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An integral transform and its application in the propagation of Lorentz-Gaussian beams

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Abstract. The aim of the present note is to derive an integral transform

$$I = \int_0^\infty x^{s+1} e^{-\beta x^2 + \gamma x} M_{k,\nu}\left(2\zeta x^2\right) J_\mu(\chi x) dx,$$

involving the product of the Whittaker function $M_{k,\nu}$ and the Bessel function of the first kind J_μ of order μ. As a by-product, we also derive certain new integral transforms as particular cases for some special values of the parameters k and ν of the Whittaker function. Eventually, we show the application of the integral in the propagation of hollow higher-order circular Lorentz-cosh-Gaussian beams through an ABCD optical system (see, for details [13], [3]).

1 Introduction

Integral transforms involving Whittaker and Bessel functions have given considerable attention in the litterature. In the last decade, many papers studied a considerable number of cases of integral transforms involving the product of Bessel and special functions [6], [2], [5]. The used weight in the integrand is $e^{\gamma x}$ or $e^{-\beta x^2}$. In view of this, it is worth investigating a general integral transform involving Whittaker and Bessel functions with a weight $e^{-\beta x^2+\gamma x}$. This integral is important to evaluate the propagation of some laser beams in the space. A closed form of the considered integral will be derived. To the best of our knowledge, the results of the

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present contribution have not been previously published. The folowing definitions are essential for the present investigation:

The classical Gauss's hypergeometric function is defined by

$$ {}_2F_1(a,b;c;\,x) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{x^n}{n!}, \tag{1} $$

where, $(a)_n$ $(a \in \mathbb{C})$ is the well known Pochhammer symbol (see [1], [8]). The Kummer confluent hypergeometric function ${}_1F_1$ (see [1], [8]) is defined by the series representation

$$ {}_1F_1(a;b;\,x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\,\frac{x^k}{k!}, \tag{2} $$

Let us consider the following second-order linear homogenous differential equation given by

$$ x^2\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (\alpha^2 x^{2r} + \beta^2 y) = 0, $$

whose solution is

$$ y = x^{-p}\left[C_1 J_{q/r}\left(\frac{\alpha}{r}x^r\right) C_2 Y_{q/r}\left(\frac{\alpha}{r}x^r\right)\right] \quad (q = \sqrt{p^2} - \beta^2), $$

where, $J_n(x)$ and $Y_n(x)$ are the Bessel functions of first and second kind, respectively. Indeed we have the following series representation for the Bessel function of the first kind of order μ (see [4], [10]):

$$ J_\mu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,(\frac{x}{2})^{\mu+2k}}{k!\Gamma(\mu+k+1)}; \quad \forall x \in \mathbb{C} \setminus (-\infty, 0). \tag{3} $$

It is well known that

$$ J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x, $$

and

$$ J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\cos x. $$

The Whittaker function $M_{k,\nu}(x)$ and $W_{k,\nu}(x)$ (see, [11], [12]), in terms of Kummer confluent hypergeometric function defined by (2) are given, respectively by

$$ M_{k,\nu}(x) = x^{\nu+\frac{1}{2}} e^{-\frac{x}{2}}\,{}_1F_1\left(\frac{1}{2}+\nu-k,\, 2\nu+1;\; x\right), \tag{4} $$

and

$$ W_{k,\nu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{x}{2}}\,U\left(\frac{1}{2}+\nu-k,\, 2\nu+1;\; x\right). \tag{5} $$

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ can be defined as

$$ L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n;\,1+\alpha;\,x). \tag{6} $$

From (6), it follows that

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^n}{k!(n-k)!(1+\alpha)_k}. \tag{7}$$

2 Main results

In this section, we compute certain integral transforms involving the product of Bessel function and Whittaker function given in (3) and (4), respectively. These integral formulas are expressed in terms of Kummer confluent hypergeometric function as given in Theorem 1.

Theorem 1. *Let* $\Re(\mu) > -1$, $\Re(s+\nu+\frac{\mu}{2}) > -1$, *and* $\Re(\beta+\zeta) > 0$. *Consider the integral* I

$$I = \int_0^\infty x^{s+1} e^{-\beta x^2 + \gamma x} M_{k,\nu}\left(2\zeta x^2\right) J_\mu(\chi x) dx = (2\zeta)^{\nu+1/2} \delta^{2\nu+s+2} e^{\frac{\gamma^2}{4\varepsilon^2}} \times$$
$$\times \sum_{r=0}^{+\infty} \frac{(\nu+1/2-k)_r (2\zeta\delta^2)^r}{r!(2\nu+1)_r} \sum_{q=0}^{\vartheta} \frac{C_\vartheta^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \tag{8}$$

where

$$C_{mq} = \left\{ \begin{array}{l} \frac{\left(\frac{\chi}{2\varepsilon}\right)^m}{2m!} \frac{\Gamma\left(\frac{m+q+1}{2}\right)}{\varepsilon^{q+1}} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\chi^2}{4\varepsilon^2}\right) + \\ +(-1)^{q+m}\left(\frac{\chi\delta^{1/4}}{2}\right)^m \delta^{\frac{q+1}{4}} e^{-\varepsilon^2\sqrt{\delta}} \sum_{p=0}^{+\infty} \frac{\left(\frac{-\chi^2\sqrt{\delta}}{4}\right)^p}{p!(2p+q+m+1)\Gamma(m+p+1)} \times \\ \times {}_1F_1\left(1; 1+p+\frac{q+m+1}{2}; \varepsilon^2\sqrt{\delta}\right) \end{array} \right\},$$

$$\varepsilon^2 = \beta+\zeta, \delta = \frac{\gamma}{2\varepsilon^2} \text{ and } \vartheta = 2\nu+2r+s+2. \tag{9}$$

Proof. To derive (8), we make use of (2) and (4), and consequently, the expression of the integral I becomes

$$I = (2\zeta)^{\nu+1/2} e^{\frac{\gamma^2}{4(\beta+\zeta)}} \sum_{r=0}^{+\infty} \frac{(\nu+1/2-k)_r (2\zeta)^r}{(2\nu+1)_r r!} I_r, \tag{10}$$

where $I_r = I_r' + I_r''$, with

$$I_r' = \int_0^\infty (t+\delta)^{2\nu+2r+s+2} e^{-(\beta+\zeta)t^2} J_\mu(\chi t + \chi\delta) dt, \tag{11}$$

and

$$I_r'' = \int_{-\delta}^0 (t+\delta)^{2\nu+2r+s+2} e^{-(\beta+\zeta)t^2} J_\mu(\chi t + \chi\delta) dt. \tag{12}$$

Making use of the following identities (see [4])

$$(t+\delta)^n = \sum_{p=0}^{n} C_n^p \delta^{n-p} t^p, \tag{13}$$

with $C_n^p = \frac{n!}{p!(n-p)!}$ is the binomial coefficient,
and

$$J_\mu(\chi t + \chi\delta) = \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) J_m(\chi t), \tag{14}$$

and using the integral formulas (see [4])

$$\int_0^\infty t^q e^{-\eta t^2} J_m(\chi t) dt = \frac{\left(\frac{\chi}{2\sqrt{\eta}}\right)^m \Gamma\left(\frac{m+q+1}{2}\right)}{2(\sqrt{\eta})^{q+1} m!} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\chi^2}{4\eta}\right), \tag{15}$$

and

$$\int_0^u t^{\nu-1} e^{-\mu t} dt = \frac{1}{\mu^\nu} \gamma(\nu, u\mu), \tag{16}$$

where γ is the incomplet Gamma function given as

$$\gamma(\alpha, x) = \alpha^{-1} x^\alpha {}_1F_1(\alpha; 1+\alpha; -x),$$

with $\Re(q+m) > -1$ and $\Re(\eta) > 0$, and Eqs. (11) and (12) can be rearranged as

$$I'_r = \sum_{q=0}^{\vartheta} C_\vartheta^q \delta^{\vartheta-q} \times \\ \times \sum_{m=-\infty}^{+\infty} \frac{J_{\mu-m}(\chi\delta)\chi^m \Gamma\left(\frac{m+q+1}{2}\right)}{2^{m+1} m! (\beta+\zeta)^{\frac{m+q+1}{2}}} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\chi^2}{4(\zeta+\beta)}\right), \tag{17}$$

and

$$I''_r = \sum_{q=0}^{\vartheta} C_\vartheta^q \delta^{\vartheta-q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta)(-1)^{q+m} \left(\frac{\chi}{2}\right)^m \delta^{\frac{q+m+1}{4}} e^{-\varepsilon^2\sqrt{\delta}} \times \\ \times \sum_{p=0}^{+\infty} \frac{\left(\frac{-\chi^2\sqrt{\delta}}{4}\right)^p}{p!(2p+q+m+1)\Gamma(m+p+1)} {}_1F_1\left(1; 1+p+\frac{q+m+1}{2}; \varepsilon^2\sqrt{\delta}\right), \tag{18}$$

with $\vartheta = 2\nu + 2r + s + 2$, and the required result easily follows. This completes the proof. □

Remark

For simulations, Eq. (14) can be replaced by

$$J_\mu(\chi t+\chi\delta)=\sum_{m=0}^{\mu}J_{\mu-m}(\chi\delta)J_m(\chi t)+\sum_{m=1}^{+\infty}(-1)^m\left[J_{\mu+m}(\chi\delta)J_m(\chi t)+J_m(\chi\delta)J_{\mu+m}(\chi t)\right],$$

3 Special cases

In this section, we derive certain new formulas by recalling relations of the Whittaker function with some other special functions as the exponential, the Modified Bessel, the sine hyperbolic, the generalized Laguerre polynomial, the Hermite polynomial, the Whittaker and erf functions. By taking some particular values of the index k and ν of this last function, we establish the following corollaries.

Corollary 1. *For $\Re(\mu)>-1$, $\Re(s+\nu+\frac{\mu}{2})>-1$, $\Re(\beta+\zeta)>0$, the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+1}e^{-(\beta-\varepsilon)x^2+\gamma x}J_\mu(\chi x)dx=\delta^{s+1-2k}e^{\frac{\gamma^2}{4\varepsilon^2}}\times\sum_{r=0}^{+\infty}\frac{(2\zeta\delta^2)^r}{r!}\sum_{q=0}^{\vartheta_1}\frac{C_{\vartheta_1}^q}{\delta^q}\sum_{m=-\infty}^{+\infty}J_{\mu-m}(\chi\delta)C_{mq},\quad(19)$$

where

$$\vartheta_1=2r+s-2k+1.$$

For $\nu=-(k+\frac{1}{2})$, the Whittaker function in (4) *can be expressed in terms of exponential function (see [7], [9]) as*

$$M_{k,-k-\frac{1}{2}}(z)=e^{\frac{z}{2}}z^{-k}.\qquad(20)$$

Now in view of the (20) *and* (8)*, it is easy to establish the result in* (19).

Corollary 2. *With the conditions of* (8)*, the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+2}e^{-\beta x^2+\gamma x}I_\nu(\zeta x^2)J_\mu(\chi x)dx=\frac{(2\zeta)^{\nu+1/2}}{2^{2\nu+3/2}\varepsilon\sqrt{2\zeta}\nu!}\delta^{2\nu+s+2}e^{\frac{\gamma^2}{4\varepsilon^4}}\times\sum_{r=0}^{+\infty}\frac{(\nu+1/2)_r}{(2\nu+1)_r}\frac{(2\zeta\delta^2)^r}{r!}\sum_{q=0}^{\vartheta_2}\frac{C_{\vartheta_2}^q}{\delta^q}\sum_{m=-\infty}^{+\infty}J_{\mu-m}(\chi\delta)C_{mq},\quad(21)$$

where

$$\vartheta_2=2\nu+2r+s+2.\qquad(22)$$

With the help of (8) *and by expressing the Whittaker function in terms of the modified Bessel function of the first kind* I_ν *(see [9], [10]), with* $k = 0$*, as follows*

$$M_{0,\nu}(2z) = 2^{2\nu+\frac{1}{2}}\nu!\sqrt{z}I_\nu(z), \tag{23}$$

the required result (21) *directly follows.*

Corollary 3. *For* $\Re(\mu) > -1,\ \Re(s+\frac{\mu+1}{2}) > -1,\ \Re(\beta+\zeta) > 0$*, the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+1}e^{-\beta x^2+\gamma x}\sinh\left(\zeta x^2\right)J_\mu(\chi x)dx = \frac{(2\zeta)}{4}\delta^{s+3}e^{\frac{\gamma^2}{4\varepsilon^4}}\times$$
$$\times\sum_{r=0}^{+\infty}\frac{(2\zeta\delta^2)^r}{(2)_r r!}\sum_{q=0}^{\vartheta_3}\frac{C_{\vartheta_3}^q}{\delta^q}\sum_{m=-\infty}^{+\infty}J_{\mu-m}(\chi\delta)C_{mq}, \tag{24}$$

where

$$\vartheta_3 = 2r+s+3. \tag{25}$$

If we take $k = 0$ *and* $\nu = \frac{1}{2}$*, the Whittaker function in* (4) *can be expressed in terms of sine hyperbolic function (see [7], [9]) as*

$$M_{0,\frac{1}{2}}(z) = 2\sinh\left(\frac{z}{2}\right), \tag{26}$$

from which the required result (24) *easily follows.*

Corollary 4. *Let* $\Re(\mu) > -1,\ \Re(s+\frac{\mu+n}{2}) > -1,\ \Re(\beta+\zeta) > 0$*, the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+n+2}e^{-(\beta+\zeta)x^2+\gamma x}L_s^n(2\zeta x^2)J_\mu(\chi x)dx = \frac{(n+1)_s}{s!}\delta^{s+n+2}e^{\frac{\gamma^2}{4\varepsilon^4}}\times$$
$$\times\sum_{r=0}^{+\infty}\frac{(-s)r(2\zeta\delta^2)^r}{(n+1)_r r!}\sum_{q=0}^{\vartheta_4}\frac{C_{\vartheta_4}^q}{\delta^q}\sum_{m=-\infty}^{+\infty}J_{\mu-m}(\chi\delta)C_{mq}, \tag{27}$$

where

$$\vartheta_4 = n+2r+s+2. \tag{28}$$

By setting $k = \frac{n+1}{2}+s$ *and* $\nu = \frac{n}{2}$ *in* (4)*, and using* (6) *we have the relation (see [7], [9])*

$$M_{\frac{n+1}{2}+s,\frac{n}{2}}(z) = \frac{s!}{(n+1)_s}e^{\frac{-z}{2}}z^{\frac{n+1}{2}}L_s^{(n)}(z), \tag{29}$$

Now in view of (29)*, the assertion* (27) *easily follows.*

Corollary 5. $\Re(\mu) > -1,\ \Re(s+\frac{2\mu-1}{4}) > -1,\ \Re(\beta+\zeta) > 0$*, the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+3/2}e^{-(\beta+\zeta)x^2+\gamma x}H_{2p}(\sqrt{2\zeta}x)J_\mu(\chi x)dx = \frac{2p!}{(-1)^p p!}\times$$
$$\times\,\delta^{s+3/2}e^{\frac{\gamma^2}{4\varepsilon^4}}\sum_{r=0}^{+\infty}\frac{(-p)_r(2\zeta\delta^2)^r}{(1/2)_r r!}\sum_{q=0}^{\vartheta_5}\frac{C_{\vartheta_5}^q}{\delta^q}\sum_{m=-\infty}^{+\infty}J_{\mu-m}(\chi\delta)C_{mq}, \tag{30}$$

where

$$\vartheta_5 = 2r + s + 3/2. \tag{31}$$

The above corollary can easily be established by setting $k = p + \frac{1}{4}$ *and* $\nu = \frac{-1}{4}$ *in* (4). *This special case corresponds to the Hermite polynomial* H_{2p} *(see [7], [9]) with the odd integer order and the relation betweeen* M *and* H_{2p} *is given by*

$$M_{p+\frac{1}{4},-\frac{1}{4}}\left(z^2\right) = (-1)^p \frac{p!}{(2p)!} e^{\frac{-z^2}{2}} \sqrt{z} H_{2p}(z). \tag{32}$$

Corollary 6. $\Re(\mu) > -1,\ \Re(s + \frac{2\mu+1}{4}) > -1,\ \Re(\beta + \zeta) > 0,$ *the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+3/2} e^{-(\beta+\zeta)x^2 - \gamma x} H_{2p+1}(\sqrt{2\zeta}x) J_\mu(\chi x) dx = \frac{(2p+1)!\sqrt{2\zeta}}{(-1)^p p!} \times$$
$$\times\, \delta^{s+5/2} e^{\frac{\gamma^2}{4\varepsilon^4}} \sum_{r=0}^{+\infty} \frac{(3/4-k)_r (2\zeta\delta^2)^r}{(3/2)_r r!} \sum_{q=0}^{\vartheta_6} \frac{C^q_{\vartheta_6}}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \tag{33}$$

where

$$\vartheta_6 = 2r + s + 5/2. \tag{34}$$

This corollary is established by setting $k = p + \frac{3}{4}$ *and* $\nu = \frac{1}{4}$ *in* (4), *which consequently gives the relation between the Whittaker function and the Hermite polynomial* H_{2p+1} *(see [7], [9]) of an even order as*

$$M_{p+\frac{3}{4},\frac{1}{4}}\left(z^2\right) = (-1)^p \frac{p!}{(2p+1)!} \frac{e^{\frac{-z^2}{2}}\sqrt{z}}{2} H_{2p+1}(z), \tag{35}$$

and by direct application of (8).

Corollary 7. *Let* $\Re(\mu) > -1,\ \Re(s + \frac{\mu+n}{2}) > -1,\ \Re(\beta+\zeta) > 0,$ *the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+3} e^{-\beta x^2 + \gamma x} W_{n+\frac{1+n}{2},n/2}(2\zeta x^2) J_\mu(\chi x) dx = \frac{(n+1)_s}{(-1)^s p! (\sqrt{2\zeta})^{\frac{1-n}{2}}} \times$$
$$\times\, \delta^{s+n+2} e^{\frac{\gamma^2}{4\varepsilon^4}} \sum_{r=0}^{+\infty} \frac{(-s)_r (2\zeta\delta^2)^r}{(n+1)_r r!} \sum_{q=0}^{\vartheta_7} \frac{C^q_{\vartheta_7}}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \tag{36}$$

where

$$\vartheta_7 = n + 2r + s + 2, \tag{37}$$

The assertion in (36) *directly follows from the relation between the Whittaker function and the generalized Laguerre polynomial (see [7], [9]) given as*

$$L_s^{(n)}(z) = \frac{(-1)^s}{s!} e^{\frac{z}{2}} z^{\frac{1-n}{2}} W_{n+\frac{1+n}{2},n/2}(z), \tag{38}$$

Corollary 8. *Let* $\Re(\mu) > -1$, $\Re(s+\frac{2\mu+1}{4}) > -1$, $\Re(\beta+\zeta) > 0$, *the undermentioned integral transform holds true:*

$$\int_0^\infty x^{s+3/2} e^{-(\beta-\zeta)x^2+\gamma x} erf(\sqrt{2\zeta}x) J_\mu(\chi x) dx = \frac{4\sqrt{2\zeta}}{\sqrt{\pi}} \delta^{s+5/2} e^{\frac{\gamma^2}{4\varepsilon^4}} \times$$
$$\times \sum_{r=0}^{+\infty} \frac{(1)_r (2\zeta\delta^2)^r}{(3/2)_r r!} \sum_{q=0}^{\vartheta_8} \frac{C_{\vartheta_8}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \quad (39)$$

where

$$\vartheta_8 = 2r + s + 5/2, \quad (40)$$

Using the fact that by taking $k = \frac{-1}{4}$ *and* $\nu = \frac{1}{4}$, *the Whittaker function becomes the error function* erf *(see [7], [9]) as*

$$M_{\frac{-1}{4},\frac{1}{4}}\left(z^2\right) = \frac{1}{2} e^{\frac{z^2}{2}} \sqrt{\pi z}\, erf(z), \quad (41)$$

and in view of (8), *it is straightforward to establish* (39).

4 Application

In this section we consider the propagation of hollow higher-order circular Lorentz-cosh-Gaussian beams [13] through an ABCD optical system, expressed as [3]

$$E_{p,l}^\alpha(\rho, z) = E_0 \sum_{n=0}^N a_{2n}(\alpha) \sum_{m=0}^p \frac{p!}{m!(p-m)!} e^{(m-p/2)^2\delta} I_n, \quad (42)$$

where

$$I_n = e^{-\frac{a^2}{w_0}} \int_0^\infty x^{l+1} e^{-\chi x^2+\zeta x} H_{2n}(\sqrt{2\alpha}x) J_0(\beta x) dx. \quad (43)$$

With the help of (30), I_n can be written as

$$I_n = e^{-\frac{a^2}{w_0}} \frac{2n!}{(-1)^n n! \sqrt{2\alpha}} \frac{(2\alpha)^{1/4}}{2\varepsilon} \delta^{l+1/2} e^{\frac{\zeta^2}{4\varepsilon^2}} \sum_{r=0}^n \frac{(-n)_r}{(1/2)_r} \frac{(2\alpha\delta^2)^r}{r!} \sum_{q=0}^{\vartheta_6} \frac{C_{\vartheta_6}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_m(\chi\delta) \times$$
$$\times \left\{ \begin{array}{l} \frac{\left(\frac{\beta}{2\varepsilon}\right)^m}{2m!} \frac{\Gamma\left(\frac{m+q+1}{2}\right)}{\varepsilon^{q+1}} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\beta^2}{4\varepsilon^2}\right) + \\ +(-1)^q \left(\frac{\beta\delta^{1/4}}{2}\right)^m \delta^{\frac{q+1}{4}} e^{-\varepsilon^2\sqrt{\delta}} \sum_{p=0}^{+\infty} \frac{\left(\frac{-\beta^2\sqrt{\delta}}{4}\right)^p}{p!(2p+q+m+1)\Gamma(m+p+1)} \times \\ \times {}_1F_1\left(1; 1+p+\frac{q+m+1}{2}; \varepsilon^2\sqrt{\delta}\right) \end{array} \right\} \quad (44)$$

Substituting (44) in (42), one finds the output field of the considered beams family and the behaviour of this field in the space.

5 Conclusion

In the present investigation, we have elaborated an integral involving the product of the Whittaker and Bessel functions. The results obtained in this document have a general nature and likely to discover certain applications in the theory of beams propagation. Some corollaries are derived from the main theorem as particular cases as they give multifarious representations involving different special functions. An application of our main result is given to derive and investigate an expression of Hollow higher-order circular Lorentz-cosh-Gaussian beams propagating through an ABCD optical system. Finally, the interesting integral derived in the present investigation may be useful in many fields of physics as plasma, radio and laser physics.

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