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Some type of semisymmetry on two classes of almost Kenmotsu manifolds

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Abstract. The object of the present paper is to study some types of semisymmetry conditions on two classes of almost Kenmotsu manifolds. It is shown that a (k, μ)-almost Kenmotsu manifold satisfying the curvature condition $Q \cdot R = 0$ is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Also in (k, μ)-almost Kenmotsu manifolds the following conditions: (1) local symmetry ($\nabla R = 0$), (2) semisymmetry ($R \cdot R = 0$), (3) $Q(S, R) = 0$, (4) $R \cdot R = Q(S, R)$, (5) locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ are equivalent. Further, it is proved that a $(k, \mu)'$-almost Kenmotsu manifold satisfying $Q \cdot R = 0$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ and a $(k, \mu)'$-almost Kenmotsu manifold satisfying any one of the curvature conditions $Q(S, R) = 0$ or $R \cdot R = Q(S, R)$ is either an Einstein manifold or locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Finally, an illustrative example is presented.

1 Introduction

In the present time, the study of almost Kenmotsu manifolds with nullity distributions is a very interesting topic in contact geometry. The notion of the (k, μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ was introduced by Blair, Koufogiorgos and Papantoniou [3], which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k, \mu) = \{Z \in T_p(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}, \qquad (1)$$

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for any $X, Y \in T_p(M)$, where $T_p(M)$ denotes the tangent space of M at any point $p \in M$, R denotes the Riemannian curvature tensor of type $(1,3)$, $h = \frac{1}{2}\pounds_\xi \phi$ and $\pounds$ denotes the Lie differentiation.

In [6], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' = \{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \end{aligned} \tag{2}$$

where $h' = h \circ \phi$.

We define an endomorphism $X \wedge_A Y$ of $T(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{3}$$

where A is a symmetric $(0,2)$-tensor and $X, Y, Z \in T(M)$.

For a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field A on M we define the tensors $R \cdot T$ and $Q(A,T)$ respectively [12] by

$$\begin{aligned} (R \cdot T)(X_1, X_2, \ldots, X_k; X, Y) = -T(R(X,Y)X_1, X_2, \ldots, X_k) \\ - \cdots - T(X_1, X_2, \ldots, R(X,Y)X_k) \end{aligned} \tag{4}$$

and

$$\begin{aligned} Q(A,T)(X_1, X_2, \ldots, X_k; X, Y) = -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) \\ - \cdots - T(X_1, X_2, \ldots, (X \wedge_A Y)X_k). \end{aligned} \tag{5}$$

A Riemannian manifold M is said to be semisymmetric if $R \cdot R = 0$ holds on M, where R is the Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{6}$$

An example of a curvature condition of semisymmetry type is $Q \cdot R = 0$, where Q is the Ricci operator of type (1,1) and is defined by $S(X,Y) = g(QX,Y)$. A natural extension of such curvature conditions form curvature conditions of pseudosymmetry type. The curvature condition $Q \cdot R = 0$ have been studied by Verstraelen et al. in [11].

If the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent on M, then M is called Ricci generalized pseudosymmetric [12], where S is the Ricci tensor of type $(0,2)$. This is equivalent to

$$R \cdot R = fQ(S,R), \tag{7}$$

holding on the set $\mathcal{U}_R = \{x \in M : R \neq 0 \text{ at } x\}$, where f is some function on $\mathcal{U}_R$. A very important subclass of this class of manifolds realizing the condition is

$$R \cdot R = Q(S,R). \tag{8}$$

Every three dimensional manifold satisfies the above curvature condition identically. Other examples are semi-Riemannian manifolds (M, g) admitting a non-zero 1-form ω such that the equality

$$\omega(X)R(Y,Z) + \omega(Y)R(Z,X) + \omega(Z)R(X,Y) \equiv 0$$

holds on M. The condition $R \cdot R = Q(S, R)$ also appears in the theory of plane gravitational waves. Recently, Kowalczyk [9] studied semi-Riemannian manifolds satisfying $Q(S, R) = 0$, where S and R are the Ricci tensor and the Riemannian curvature tensor respectively.

Almost Kenmotsu manifolds have been studied by several authors. In [5], Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds. Also almost Kenmotsu manifolds with nullity distributions were investigated by Dileo and Pastore [6]. Wang and Wang [15] studied pseudosymmetric and quasi weakly symmetric almost Kenmotsu manifolds with generalized nullity distributions. Further in [13], Wang and Liu studied ξ-Riemannian semisymmetric almost Kenmotsu manifolds with nullity distributions. Ghosh et. al [7] classified almost Kenmotsu manifolds with generalized $(k, \mu)'$-nullity distribution satisfying certain curvature condition. In [4], Dey and Majhi studied Quasi-conformal curvature tensor on almost Kenmotsu manifolds.

Motivated by the above studies, in the present paper we characterize two classes of almost Kenmotsu manifolds satisfying certain semisymmetry type curvature conditions.

The paper is organized as follows:

In section 2, some preliminaries on almost Kenmotsu manifolds are discussed. Section 3 is devoted to study (k, μ)-almost Kenmotsu manifolds with the properties $Q \cdot R = 0$, $Q(S, R) = 0$, $R \cdot R = Q(S, R)$. In Section 4, we characterize $(k, \mu)'$-almost Kenmotsu manifolds satisfying the curvature properties $Q \cdot R = 0$, $Q(S, R) = 0$, $R \cdot R = Q(S, R)$. Finally, in Section 5, an illustrative example is presented.

2 Almost Kenmotsu manifolds

A $(2n+1)$-dimensional differentiable manifold M is said to have a (ϕ, ξ, η)-structure or an almost contact structure, if it admits a $(1, 1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([1], [2]),

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{9}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (9) easily.

If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y of $T(M)$, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any X, Y of $T(M)$. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1, 2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [1]. Recently in ([6],[5],[10]), almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X,Y. It is well known [8] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [10]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{10}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{11}$$

for any vector field X. The $(1, 1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([6], [13])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2). \tag{12}$$

3 (k, μ)-almost Kenmotsu manifolds

In this section we study almost Kenmotsu manifolds with ξ belonging to the (k, μ)--nullity distribution.

From (1) we obtain

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{13}$$

where $k, \mu \in \mathbb{R}$. Before proving our main results in this section we first state the following:

Lemma 1. *[15] Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a (k, μ)-almost Kenmotsu manifold. Then M^{2n+1} is semisymmetric if and only if it is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Lemma 2. *[6] Let M^{2n+1} be an almost Kenmotsu manifold of dimension $(2n + 1)$. Suppose that the characteristic vector field ξ belonging to the (k, μ)-nullity distribution. Then $k = -1$, $h = 0$ and M^{2n+1} is locally a wrapped product of an open interval and an almost Kähler manifold.*

In view of Lemma 2 it follows from (13),

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{14}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{15}$$

$$S(X,\xi) = -2n\eta(X), \tag{16}$$

$$Q\xi = -2n\xi, \tag{17}$$

for any vector fields X, Y on M^{2n+1}. Also for an (k,μ)-almost Kenmotsu manifold

$$\nabla_X \xi = X - \eta(X)\xi \tag{18}$$

and

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{19}$$

We now prove our main results.

Theorem 1. *If a (k,μ)-almost Kenmotsu manifold satisfies the semisymmetry type curvature condition $Q \cdot R = 0$, then the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof. Let us suppose that the (k,μ)-almost Kenmotsu manifold satisfies the semisymmetry type curvature condition $Q \cdot R = 0$. Then we have

$$Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ = 0, \tag{20}$$

for all vector fields $X,\ Y,\ Z$ in M^{2n+1}.

Setting $Z = \xi$ in the above equation and using (14) we obtain

$$\eta(X)Y - \eta(Y)X = 0,$$

which implies

$$R(X,Y)\xi = 0. \tag{21}$$

Taking Covariant derivative of (21) in the direction of any vector field Z and using (14) and (18) we get

$$((\nabla_Z \eta)X)Y - ((\nabla_Z \eta)Y)X - R(X,Y)Z + \eta(Z)(\eta(X)Y - \eta(Y)X) = 0. \tag{22}$$

Using (19) in (22) we have

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y].$$

This shows that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Hence, the theorem is proved. □

Proposition 1. *In a (k,μ)-almost Kenmotsu manifold, the curvature condition $Q(S,R) = 0$ holds if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof. Let us first suppose that the curvature condition $Q(S,R) = 0$ holds on M^{2n+1}. Then we have

$$((X \wedge_S Y) \cdot R)(U,V)W = 0, \tag{23}$$

which implies

$$(X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V)W - R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W = 0. \quad (24)$$

Using (3) in (24) we get

$$S(Y,R(U,V)W)X - S(X,R(U,V)W)Y - R(S(Y,U)X - S(X,U)Y,V)W - R(U,S(Y,V)X - S(X,V)Y)W - R(U,V)(S(Y,W)X - S(X,W)Y) = 0. \quad (25)$$

Setting $W = \xi$ in the above equation and using (14) and (16) we obtain

$$S(X,U)\eta(Y)V - S(Y,U)\eta(X)V + S(Y,V)\eta(X)U - S(X,V)\eta(Y)U + 2n\eta(Y)R(U,V)X - 2n\eta(X)R(U,V)Y = 0. \quad (26)$$

Putting $Y = \xi$ in the previous equation and using (14) and (16) yields

$$S(X,U)V - S(X,V)U + 2nR(U,V)X = 0. \quad (27)$$

Replacing U by ξ in the foregoing equation and using (15) and (16)we have

$$S(X,V)\xi + 2ng(X,V)\xi = 0. \quad (28)$$

Taking inner product with ξ we get

$$S(X,V) = -2ng(X,V). \quad (29)$$

Using (29) in (27) we obtain

$$R(U,V)X = -[g(V,X)U - g(U,X)V], \quad (30)$$

which implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.

Conversely, if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$, then

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y], \quad (31)$$

which implies

$$S(Y,Z) = -2ng(Y,Z). \quad (32)$$

Now,

$$\begin{aligned} Q(S,R)(U,V,W;X,Y) &= (X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V)W \\ &\quad - R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W, \\ &= S(Y,R(U,V)W)X - S(X,R(U,V)W)Y \\ &\quad - R(S(Y,U)X - S(X,U)Y,V)W \\ &\quad - R(U,S(Y,V)X - S(X,V)Y)W \\ &\quad - R(U,V)(S(Y,W)X - S(X,W)Y). \end{aligned} \quad (33)$$

Using (31) and (32) in (33) yields

$$Q(S,R)(U,V,W;X,Y)=0. \tag{34}$$

This completes the proof. □

Proposition 2. *In a (k,μ)-almost Kenmotsu manifold the curvature condition*

$$R\cdot R=Q(S,R)$$

holds if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.

Proof. Let the curvature condition $R\cdot R=Q(S,R)$ holds on M^{2n+1}. Now,

$$\begin{aligned}(R(X,Y)\cdot R)(U,V)W&=R(X,Y)R(U,V)W-R(R(X,Y)U,V)W\\&\quad-R(U,R(X,Y)V)W-R(U,V)R(X,Y)W.\end{aligned} \tag{35}$$

Setting $Y=U=\xi$ in the above equation we get

$$\begin{aligned}(R(X,\xi)\cdot R)(\xi,V)W&=R(X,\xi)R(\xi,V)W-R(R(X,\xi)\xi,V)W\\&\quad-R(\xi,R(X,\xi)V)W-R(\xi,V)R(X,\xi)W.\end{aligned} \tag{36}$$

With the help of (14)–(16) we compute the terms of the above expression as below

$$R(X,\xi)R(\xi,V)W=g(V,W)[X-\eta(X)\xi]+\eta(W)[g(X,V)\xi-\eta(V)X], \tag{37}$$

$$R(R(X,\xi)\xi,V)W=-g(V,W)\eta(X)\xi+\eta(X)\eta(W)V-R(X,V)W, \tag{38}$$

$$R(\xi,R(X,\xi)V)W=g(X,W)\eta(V)\xi-\eta(V)\eta(W)X \tag{39}$$

and

$$\begin{aligned}R(\xi,V)R(X,\xi)W&=g(X,W)V-g(X,W)\eta(V)\xi\\&\quad+g(V,X)\eta(W)\xi-\eta(X)\eta(W)V.\end{aligned} \tag{40}$$

Substituting (37)–(40) in (36) we obtain

$$(R(X,\xi)\cdot R)(\xi,V)W=R(X,V)W+g(V,W)X-g(X,W)V. \tag{41}$$

Again

$$\begin{aligned}Q(S,R)(U,V,W;X,Y)&=S(Y,R(U,V)W)X-S(X,R(U,V)W)Y\\&\quad-R(S(Y,U)X-S(X,U)Y,V)W\\&\quad-R(U,S(Y,V)X-S(X,V)Y)W\\&\quad-R(U,V)(S(Y,W)X-S(X,W)Y).\end{aligned} \tag{42}$$

Using (14)–(16), from above we have

$$\begin{aligned}Q(S,R)(\xi,V,W;X,\xi) = & \, 2ng(V,W)X - \eta(W)S(X,V)\xi + 2nR(X,V)W \\ & - 2n\eta(W)g(X,V)\xi + S(X,W)V - \eta(V)S(X,W)\xi \\ & - 2n\eta(V)g(X,W)\xi. \end{aligned} \tag{43}$$

Since, the condition $R \cdot R = Q(S,R)$ is realized on M^{2n+1} we obtain

$$\begin{aligned}R(X,V)W + g(V,W)X - g(X,W)V = & \, 2ng(V,W)X - \eta(W)S(X,V)\xi \\ & + 2nR(X,V)W - 2n\eta(W)g(X,V)\xi + S(X,W)V - \eta(V)S(X,W)\xi \\ & - 2n\eta(V)g(X,W)\xi. \end{aligned} \tag{44}$$

Setting $W = \xi$ in the foregoing equation and taking inner product with ξ yields

$$S(X,V) = -2ng(X,V). \tag{45}$$

Using (45) in (44) we get

$$R(X,V)W = -[g(V,W)X - g(X,W)V].$$

The converse follows from Lemma 1 and Proposition 1. □

Theorem 2. *In a (k,μ)-almost Kenmotsu manifold the following conditions:*

(1) local symmetry ($\nabla R = 0$),

(2) semisymmetry ($R \cdot R = 0$),

(3) $Q(S,R) = 0$,

(4) $R \cdot R = Q(S,R)$,

(5) locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$

are equivalent.

Proof. It is known that local symmetry implies semisymmetry and the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ is locally symmetric. The remaining proof follows directly from Lemma 1, Proposition 1 and Proposition 2. □

4 $(k,\mu)'$-almost Kenmotsu manifolds

Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (12) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigen spaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. Throughout this section we consider $h' \neq 0$. Before presenting our main theorems we recall some results:

Lemma 3. *[6, Prop. 4.1 and Prop. 4.3] Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and $Spec(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given by the following:*

(a) $K(X, \xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X, \xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$,

(b) $K(X, Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$;
$K(X, Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and*
$K(X, Y) = -(k + 2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k - 2n)$.

Lemma 4. *[14, Lemma 3] Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution. If $h' \neq 0$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k + 1)\eta \otimes \xi - 2nh'. \tag{46}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k - 2n)$.

Lemma 5. *[6, Prop. 4.2] Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belonging to the $(k, -2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemann curvature tensor satisfies:*

$$\begin{gathered}
R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0, \\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0, \\
R(X_\lambda, Y_{-\lambda})Z_\lambda = (k + 2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k + 2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\
R(X_\lambda, Y_\lambda)Z_\lambda = (k - 2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k + 2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{gathered}$$

From (2), we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{47}$$

where $k, \mu \in \mathbb{R}$. Also we get from (47)

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \tag{48}$$

Contracting X in (47), we have

$$S(Y, \xi) = 2nk\eta(Y). \tag{49}$$

Theorem 3. *If a $(k,\mu)'$-almost Kenmotsu manifold satisfies the semisymmetry type curvature condition $Q\cdot R=0$, then the manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. Let us suppose that the semisymmetry type curvature condition $Q\cdot R=0$ holds on M^{2n+1}. Then we have

$$Q(R(X,Y)Z)-R(QX,Y)Z-R(X,QY)Z-R(X,Y)QZ=0, \tag{50}$$

for all vector fields $X,\ Y,\ Z$ in M^{2n+1}.

Setting $X=Z=\xi$ in the above equation and using (14) we get

$$\mu[h'Q-Qh']Y=4nk^2[\eta(Y)\xi-Y]-4nk\mu h'Y. \tag{51}$$

Now from (46) we observe that $h'Q=Qh'$. Applying this in (51) we obtain

$$h'Y=\frac{k}{2}[Y-\eta(Y)\xi]. \tag{52}$$

Substituting Y by $h'Y$ in the foregoing equation we have

$$\phi^2Y=-\frac{k^2}{4(k+1)}[-Y+\eta(Y)\xi]. \tag{53}$$

With the help of (9) we get $-\frac{k^2}{4(k+1)}=1$, which implies $k=-2$.

Since $\lambda^2=-k-1$, $k=-2$ implies $\lambda^2=1$. Without loss of generality we assume that $\lambda=-1$.

Now letting $X,\ Y,\ Z\in[\lambda]'$ and noticing that $k=-2$, $\lambda=-1$, from Lemma 5 we have

$$R(X_\lambda,Y_\lambda)Z_\lambda=0,$$

and

$$R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda}=-4[g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda}-g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}],$$

for any $X_\lambda,Y_\lambda,Z_\lambda\in[\lambda]'$ and $X_{-\lambda},Y_{-\lambda},Z_{-\lambda}\in[-\lambda]'$. Also noticing $\mu=-2$ it follows from Lemma 4.1 that $K(X,\xi)=-4$ for any $X\in[-\lambda]'$ and $K(X,\xi)=0$ for any $X\in[\lambda]'$. Again from Lemma 4.1 we see that $K(X,Y)=-4$ for any $X,Y\in[-\lambda]'$ and $K(X,Y)=0$ for any $X,Y\in[\lambda]'$. As is shown in [6] that the distribution $[\xi]\oplus[\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H=-(1-\lambda)\xi$, where H is the mean curvature tensor field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda=-1$, then the two orthogonal distributions $[\xi]\oplus[\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. This completes the proof of our theorem. □

Theorem 4. *If a $(k,\mu)'$-almost Kenmotsu manifold satisfies the curvature condition $Q(S,R)=0$, then the manifold is either Einstein or locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. Since, the curvature condition $Q(S,R)=0$ holds on M^{2n+1} we have

$$((X\wedge_S Y)\cdot R)(U,V)W=0, \tag{54}$$

which implies

$$(X\wedge_S Y)R(U,V)W-R((X\wedge_S Y)U,V)W \\ -R(U,(X\wedge_S Y)V)W-R(U,V)(X\wedge_S Y)W=0. \tag{55}$$

Using (3) in (55) we get

$$S(Y,R(U,V)W)X-S(X,R(U,V)W)Y-R(S(Y,U)X \\ -S(X,U)Y,V)W-R(U,S(Y,V)X-S(X,V)Y)W \\ -R(U,V)(S(Y,W)X-S(X,W)Y)=0. \tag{56}$$

Putting $W=\xi$ in (56) and using (47) and (49) we obtain

$$-2[S(h'U,Y)\eta(V)X-S(h'V,Y)\eta(U)X] \\ +2[S(h'U,X)\eta(V)Y-S(h'V,X)\eta(U)Y] \\ +kS(Y,U)\eta(X)V+2S(Y,U)[\eta(V)h'X-\eta(X)h'V] \\ -kS(X,U)\eta(Y)V-2S(X,U)[\eta(V)h'Y-\eta(Y)h'V] \\ -kS(Y,V)\eta(X)U+2S(Y,V)[\eta(X)h'U-\eta(U)h'X] \\ +kS(X,V)\eta(Y)U-2S(X,V)[\eta(Y)h'U-\eta(U)h'Y] \\ -2nk\eta(Y)R(U,V)X+2nk\eta(X)R(U,V)Y=0. \tag{57}$$

Setting $Y=\xi$ in the foregoing equation and making use of (47) and (49) we have

$$2[S(h'U,X)\eta(V)\xi-S(h'V,X)\eta(U)\xi]-kS(X,U)V+2S(X,U)h'V \\ +kS(X,V)U-2S(X,V)h'U-2nkR(U,V)X=0. \tag{58}$$

Again replacing U by ξ in the above equation and using (48) and (49) yields

$$-2S(h'V,X)\xi+kS(X,V)\xi-2nk^2g(X,V)\xi+4nkg(h'V,X)\xi=0. \tag{59}$$

Now letting $X,\ V\in[\lambda]'$, then equation (59) reduces to

$$(k-2\lambda)[S(X,V)-2nkg(X,V)]=0, \tag{60}$$

which implies that either $S(X,V)=2nkg(X,V)$ for all $X,\ V\in[\lambda]'$ or $k=2\lambda$.

If $S(X,V)=2nkg(X,V)$, then the manifold is an Einstein manifold.

If $k=2\lambda$, then from $\lambda^2=-k-1$ we get $\lambda=-1$ and hence $k=-2$.

Therefore, by the same argument as in Theorem 3 we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. □

Theorem 5. *If a $(k,\mu)'$-almost Kenmotsu manifold satisfies the curvature condition $R \cdot R = Q(S,R)$, then the manifold is either Einstein or locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. To complete the proof we first evaluate the terms $R\cdot R$ and $Q(S,R)$ as given below.

$$\begin{aligned}(R(X,Y)\cdot R)(U,V)W &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &\quad - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W. \end{aligned} \tag{61}$$

Putting $Y = U = \xi$ in the above equation we obtain

$$\begin{aligned}(R(X,\xi)\cdot R)(\xi,V)W &= R(X,\xi)R(\xi,V)W - R(R(X,\xi)\xi,V)W \\ &\quad - R(\xi,R(X,\xi)V)W - R(\xi,V)R(X,\xi)W. \end{aligned} \tag{62}$$

For the sake of completeness, we present the terms of the R.H.S of the equation (62) as follows:

$$\begin{aligned}R(X,\xi)R(\xi,V)W &= [-k^2 g(V,W)\eta(X) + k^2\eta(W)g(X,V) - 4k\eta(W)g(h'V,X) \\ &\quad + 2k\eta(X)g(h'V,W) + 4\eta(W)g(h'^2X,V)]\xi \\ &\quad + k^2 g(V,W)X - 2kg(V,W)h'X - k^2\eta(V)\eta(W)X \\ &\quad + 2k\eta(V)\eta(W)h'X - 2kg(h'V,W)X + 4g(h'V,W)h'X, \end{aligned} \tag{63}$$

$$\begin{aligned}R(R(X,\xi)\xi,V)W &= [-k^2\eta(X)g(V,W) + 2k\eta(X)g(h'V,W)]\xi + kR(X,V)W \\ &\quad - 2R(h'X,V)W + k^2\eta(X)\eta(W)V - 2k\eta(X)\eta(W)h'V, \end{aligned} \tag{64}$$

$$\begin{aligned}R(\xi,R(X,\xi)V)W &= [k^2\eta(V)g(X,W) - 4k\eta(V)g(h'X,W) + 4g(h'^2X,W)]\xi \\ &\quad - k^2\eta(V)\eta(W)X + 4k\eta(V)\eta(W)h'X - 4\eta(V)\eta(W)h'^2X \end{aligned} \tag{65}$$

and

$$\begin{aligned}R(\xi,V)R(X,\xi)W &= [-k^2\eta(V)g(X,W) + k^2\eta(W)g(X,V) - 4k\eta(W)g(h'X,V) \\ &\quad + 2k\eta(V)g(h'X,W) + \eta(W)g(h'^2X,V)]\xi + k^2g(X,W)V \\ &\quad - 2kg(X,W)h'V - k^2\eta(X)\eta(W)V + 2k\eta(X)\eta(W)h'V \\ &\quad - 2kg(h'X,W)V + 4g(h'X,W)h'V, \end{aligned} \tag{66}$$

for any vector fields X, V, W, where equations (47) and (48) have been used.

Substituting (63)-(66) in (62) we get

$$\begin{aligned}(R(X,\xi)\cdot R)(\xi,V)W &= -kR(X,V)W + 2R(h'X,V)W + k^2g(V,W)X \\ &\quad - 2kg(h'V,W)X - 2kg(V,W)h'X - 2k\eta(V)\eta(W)h'X \\ &\quad + 4\eta(V)\eta(W)h'^2X + 4g(h'V,W)h'X - k^2g(X,W)V \\ &\quad + 2kg(h'X,W)V - 4g(h'X,W)h'V + 2kg(X,W)h'V \\ &\quad - [-2k\eta(V)g(h'X,W) + 4\eta(V)g(h'^2X,W)]\xi. \end{aligned} \tag{67}$$

On the other hand

$$\begin{aligned} Q(S,R)(U,V,W;X,Y) &= ((X \wedge_S Y)\cdot R)(U,V)W \\ &= S(Y,R(U,V)W)X - S(X,R(U,V)W)Y \\ &\quad - R(S(Y,U)X - S(X,U)Y,V)W \\ &\quad - R(U,S(Y,V)X - S(X,V)Y)W \\ &\quad - R(U,V)(S(Y,W)X - S(X,W)Y). \end{aligned} \tag{68}$$

Substituting $Y = U = \xi$ and using (47)-(49) we obtain

$$\begin{aligned} Q(S,R)(\xi,V,W;X,\xi) &= k[2nkg(V,W) - 2nk\eta(V)\eta(W)]X \\ &\quad - k[2nkg(V,W)\eta(X) - \eta(W)S(X,V)]\xi \\ &\quad + 2[2nk\eta(X)g(h'V,W) - \eta(W)S(h'V,X)]\xi \\ &\quad + 2nkg(h'V,W)X - 2nkR(X,V)W \\ &\quad + 2nk\eta(X)[k\{g(V,W)\xi - \eta(W)V\} - 2\{g(h'V,W)\xi \\ &\quad - \eta(W)h'V\}] - 2nk\eta(V)[k\{g(X,W)\xi - \eta(W)X\} \\ &\quad - 2\{g(h'X,W)\xi - \eta(W)h'X\}] - 2nk\eta(W)[k\{g(V,X)\xi - \\ &\quad \eta(X)V\} - 2\{g(h'V,X)\xi - \eta(X)h'V\}] \\ &\quad + S(X,W)[k\{\eta(V)\xi - V\} + 2h'V]. \end{aligned} \tag{69}$$

Since the condition $R\cdot R = Q(S,R)$ is realized on M, equating (67), (69) and then putting $W = \xi$ we get

$$\begin{aligned} &- kR(X,V)\xi + 2R(h'X,V)\xi + k^2\eta(V)X - 4k\eta(V)h'X + 4\eta(V)h'^2X - k^2\eta(X)V \\ &+ 2k\eta(X)h'V = kS(X,V)\xi - 2S(h'V,X)\xi - 2nk^2g(X,V)\xi + 4nkg(h'V,X)\xi. \end{aligned}$$

Taking inner product of the foregoing equation with ξ yields

$$kS(X,V) - 2S(h'V,X) - 2nk^2g(X,V) + 4nkg(h'V,X) = 0. \tag{70}$$

Now letting $X,V \in [\lambda]'$, then from (70) we obtain

$$(k-2\lambda)[S(X,V) - 2nkg(X,V)] = 0, \tag{71}$$

which implies that either $S(X,V) = 2nkg(X,V)$ or $k = 2\lambda$.

If $S(X,V) = 2nkg(X,V)$, then the manifold is an Einstein manifold.

If $k = 2\lambda$, then by the same argument as in Theorem 3 we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. This completes the proof of our theorem. □

5 Example of a 3-dimensional (k,μ)-almost Kenmotsu manifold

Consider $M^3 = \{(x,y,z)\in\mathbb{R}^3 : z \neq 0\}$. The vector fields

$$e_1 = z\frac{\partial}{\partial x},\ \ e_2 = z\frac{\partial}{\partial y},\ \ e_3 = -z\frac{\partial}{\partial z}$$

are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let us consider $e_3 = \xi$. The 1-form η is defined by $\eta(X) = g(X, e_3)$ for any vector field X on M^3. The $(1,1)$-tensor field ϕ is defined by

$$\phi(e_1) = -e_2, \ \ \phi(e_2) = e_1, \ \ \phi(e_3) = 0.$$

Using linearity of ϕ and g we have

$$\eta(\xi) = 1, \ \ \phi^2 X = -X + \eta(X)\xi \ \text{ and } \ g(\phi X, \phi Y) = g(X, Y) - \eta(x)\eta(Y)$$

for any vector field X, Y on M^3. Now it is easy to see that

$$[e_1, \xi] = e_1, \ \ [e_1, e_2] = 0 \ \text{ and } \ [e_2, \xi] = 0.$$

In view of the above relations we have $h = \frac{1}{2}\pounds_\xi \phi = 0$.

The well known Koszul's formula is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using the above Koszul's formula, we obtain the Levi-Civita connection ∇ as follows:

$$\begin{array}{lll} \nabla_{e_1} e_1 = -\xi, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} \xi = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = \xi, & \nabla_{e_2} \xi = e_2, \\ \nabla_\xi e_1 = 0, & \nabla_\xi e_2 = 0, & \nabla_\xi \xi = 0. \end{array}$$

In view of the above relations we have $\nabla_X \xi = X - \eta(X)\xi$ for any vector field X on M^3. Thus (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ and hence M^3 is an almost Kenmotsu structure.

By the above relations, we can easily obtain the components of the curvature tensor R as follows:

$$\begin{array}{lll} R(e_1, e_2)e_1 = e_2, & R(e_1, e_2)e_2 = -e_1, & R(e_1, e_2)\xi = 0, \\ R(e_2, \xi)e_1 = 0, & R(e_2, \xi)e_2 = \xi, & R(e_2, \xi)\xi = -e_2, \\ R(e_1, \xi)e_1 = \xi, & R(e_1, \xi)e_2 = 0, & R(e_1, \xi)\xi = -e_1. \end{array}$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to (k, μ)-nullity distribution with $k = -1$ and $h = 0$. Also we see that

$$R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y].$$

From above we say that M^3 is locally isometric to $\mathbb{H}^3(-1)$. Also the Ricci tensor S is of Einstein type, i.e., $S(X, Y) = -2g(X, Y)$.

Since the space is of constant curvature, it is locally symmetric ($\nabla R = 0$), which imply it is semisymmetric ($R \cdot R = 0$). Since M^3 is Einstein and of constant curvature, $Q(S, R) = 0$ holds and therefore, $R \cdot R = Q(S, R)$ holds on M^3. Thus, Theorem 2 is verified.

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References

[1] D. E. Blair: *Contact Manifolds in Riemannian Geometry.* Lecture Notes on Mathematics 509. Springer-Verlag, Berlin-New York (1976).

[2] D. E. Blair: *Riemannian Geometry of Contact and Symplectic Manifolds (second edition).* Progress in Mathematics 203. Birkhäuser, Boston (2010).

[3] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou: Contact metric manifolds satisfying a nullity condition. *Israel. J. Math.* 91 (1-3) (1995) 189–214.

[4] D. Dey, P. Majhi: On the quasi-conformal curvature tensor of an almost Kenmotsu manifold with nullity distributions. *Facta Univ. Ser. Math. Inform.* 33 (2) (2018) 255–268.

[5] G. Dileo, A. M. Pastore: Almost Kenmotsu manifolds and local symmetry. *Bull. Belg. Math. Soc. Simon Stevin* 14 (2) (2007) 343–354.

[6] G. Dileo, A. M. Pastore: Almost Kenmotsu manifolds and nullity distributions. *J. Geom.* 93 (1-2) (2009) 46–61.

[7] G. Ghosh, P. Majhi, U. C. De: On a classification of almost Kenmotsu manifolds with generalized $(k, \mu)'$-nullity distribution. *Kyungpook Math. J.* 58 (1) (2018) 137–148.

[8] K. Kenmotsu: A class of almost contact Riemannian manifolds. *Tohoku Math. J. (2)* 24 (1972) 93–103.

[9] D. Kowalczyk: On some subclass of semisymmetric manifolds. *Soochow J. Math.* 27 (4) (2001) 445–461.

[10] A. M. Pastore, V. Saltarelli: Generalized nullity distribution on almost Kenmotsu manifolds. *Int. Elec. J. Geom.* 4 (2) (2011) 168–183.

[11] P. Verheyen, L. Verstraelen: A new intrinsic characterization of hypercylinders in Euclidean spaces. *Kyungpook Math. J.* 25 (1) (1985) 1–4.

[12] L. Verstraelen: Comments on pseudosymmetry in the sense of Ryszard Deszcz. *In: Geometry and Topology of Submanifolds, VI. River Edge. NJ: World Sci. Publishing* 6 (1994) 199–209.

[13] Y. Wang, X. Liu: Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions. *Ann. Polon. Math.* 112 (1) (2014) 37–46.

[14] Y. Wang, X. Liu: On ϕ-recurrent almost Kenmotsu manifolds. *Kuwait J. Sci.* 42 (1) (2015) 65–77.

[15] Y. Wang, W. Wang: Curvature properties of almost Kenmotsu manifolds with generalized nullity conditions. *Filomat* 30 (14) (2016) 3807–3816.

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