $E_j\colon \mathbb{R}^n \to \mathbf{M}_{n,m}$ and the projection $E^i\colon \mathbf{M}_{n,m} \to \mathbb{R}^n$, where $E_j(x) = xe_j^t$ and $E^i(X) = Xe_i$.

It is easy to show that for every linear operator $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$,

$$TX = T[X_1 \mid X_2 \mid \ldots \mid X_m] = [\sum_{j=1}^{m} T_{1j}X_j \mid \sum_{j=1}^{m} T_{2j}X_j \mid \ldots \mid \sum_{j=1}^{m} T_{mj}X_j],$$

where $T_{ij} = E^i T E_j\colon \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 4. *If $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is a linear preserver of $\prec_{gt}$, then T_{ij} preserves $\prec_{gt}$ on $\mathbb{R}^n$, for all $i, j \in \mathbb{N}_m$.*

Proof. We show that for each i, j $(1 \le i, j \le n)$ E^i and E_j preserve $\prec_{gt}$. Let $x \in \mathbb{R}^n$, $X \in \mathbf{M}_{n,m}$ and $\mu \in \mathbb{R}^{n-1}$. We see

$$E_j A_\mu x = A_\mu x e_j^t = A_\mu E_j x$$

and

$$E^i A_\mu X = A_\mu X e_i = A_\mu E^i x.$$

There E^i and E_j preserve $\prec_{gt}$.

Now, suppose that T preserves $\prec_{gt}$. Since $A_\mu E_j x \prec_{gt} E_j x$, $TA_\mu E_j x \prec_{gt} TE_j x$. So $TA_\mu E_j x = A_{\mu'} TE_j x$, for some $\mu' \in \mathbb{R}^{n-1}$. There

$$\begin{aligned} T_{ij} A_\mu x &= E^i T E_j A_\mu x = E^i T A_\mu E_j x \\ &= E^i A_{\mu'} T E_j x = A_{\mu'} E^i T E_j x = A_{\mu'} T_{ij} x. \end{aligned}$$

Hence, T_{ij} preserves $\prec_{gt}$. □

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Let us first prove the sufficiency of the conditions. At first, let $X, Y \in \mathbf{M}_{n,m}$ such that $X \prec_{gt} Y$. So there exists $\mu = (\mu_1, \mu_2, \ldots, \mu_{n-1}) \in \mathbb{R}^{n-1}$ such that $X = A_\mu Y$. If T has the form (I); Then

$$TX = \sum_{j=1}^{m} (\sum_{i=1}^{n} x_{ij}) A_j = \sum_{j=1}^{m} (\sum_{i=1}^{n} y_{ij}) A_j = TY,$$

because of $X_j \prec_{gt} Y_j$, $\forall j \in \mathbb{N}_m$, and hence $TX \prec_{gt} TY$.

Let T have the form (II). Then

$$TX = T(A_\mu Y) = A_\mu YR + JA_\mu YS = A_\mu YR + A_\mu JYS = A_\mu TY.$$

It follows that $TX \prec_{gt} TY$. If T has the form (III); Then

$$TX = T(A_\mu Y) = PA_\mu YR + JA_\mu YS = (PA_\mu P)PYR + (PA_\mu P)JYS = A_{\mu'} TY,$$

where $\mu' = (\mu_{n-1}, \mu_{n-2}, \ldots, \mu_1) \in \mathbb{R}^{n-1}$. Thus, $TX \prec_{gt} TY$.

It remains to prove the converse implication of the theorem. Assume that T preserves $\prec_{gt}$. Here, $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{R}^m$. We show that all of T_{ij} have the same form in Theorem 1, for all $i, j \in \mathbb{N}_m$. That is, all of T_{ij} satisfy (i), all of T_{ij} satisfy (ii), or all of T_{ij} satisfy (iii), for all $i, j \in \mathbb{N}_m$. Otherwise, if there exist $(r, s) \neq (k, l)$ such that T_{rs} and T_{kl} do not satisfy the same form; The case $n = 1$ is trivial. Consider three following cases.

(a) $n \ge 2$, T_{rs} has the form (i) and T_{kl} has the form (ii).

(b) $n \geq 3$, T_{rs} has the form (i) and T_{kl} has the form (iii).

(c) $n \geq 3$, T_{rs} has the form (ii) and T_{kl} has the form (iii).

We proceed by considering three steps.

Step 1. $l = s$. In this case $k \neq r$, also, T_{rs} and T_{ks} do not satisfy the same form. Let

$$X = [X_1 \mid X_2 \mid \ldots \mid X_m] \text{ and } Y = [Y_1 \mid Y_2 \mid \ldots \mid X_m]$$

be two matrices in M_{nm} such that $X_i = Y_i = \mathbf{0}$ for all $i \neq s$, $1 \leq i \leq m$. If $X \prec_{gt} Y$, then $TX \prec_{gt} TY$, and so there is some $\mu \in \mathbb{R}^{n-1}$ such that $TX = A_\mu TY$. Therefore

$$\begin{aligned} TX &= [T_{1s}X_s| \ldots |T_{ms}X_s] = A_\mu[T_{1s}Y_s| \ldots |T_{ms}Y_s] \\ &= [A_\mu T_{1s}Y_s| \ldots |A_\mu T_{ms}Y_s]. \end{aligned}$$

It shows that

$$T_{rs}X_s = A_\mu T_{rs}Y_s, \tag{1}$$

and

$$T_{ks}X_s = A_\mu T_{ks}Y_s. \tag{2}$$

(a) If $n \geq 2$, T_{rs} has the form (i) and T_{ks} has the form (ii), then $T_{rs}x = tr(x)a$, $a \notin Span\{e\}$, and $T_{ks}x = \alpha x + \beta Jx$, $\alpha \neq 0$, $\forall x \in \mathbb{R}^n$. For each $i \in \mathbb{N}_{n-1}$ let $X = E_{is}$ and $Y = E_{(i+1)s}$. From $X \prec_{gt} Y$ and 2 conclude that for each $i \in \mathbb{N}_{n-1}$ there is some $\mu \in \mathbb{R}^{n-1}$ such that $\alpha e_i + \beta e = A_\mu(\alpha e_{i+1} + \beta e)$. Let $\mu_0 = 0$. So

$$\alpha + \beta = \mu_{i-1}\beta + (1 - \mu_{i-1} - \mu_i)\beta + \mu_i(\alpha + \beta),$$

and

$$\beta = \mu_i\beta + (1 - \mu_i - \mu_{i+1})(\alpha + \beta) + \mu_{i+1}\beta.$$

Hence $\mu_i = 1$ and $\mu_{i+1} = 0$. The relation 1 ensures that $a = A_\mu a$. This means that

$$a_{i+1} = \mu_i a_i + (1 - \mu_i - \mu_{i+1})a_{i+1} + \mu_{i+1}a_{i+2},$$

and so $a_{i+1} = a_i$. We see that $a \in Span\{e\}$, which is a contradiction.

(b) If $n \geq 3$, T_{rs} has the form (i) and T_{ks} has the form (iii), then $T_{rs}x = tr(x)a$, $a \notin Span\{e\}$, and $T_{ks}x = \alpha Px + \beta Jx$, $\alpha \neq 0$, $\forall x \in \mathbb{R}^n$. For each $i \in \mathbb{N}_{n-1}$ $E_{is} \prec_{gt} E_{(i+1)s}$. So for every $i \in \mathbb{N}_{n-1}$ there exists some $\mu \in \mathbb{R}^{n-1}$ such that

$$T_{rs}e_i = A_\mu T_{rs}e_{i+1}, \tag{3}$$

and

$$T_{ks}e_i = A_\mu T_{ks}e_{i+1}. \tag{4}$$

The relation (4) ensures that $\mu_{n-i-1} = 0$ and $\mu_{n-i} = 1$. By applying (3), observe that $a_{n-i} = a_{n-i+1}$. It deduces that $a \in Span\{e\}$, a contradiction. (c) If $n \geq 3$, T_{rs} has the form (ii) and T_{ks} has the form (iii), then $T_{rs}x = \alpha_1 x + \beta_1 Jx$, $\alpha_1 \neq 0$, and $T_{ks}x = \alpha_2 Px + \beta_2 Jx$, $\alpha_2 \neq 0$, $\forall x \in \mathbb{R}^n$. Put $X = \sum_{i=1}^{n} iE_{is}$ and $Y = 2E_{1s} + E_{2s} + \sum_{i=3}^{n} iE_{is}$, of the Relations (1) and (2) we have

$$T_{rs}(\sum_{i=1}^{n} ie_i) = A_\mu T_{rs}(2e_1 + e_2 + \sum_{i=3}^{n} ie_i) \tag{5}$$

$$T_{ks}(\sum_{i=1}^{n} ie_i) = A_\mu T_{ks}(2e_1 + e_2 + \sum_{i=3}^{n} ie_i), \tag{6}$$

for some $\mu \in \mathbb{R}^{n-1}$. From (5) and (6),

$$\sum_{i=1}^{n} ie_i = A_\mu(2e_1 + e_2 + \sum_{i=3}^{n} ie_i), \tag{7}$$

$$\sum_{i=1}^{n}(n+1-i)e_i = A_\mu(\sum_{i=1}^{n-2}(n+1-i)e_i + e_{n-1} + 2e_n). \tag{8}$$

The Relation (7) yields that $\mu_1 = 1$ and The Relation (8) yields that $\mu_1 = 0$. This is a contradiction.

Step 2. $k = r$. We observe that $l \neq s$. Also, T_{rs} and T_{rl} do not satisfy the same form.

(a) If $n \geq 2$, T_{rs} has the form (i) and T_{rl} has the form (ii), then $T_{rs}x = tr(x)a$, $a \notin Span\{e\}$, and $T_{rl}x = \alpha x + \beta Jx$, $\alpha \neq 0$, $\forall x \in \mathbb{R}^n$. Since $a \notin Span\{e\}$, there is some i ($i \in \mathbb{N}_{n-1}$) such that $a_i \neq a_{i+1}$. Let $c = \frac{a_i - a_{i+1}}{\alpha}$, $X = E_{is} + cE_{il}$, and $Y = E_{(i+1)s} + cE_{(i+1)l}$. In this case, we have $X \prec_{gt} Y$. So $TX \prec_{gt} TY$, and then $(TX)_r \prec_{gt} (TY)_r$. So, there exists some $\mu \in \mathbb{R}^n$ such that $(TX)_r = A_\mu(TY)_r$. On the other hand, since

$$(TX)_r = T_{rs}X_s + T_{rl}X_l = a + c\alpha e_i + c\beta e,$$

and

$$(TY)_r = T_{rs}Y_s + T_{rl}Y_l = a + c\alpha e_{i+1} + c\beta e,$$

we conclude that $a + c\alpha e_i + c\beta e = A_\mu(a + c\alpha e_{i+1} + c\beta e)$. Therefore, $a + c\alpha e_i = A_\mu(a + c\alpha e_{i+1})$. It follows that $\mu_j(a_{j+1} - a_j) = 0$, for all $1 \leq j \leq i-1$, and $\alpha c = \mu_{i-1}(a_{i-1} - a_i) + \mu_i(a_{i+1} - a_i + \alpha c)$. Hence, $a_i - a_{i+1} = \alpha c = 0$. Therefore $(TX)_r \not\prec_{gt} (TY)_r$, which is a Contradiction.

(b) If $n \geq 3$, T_{rs} has the form (i) and T_{rl} has the form (iii), then $T_{rs}x = tr(x)a$, $a \notin Span\{e\}$, and $T_{rl}x = \alpha Px + \beta Jx$, $\alpha \neq 0$, $\forall x \in \mathbb{R}^n$. As $a \notin Span\{e\}$, we conclude that there is some i ($i \in \mathbb{N}_{n-1}$) such that $a_i \neq a_{i+1}$. Let $c = \frac{a_i - a_{i+1}}{\alpha}$, $X = E_{(n-i+1)s} + cE_{il}$, and $Y = E_{(n-i)s} + cE_{(n-i)l}$. We obtain a contradiction.

(c) If $n \geq 3$, T_{rs} has the form (ii) and T_{rl} has the form (iii), then $T_{rs}x = \alpha_1 x + \beta_1 Jx$, $\alpha_1 \neq 0$, and $T_{rl}x = \alpha_2 Px + \beta_2 Jx$, $\alpha_2 \neq 0$, $\forall x \in \mathbb{R}^n$. Consider

$$X = 2(\alpha_2 E_{1s} + \alpha_1 E_{1l}) + \alpha_2 E_{2s} + \alpha_1 E_{2l} + \sum_{i=3}^{n} i(\alpha_2 E_{is} + \alpha_1 E_{il}),$$

and

$$Y = \sum_{i=1}^{n} i(\alpha_2 E_{is} + \alpha_1 E_{il}).$$

The relation $X \prec_{gt} Y$ shows that $TX \prec_{gt} TY$, and than $(TX)_r \prec_{gt} (TY)_r$. But

$$\begin{aligned}(TX)_r &= \alpha_1\alpha_2(2e_1 + e_2 + \textstyle\sum_{i=3}^{n} ie_i) + \alpha_2\beta_1\frac{n(n+1)}{2}e \\ &+ \alpha_1\alpha_2(\textstyle\sum_{i=1}^{n-2}(n+1-i)e_i + e_{n-1} + 2e_n) + \alpha_1\beta_2\frac{n(n+1)}{2}e.\end{aligned}$$

and

$$\begin{aligned}(TY)_r &= \alpha_1\alpha_2\textstyle\sum_{i=1}^{n} ie_i + \alpha_2\beta_1\frac{n(n+1)}{2}e \\ &+ \alpha_1\alpha_2\textstyle\sum_{i=1}^{n}(n+1-i)e_i + \alpha_1\beta_2\frac{n(n+1)}{2}e,\end{aligned}$$

We see that $(TY)_r \in span\{e\}$ but $(TX)_r \notin span\{e\}$, we conclude that $(TX)_r \not\prec_{gt} (TY)_r$, which would be a contradiction.

Step 3. $l \neq s$ and $k \neq r$. By Step 1, T_{rs} and T_{ks} have the same form. Also, about T_{rl} and T_{kl}. From Step 2, T_{rs} and T_{rl} have the same form, and T_{ks} and T_{kl} satisfy the same form, too. So, T_{rl}, T_{ks} satisfies Lemma 3 in all cases (a), (b) and (c). Thus, there are some $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $T_{rl}(x) = \gamma_1 tr(x)e$ and $T_{ks}(x) = \gamma_2 tr(x)e$, $\forall x \in \mathbb{R}^n$.

(a) If $n \geq 2$, T_{rs} has the form (i) and T_{kl} has the form (ii), then $T_{rs}x = tr(x)a$, $a \notin Span\{e\}$, and $T_{kl}x = \alpha x + \beta Jx$, $\alpha \neq 0$, $\forall x \in \mathbb{R}^n$. Fix i ($i \in \mathbb{N}_{n-1}$). Select $X = E_{is} + E_{il}$ and $Y = E_{(i+1)s} + E_{(i+1)l}$. As $X \prec_{gt} Y$, we see $TX \prec_{gt} TY$. So there exists $\mu \in \mathbb{R}^{n-1}$ such that $TX = A_\mu TY$, and then $(TX)_r = A_\mu(TY)_r$ and $(TX)_k = A_\mu(TY)_k$. It shows that

$$T_{rs}X_s + T_{rl}X_l = A_\mu(T_{rs}Y_s + T_{rl}Y_l),$$

and

$$T_{ks}X_s + T_{kl}X_l = A_\mu(T_{ks}Y_s + T_{kl}Y_l).$$

Thus,

$$T_{rs}e_i + T_{rl}e_i = A_\mu(T_{rs}e_{i+1} + T_{rl}e_{i+1}),$$

and

$$T_{ks}e_i + T_{kl}e_i = A_\mu(T_{ks}e_{i+1} + T_{kl}e_{i+1}).$$

It means that

$$a + \gamma_1 e = A_\mu(a + \gamma_1 e),$$

and

$$\gamma_2 e + \alpha e_i + \beta e = A_\mu(\gamma_2 e + \alpha e_{i+1} + \beta e).$$

Observe that

$$a = A_\mu a, \tag{9}$$

and

$$e_i = A_\mu e_{i+1}. \tag{10}$$

From the relation 10, conclude that $\mu_i = 1$ and $\mu_{i+1} = 0$. The relation 9 ensures that $a_i = a_{i+1}$. Since $i(i \in \mathbb{N}_{n-1})$ is arbitrary, $a \in Span\{e\}$. This is a contradiction.

(b) If $n \geq 3$, T_{rs} has the form (i) and T_{kl} has the form (iii), then $T_{rs}x = tr(x)a$, $a \notin Span\{e\}$, and $T_{kl}x = \alpha Px + \beta Jx$, $\alpha \neq 0$, $\forall x \in \mathbb{R}^n$. By choosing $X = E_{is} + E_{il}$ and $Y = E_{(i+1)s} + E_{(i+1)l}$, obtain a contradiction.

(c) If $n \geq 3$, T_{rs} has the form (ii) and T_{kl} has the form (iii), then $T_{rs}x = \alpha_1 x + \beta_1 Jx$, $\alpha_1 \neq 0$, and $T_{kl}x = \alpha_2 Px + \beta_2 Jx$, $\alpha_2 \neq 0$, $\forall x \in \mathbb{R}^n$. Choose

$$X = \sum_{i=1}^{n} i(E_{is} + E_{il}),$$

and

$$Y = 2(E_{1s} + E_{1l}) + E_{2s} + E_{2l} + \sum_{i=3}^{n} i(E_{is} + E_{il}).$$

As $X \prec_{gt} Y$, observe that $TX \prec_{gt} TY$. So there exists some $\mu \in \mathbb{R}^{n-1}$ such that $TX = A_\mu TY$, and thus, $(TX)_r = A_\mu (TY)_r$ and $(TX)_k = A_\mu (TY)_k$. It deduces that

$$\begin{aligned} &\alpha_1(\sum_{i=1}^{n} ie_i) + \frac{n(n+1)}{2}\beta_1 e + \gamma_1 \frac{n(n+1)}{2} e_1 \\ &\qquad = A_\mu \left[\alpha_1(2e_1 + e_2 + \sum_{i=3}^{n} ie_i) + \frac{n(n+1)}{2}\beta_1 e + \gamma_1 \frac{n(n+1)}{2} e\right], \end{aligned}$$

and

$$\begin{aligned} &\gamma_2 \frac{n(n+1)}{2} e + \alpha_2 \sum_{i=1}^{n} (n+1-i)e_i + \beta_2 \frac{n(n+1)}{2} e \\ &= A_\mu \left[\gamma_2 \frac{n(n+1)}{2} e + \alpha_2 \left(\sum_{i=1}^{n-2} (n+1-i)e_i + e_{n-1} + 2e_n\right) + \beta_2 \frac{n(n+1)}{2} e\right]. \end{aligned}$$

It implies that

$$\sum_{i=1}^{n} ie_i = A_\mu[2e_1 + e_2 + \sum_{i=3}^{n} ie_i],$$

and

$$\sum_{i=1}^{n} (n+1-i)e_i = A_\mu(\sum_{i=1}^{n-2} (n+1-i)e_i + e_{n-1} + 2e_n).$$

We conclude that $\mu_1 = 1$ and $\mu_1 = 0$, which is a contradiction.

Now we finish the proof in three cases.

Case 1. For each $i, j \in \mathbb{N}_m$, T_{ij} satisfies (i). That is, for every $i, j \in \mathbb{N}_m$ there exists some $A_{ij} \in \mathbb{R}^n$ such that $T_{ij}x = tr(x)A_{ij}$, $\forall x \in \mathbb{R}^n$. Set

$$A_j = [A_{1j} \mid A_{2j} \mid \ldots \mid A_{mj}], \quad \forall j \in \mathbb{N}_m.$$

Then

$$\begin{aligned}
TX &= T[X_1 \mid X_2 \mid \ldots \mid X_m] \\
&= [\textstyle\sum_{j=1}^m T_{1j}X_j \mid \sum_{j=1}^m T_{2j}X_j \mid \ldots \mid \sum_{j=1}^m T_{mj}X_j] \\
&= [\textstyle\sum_{j=1}^m tr(X_j)A_{1j} \mid \sum_{j=1}^m tr(X_j)A_{2j} \mid \ldots \mid \sum_{j=1}^m tr(X_j)A_{mj}] \\
&= \textstyle\sum_{j=1}^m tr(X_j)[A_{1j} \mid A_{2j} \mid \ldots \mid A_{mj}] \\
&= \textstyle\sum_{j=1}^m tr(X_j)A_j \\
&= \textstyle\sum_{j=1}^m (\sum_{i=1}^n x_{ij})A_j.
\end{aligned}$$

Case 2. For each $i, j \in \mathbb{N}_m$, T_{ij} satisfies (ii). It means that for each $i, j \in \mathbb{N}_m$ $T_{ij}x = r_{ij}x + s_{ij}Jx$, for some $r_{ij}, s_{ij} \in \mathbb{R}$ and $\forall x \in \mathbb{R}^n$. Put $R = [r_{ij}] \in \mathbf{M}_m$, $S = [s_{ij}] \in \mathbf{M}_m$. Then

$$\begin{aligned}
TX &= T[X_1 \mid X_2 \mid \ldots \mid X_m] \\
&= [\textstyle\sum_{j=1}^m T_{1j}X_j \mid \sum_{j=1}^m T_{2j}X_j \mid \ldots \mid \sum_{j=1}^m T_{mj}X_j] \\
&= [\textstyle\sum_{j=1}^m (r_{1j}X_j + s_{1j}JX_j) \mid \sum_{j=1}^m (r_{2j}X_j + s_{2j}JX_j) \\
&\quad \mid \ldots \mid \textstyle\sum_{j=1}^m (r_{mj}X_j + s_{mj}JX_j)] \\
&= [\textstyle\sum_{j=1}^m r_{1j}X_j \mid \ldots \mid \sum_{j=1}^m r_{mj}X_j] + [\sum_{j=1}^m s_{1j}JX_j \\
&\quad \mid \ldots \mid \textstyle\sum_{j=1}^m s_{mj}JX_j] \\
&= [X_1 \mid X_2 \mid \ldots \mid X_m][r_{ij}] + J[X_1 \mid X_2 \mid \ldots \mid X_m][s_{ij}] \\
&= XR + JXS.
\end{aligned}$$

Case 3. For each $i, j \in \mathbb{N}_m$, T_{ij} satisfies (iii). In a similar fashion one can prove it. $\square$

We need the following lemma to prove the next theorem.

Lemma 5. *Let $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. If T strongly preserves $\prec_{gt}$, then T is invertible.*

Proof. Let $X \in \mathbf{M}_{n,m}$, and let $TX = 0$. Since $TX = T0$ and T strongly preserves $\prec_{gt}$, this implies that $X \prec_{gt} 0$. So $X = 0$, and thus T is invertible. $\square$

Now we bring proof of Theorem 3.

Proof of Theorem 3. (a) $\Rightarrow$ (b): By the hypothesis, T has one of the forms of Theorem 2. Consider two following case:

Case $n = 1$: We claim that if T satisfies (I), then T satisfies (II), too. If T has the form (I), then there exist some $A_j = [a_{j1}a_{j2}\cdots a_{jm}]$, $j \in \mathbb{N}_m$, such that

$$TX = \sum_{j=1}^m x_{1j}A_j, \quad \forall X = [x_{11}x_{12}\cdots x_{1m}] \in \mathbf{M}_{1,m}.$$

So $TX = XA$, where $A = [a_{ij}] \in \mathbf{M}_m$. We see that T satisfies (II) with $R = A$ and $S = 0$. So if (a) holds, then T has one of the forms of (II) or (III). If T has the form (II) or (III), then $TX = XR + XS$. Therefore $TX = X(R + S)$. Since T is invertible, $R + S$ is invertible, too. We have $TX = X(R + S) + X0 = XR' + XS'$ where $R' = R + S$, $S' = 0$ and the matrix $R'(R' + 0) = {R'}^2$ is invertible.

Case $n \geq 2$: The case (I) can not occur, because of $T(E_{11} - E_{21}) = 0$. It is enough to show that for $n \geq 2$ the matrices R and $R + nS$ are invertible. If R is not invertible, then there exists some $X_1 \in \mathbb{R}_m \setminus \{0\}$ such that $X_1 R = 0$. Define $X \in \mathbf{M}_{n,m}$ such that all its row are X_1 and $Y \in \mathbf{M}_{n,m}$ such that the first row is nX_1 and the other rows are zero. See $YR = 0 = XR$ and $JY = JX$. Then

$$TX = QXR + JXS = JXS = JYS = QYR + JYS = TY,$$

where $Q = I$ or P. We observe that $X \neq Y$, but $TX = TY$, a contradiction. So R is invertible. If $R + nS$ is not invertible, then there is some $Z_1 \in \mathbb{R}_m \setminus \{0\}$ such that $Z_1(R + nS) = 0$. Let $Z \in \mathbf{M}_{n,m}$ such that all its row are Z_1. We deduce that $TZ = 0$, but $Z \neq 0$. It is a contradiction. Hence $R + nS$ is invertible.

(b) $\Rightarrow$ (c): Suppose that $TX = QXR + JXS$, where $Q = I$ or P, and $R(R+nS)$ is invertible. Define $T' \colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ by $T'X = QXR' + JXS'$, where $R' = R^{-1}$ and $S' = -(R + nS)^{-1} S R^{-1}$. It is easy to see $(T'T)X = X$, $\forall X \in \mathbf{M}_{n,m}$. So $T' = T^{-1}$. As T^{-1} preserves $\prec_{gt}$, we conclude that T strongly preserves $\prec_{gt}$. Therefore, (c) holds.

(c) $\Rightarrow$ (a): This follows immediately from Lemma 4. $\square$

References

[1] A. Armandnejad, Z. Gashool: Strong linear preservers of g-tridiagonal majorization on $\mathbb{R}^n$. *Electronic Journal of Linear Algebra* 123 (2012) 115–121.

[2] A. Armandnejad, S. Mohtashami, M. Jamshidi: On linear preservers of g-tridiagonal majorization on $\mathbb{R}^n$. *Linear Algebra and its Applications* 459 (2014) 145–153.

[3] L.B. Beasley, S.-G. Lee, Y.-H. Lee: A characterization of strong preservers of matrix majorization. *Linear Algebra and its Applications* 367 (2003) 341–346.

[4] R. Bahatia: *Matrix Analysis*. Springer-Verlag, New York (1997).

[5] R.A. Brualdi, G. Dahl: An extension of the polytope of doubly stochastic matrices. *Linear and Multilinear Algebra* 6 (3) (2013) 393–408.

[6] H. Chiang, C.K. Li: Generalized doubly stochastic matrices and linear preservers. *Linear and Multilinear Algebra* 53 (2005) 1–11.

[7] A.M. Hasani, M. Radjabalipour: The structure of linear operators strongly preserving majorizations of matrices. *Electronic Journal of Linear Algebra* 15 (2006) 260–268.

[8] A.M. Hasani, M. Radjabalipour: On linear preservers of (right) matrix majorization. *Linear Algebra and its Applications* 423 (2007) 255–261.

[9] A.W. Marshall, I. Olkin, B.C. Arnold: *Inequalities: Theory of majorization and its applications*. Springer, New York (2011).

Received: 3 July 2019
Accepted for publication: 15 October 2019
Communicated by: Diana Barseghyan