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A non-linear discrete-time dynamical system related to epidemic SISI model

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Abstract. We consider SISI epidemic model with discrete-time. The crucial point of this model is that an individual can be infected twice. This non-linear evolution operator depends on seven parameters and we assume that the population size under consideration is constant, so death rate is the same with birth rate per unit time. Reducing to quadratic stochastic operator (QSO) we study the dynamical system of the SISI model.

1 Introduction

In [3] SISI model is considered in continuous time as a spread of bovine respiratory syncytial virus (BRSV) amongst cattle. They performed an equilibrium and stability analysis and considered an applications to Aujesky's disease (pseudorabies virus) in pigs. In [5] SISI model was considered as an example and characterised the conditions for fixed point equation. In the both these works it was assumed that the population size under consideration is a constant, so the per capita death rate is equal to per capita birth rate. In epidemiology, a *susceptible* individual (sometimes known simply as a susceptible) is a member of a population who is at risk of becoming infected by a disease. A *susceptibility* only refers to the fact that the virus is able to get into the cell, via having the proper receptor(s), and as a result, despite the fact that a host may be susceptible, the virus may still not be able to cause any pathologies within the host[1]. In epidemiology, *infectivity* is the ability of a pathogen to establish an infection. In biology, a *pathogen* in the oldest and broadest sense, is any organism that can produce disease. A pathogen may also be referred to as an infectious agent, or simply a germ[2].

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[1]https://en.wikipedia.org/wiki/Susceptible
[2]https://en.wikipedia.org/wiki/Infectivity

Let us consider SISI model [5]:

$$\begin{cases} \dfrac{dS}{dt} = b(S+I+S_1+I_1) - \mu S - \beta_1 A(I,I_1)S \\ \dfrac{dI}{dt} = -\mu I + \beta_1 A(I,I_1)S - \alpha I \\ \dfrac{dS_1}{dt} = -\mu S_1 + \alpha I - \beta_2 A(I,I_1)S_1 \\ \dfrac{dI_1}{dt} = -\mu I_1 + \beta_2 A(I,I_1)S_1 \end{cases} \tag{1}$$

where $S-$ density of susceptibles who did not have the disease before, $I-$ density of first time infected persons, S_1- density of recovereds, I_1- density of second time infected persons, $b-$ birth rate, $\mu-$ death rate, $\alpha-$ recovery rate, β_1- susceptibility of persons in S, β_2- susceptibility of persons in S_1, k_1- infectivity of persons in I, k_2- infectivity of persons in I_1. Moreover, $A(I,I_1)$ denotes the so-called force of infection,

$$A(I,I_1) = \frac{k_1 I + k_2 I_1}{P}$$

and $P = S + I + S_1 + I_1$ denotes the total population size. Here we do some replacements:

$$x = \frac{S}{P}, u = \frac{I}{P}, y = \frac{S_1}{P}, v = \frac{I_1}{P}$$

In (1) we assume that $b = \mu$ and by substituting x, u, y, v we have

$$\begin{cases} \dfrac{dx}{dt} = b - bx - \beta_1 A(u,v)x \\ \dfrac{du}{dt} = -bu + \beta_1 A(u,v)x - \alpha u \\ \dfrac{dy}{dt} = -by + \alpha u - \beta_2 A(u,v)y \\ \dfrac{dv}{dt} = -bv + \beta_2 A(u,v)y \end{cases} \tag{2}$$

where all parameters are non-negative. We notice that $\frac{d}{dt}(x+u+y+v) = 0$, from this we deduce that the total population size is constant over time and therefore we assume $x + u + y + v = 1$.

2 Quadratic Stochastic Operators

The quadratic stochastic operator (QSO) [2], [4] is a mapping of the standard simplex

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1\} \tag{3}$$

into itself, of the form

$$V : x'_k = \sum_{i=1}^m \sum_{j=1}^m P_{ij,k} x_i x_j, \qquad k = 1, \dots, m, \tag{4}$$

where the coefficients $P_{ij,k}$ satisfy the following conditions

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1, \qquad (i,j,k = 1,\dots,m). \tag{5}$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\mathbb{P} = (P_{ij,k})_{i,j,k=1}^{m}$ with conditions (5).

Note that each element $x \in S^{m-1}$ is a probability distribution and each such distribution can be interpreted as a state of the corresponding biological system.

For a given $\lambda^{(0)} \in S^{m-1}$ the *trajectory* (orbit) $\{\lambda^{(n)}; n \geq 0\}$ of $\lambda^{(0)}$ under the action of QSO (4) is defined by

$$\lambda^{(n+1)} = V(\lambda^{(n)}),\ n = 0, 1, 2, \dots$$

The main problem in mathematical biology consists in the study of the asymptotical behaviour of the trajectories. The difficulty of the problem depends on given matrix $\mathbb{P}$.

Definition 1. A QSO V is called regular if for any initial point $\lambda^{(0)} \in S^{m-1}$, the limit

$$\lim_{n\to\infty} V^n(\lambda^{(0)})$$

exists, where V^n denotes n-fold composition of V with itself (i.e. n time iterations of V).

3 Reduction to QSO

In this paper we study the discrete time dynamical system associated to the system (2).

Define the evolution operator $V : S^3 \to \mathbb{R}^4, \quad (x,u,y,v) \mapsto \left(x^{(1)}, u^{(1)}, y^{(1)}, v^{(1)}\right)$

$$V : \begin{cases} x^{(1)} = x + b - bx - \beta_1 A(u,v)x \\ u^{(1)} = u - bu + \beta_1 A(u,v)x - \alpha u \\ y^{(1)} = y - by + \alpha u - \beta_2 A(u,v)y \\ v^{(1)} = v - bv + \beta_2 A(u,v)y \end{cases} \tag{6}$$

where $A(u,v) = k_1 u + k_2 v$. Note that if $k_1 = k_2 = 0$ then $A(u,v) = 0$ and operator (6) becomes linear operator which is well studied.

By definition the operator V has a form of QSO, but the parameters of this operator are not related to $P_{ij,k}$. Here to make some relations with $P_{ij,k}$ we find conditions on parameters of (6) rewriting it in the form (4) (as in [7], [8]). Using $x + u + y + v = 1$ we change the form of the operator (6) as following:

$$V : \begin{cases} x^{(1)} = x(1-b)(x+u+y+v) + b(x+u+y+v)^2 - \beta_1(k_1 u + k_2 v)x \\ u^{(1)} = u(1-b-\alpha)(x+u+y+v) + \beta_1(k_1 u + k_2 v)x \\ y^{(1)} = y(1-b)(x+u+y+v) + \alpha u(x+u+y+v) - \beta_2(k_1 u + k_2 v)y \\ v^{(1)} = v(1-b)(x+u+y+v) + \beta_2(k_1 u + k_2 v)y \end{cases}$$

From this system and QSO (4) for the case $m = 4$ we obtain the following relations:

$$
\begin{array}{lll}
P_{11,1} = 1, & 2P_{12,1} = 1 + b - \beta_1 k_1, & 2P_{13,1} = 1 + b, \\
2P_{14,1} = 1 + b - \beta_1 k_2, & P_{22,1} = b, & 2P_{23,1} = 2b, \\
2P_{24,1} = 2b, & P_{33,1} = b, & 2P_{34,1} = 2b, \\
P_{44,1} = b, & 2P_{12,2} = 1 - b - \alpha + \beta_1 k_1, & 2P_{14,2} = \beta_1 k_2, \\
P_{22,2} = 1 - b - \alpha, & 2P_{23,2} = 1 - b - \alpha, & 2P_{24,2} = 1 - b - \alpha, \\
2P_{12,3} = \alpha, & 2P_{13,3} = 1 - b, & P_{22,3} = \alpha, \\
2P_{23,3} = 1 - b + \alpha - \beta_2 k_1, & 2P_{24,3} = \alpha, & P_{33,3} = 1 - b, \\
2P_{34,3} = 1 - b - \beta_2 k_2, & 2P_{14,4} = 1 - b, & 2P_{23,4} = \beta_2 k_1, \\
2P_{24,4} = 1 - b, & 2P_{34,4} = 1 - b + \beta_2 k_2, & P_{44,4} = 1 - b,
\end{array} \tag{7}
$$

other $P_{ij,k} = 0$.

Proposition 1. *We have $V\left(S^3\right) \subset S^3$ if and only if the non-negative parameters $b, \alpha, \beta_1, \beta_2, k_1, k_2$ satisfy the following conditions*

$$
\begin{array}{ccc}
\alpha + b \leq 1, & \beta_1 k_2 \leq 2, & \beta_2 k_1 \leq 2, \\
b + \beta_2 k_2 \leq 1, & |b - \beta_1 k_1| \leq 1, & |b - \beta_2 k_2| \leq 1, \\
|b - \beta_1 k_2| \leq 1, & |\alpha + b - \beta_1 k_1| \leq 1, & |\alpha - b - \beta_2 k_1| \leq 1.
\end{array} \tag{8}
$$

Moreover, under conditions (8) *the operator V is a QSO.*

Proof. The proof can be obtained by using equalities (7) and solving inequalities $0 \leq P_{ij,k} \leq 1$ for each $P_{ij,k}$. □

Remark 1. In the sequel of the paper we consider operator (6) with parameters $b, \alpha, \beta_1, \beta_2, k_1, k_2$ which satisfy conditions (8). This operator maps S^3 to itself and we are interested to study the behaviour of the trajectory of any initial point $\lambda \in S^3$ under iterations of the operator V.

4 Fixed points of the operator (6)

Recall that a fixed point of the operator V is a solution of $V(\lambda) = \lambda$.

4.1 Finding fixed points of the operator (6)

Denote

$$\lambda_1=(1,0,0,0)\,, \quad \lambda_2=(0,1,0,0)\,, \quad \lambda_3=(0,0,1,0)\,, \quad \lambda_4=(0,0,0,1)\,,$$

$$\Lambda_5=\{\lambda=(x,u,y,v)\in S^3 : x=u=0\},$$

$$\Lambda_6=\{\lambda=(x,u,y,v)\in S^3 : x=y=0\},$$

$$\Lambda_7=\{\lambda=(x,u,y,v)\in S^3 : x=v=0\},$$

$$\Lambda_8=\{\lambda=(x,u,y,v)\in S^3 : u=y=0\},$$

$$\Lambda_9=\{\lambda=(x,u,y,v)\in S^3 : u=v=0\},$$

$$\Lambda_{10}=\{\lambda=(x,u,y,v)\in S^3 : y=v=0\},$$

$$\Lambda_{11}=\{\lambda=(x,u,y,v)\in S^3 : x=0\},$$

$$\Lambda_{12}=\{\lambda=(x,u,y,v)\in S^3 : u=0\},$$

$$\Lambda_{13}=\{\lambda=(x,u,y,v)\in S^3 : y=0\},$$

$$\Lambda_{14}=\{\lambda=(x,u,y,v)\in S^3 : v=0\},$$

$$\lambda_{15}=\left(\frac{b}{\beta_1 k_1},\frac{\beta_1 k_1-b}{\beta_1 k_1},0,0\right), \quad \lambda_{16}=\left(\frac{b+\alpha}{\beta_1 k_1},\frac{b(\beta_1 k_1-b-\alpha)}{\beta_1 k_1(b+\alpha)},\frac{\alpha(\beta_1 k_1-b-\alpha)}{\beta_1 k_1(b+\alpha)},0\right),$$

$$\lambda_{17}=\left(\frac{b}{b+\beta_1 A},\frac{b\beta_1 A}{(b+\beta_1 A)(b+\alpha)},\frac{\alpha b\beta_1 A}{(b+\beta_1 A)(b+\beta_2 A)(b+\alpha)},\frac{\alpha\beta_1\beta_2 A^2}{(b+\beta_1 A)(b+\beta_2 A)(b+\alpha)}\right),$$

where A is a positive solution of the equation

$$1=\frac{b\beta_1 k_1}{(b+\beta_1 A)(b+\alpha)}+\frac{\alpha\beta_1\beta_2 k_2 A}{(b+\beta_1 A)(b+\beta_2 A)(b+\alpha)} \tag{9}$$

It is easy to see that

$$\lambda_1\in\Lambda_8,\Lambda_9,\Lambda_{10},\Lambda_{12},\Lambda_{13},\Lambda_{14}, \qquad \lambda_2\in\Lambda_6,\Lambda_7,\Lambda_{10},\Lambda_{11},\Lambda_{13},\Lambda_{14},$$

$$\lambda_3\in\Lambda_5,\Lambda_7,\Lambda_9,\Lambda_{11},\Lambda_{12},\Lambda_{14}, \qquad \lambda_4\in\Lambda_5,\Lambda_6,\Lambda_8,\Lambda_{11},\Lambda_{12},\Lambda_{13},$$

$$\Lambda_5\subset\Lambda_{11},\Lambda_{12}, \qquad \Lambda_6\subset\Lambda_{11},\Lambda_{13}, \qquad \Lambda_7\subset\Lambda_{11},\Lambda_{14},$$

$$\Lambda_8\subset\Lambda_{12},\Lambda_{13}, \qquad \Lambda_9\subset\Lambda_{12},\Lambda_{14}, \qquad \Lambda_{10}\subset\Lambda_{13},\Lambda_{14}.$$

By the following proposition we give all possible fixed points of the operator V.

Proposition 2. *Let $Fix(V)$ be set of fixed points of the operator* (6)*. Then*

$$Fix(V) = \begin{cases} \{\lambda_1\} & \\ \{\lambda_4\} \bigcup \Lambda_9, & \textit{if } b = 0 \\ \Lambda_6 \bigcup \Lambda_9, & \textit{if } b = \alpha = 0 \\ \Lambda_8 \bigcup \Lambda_9, & \textit{if } b = \beta_1 = 0 \\ \Lambda_5 \bigcup \Lambda_9, & \textit{if } b = \beta_2 = 0 \\ \{\lambda_4\} \bigcup \Lambda_9, & \textit{if } b = k_1 = 0 \\ \Lambda_5 \bigcup \Lambda_{12}, & \textit{if } b = k_2 = 0 \\ \Lambda_9 \bigcup \Lambda_{13}, & \textit{if } b = \alpha = \beta_1 = 0 \\ \Lambda_9 \bigcup \Lambda_{11}, & \textit{if } b = \alpha = \beta_2 = 0 \\ \Lambda_6 \bigcup \Lambda_{14}, & \textit{if } b = \alpha = k_1 = 0 \\ \Lambda_6 \bigcup \Lambda_{12}, & \textit{if } b = \alpha = k_2 = 0 \\ \Lambda_{12}, & \textit{if } b = \beta_1 = \beta_2 = 0 \\ \Lambda_8 \bigcup \Lambda_9, & \textit{if } b = \beta_1 = k_1 = 0 \\ \Lambda_{12}, & \textit{if } b = \beta_1 = k_2 = 0 \\ \Lambda_5 \bigcup \Lambda_9, & \textit{if } b = \beta_2 = k_1 = 0 \\ \Lambda_{12}, & \textit{if } b = \beta_2 = k_2 = 0 \\ S^3, & \textit{if } b = \alpha = k_1 = k_2 = 0 \textit{ or } b = \alpha = \beta_1 = \beta_2 = 0 \\ \{\lambda_1, \lambda_{15}\}, & \textit{if } b > 0, \alpha = 0 \textit{ and } \beta_1 k_1 > b \\ \{\lambda_1, \lambda_{16}\}, & \textit{if } b > 0,\ \alpha > 0,\ \beta_2 = 0,\ \beta_1 k_1 > b + \alpha \\ \{\lambda_1, \lambda_{17}\}, & \textit{if } \alpha b \beta_1 \beta_2 k_1 > 0 \end{cases}$$

Proof. Evidently that λ_1 is a fixed point.

1. If $b = 0$ then the operator (6) has the following representation

$$V : \begin{cases} x^{(1)} = x - \beta_1(k_1 u + k_2 v)x \\ u^{(1)} = u - \alpha u + \beta_1(k_1 u + k_2 v)x \\ y^{(1)} = y + \alpha u - \beta_2(k_1 u + k_2 v)y \\ v^{(1)} = v + \beta_2(k_1 u + k_2 v)y \end{cases} \tag{10}$$

   First, we assume that all other parameters are nonzero. From the first equation of this system we get

$$x^{(1)} = x - \beta_1(k_1 u + k_2 v)x = x,$$

   if $x = 0$ or (and) $k_1 u + k_2 v = 0$. If $x = 0$ then by second equation

$$u^{(1)} = u - \alpha u + \beta_1(k_1 u + k_2 v)x = u$$

we have $u = 0$. From this and third equation of the system

$$y^{(1)} = y + \alpha u - \beta_2(k_1u + k_2v)y = y,$$

one has $vy = 0$, i.e., $v = 0$ or (and) $y = 0$. Thus, in this case, fixed points are $(0, 0, 1, 0) = \lambda_3$ or (and) $(0, 0, 0, 1) = \lambda_4$. If $k_1u + k_2v = 0$, i.e., $u = v = 0$ then $x^{(1)} = x$, $u^{(1)} = 0$, $y^{(1)} = y$, $v^{(1)} = 0$, so in this case, we get the set of fixed points $(x, 0, y, 0) = \Lambda_9$. Since $\lambda_1, \lambda_3 \in \Lambda_9$ it follows that the fixed points of the operator (6) are $\{\lambda_4\} \bigcup \Lambda_9$.

2. If $b = 0$, $\alpha = 0$ then the operator (6) has the form

$$V : \begin{cases} x^{(1)} = x - \beta_1(k_1u + k_2v)x \\ u^{(1)} = u + \beta_1(k_1u + k_2v)x \\ y^{(1)} = y - \beta_2(k_1u + k_2v)y \\ v^{(1)} = v + \beta_2(k_1u + k_2v)y \end{cases} \tag{11}$$

Here also we assume that all other parameters are nonzero and by $x^{(1)} = x$ we get $x = 0$ or (and) $u = v = 0$. If $x = 0$ then $u^{(1)} = u$ and from $y^{(1)} = y$ one has $y = 0$, consequently, we have $v^{(1)} = v$. Thus, for $x = 0$ we have the fixed point of the form $\Lambda_6 = (0, u, 0, v)$. If $u = v = 0$ then $x^{(1)} = x$ and $y^{(1)} = y$, so the set of fixed points is $\Lambda_9 = (x, 0, y, 0)$. The other cases including the condition $b = 0$ can be handled in the same way.

3. If $b > 0$, $\alpha = 0$ then the operator (6) is as follows

$$V : \begin{cases} x^{(1)} = x + b - bx - \beta_1(k_1u + k_2v)x \\ u^{(1)} = u - bu + \beta_1(k_1u + k_2v)x \\ y^{(1)} = y - by - \beta_2(k_1u + k_2v)y \\ v^{(1)} = v - bv + \beta_2(k_1u + k_2v)y \end{cases} \tag{12}$$

From $y^{(1)} = y$ one has $by + \beta_2(k_1u + k_2v)y = 0$ which holds in the case $y = 0$, and so the equation $v^{(1)} = v - bv = v$ holds only $v = 0$. Consequently, the solutions of the equation

$$u^{(1)} = u - bu + \beta_1k_1ux = u$$

are $u = 0$ and $x = \frac{b}{\beta_1k_1}$. Therefore, if $\beta_1k_1 \le b$ then the fixed point is λ_1, if $\beta_1k_1 > b$ then the fixed points are λ_1 and

$$\lambda_{15} = \Big(\frac{b}{\beta_1k_1}; 1 - \frac{b}{\beta_1k_1}; 0; 0\Big).$$

4. If $b > 0$, $\alpha > 0$, $\beta_2 = 0$, $\beta_1k_1 > b + \alpha$ then the operator (6) has the following representation

$$V : \begin{cases} x^{(1)} = x + b - bx - \beta_1(k_1u + k_2v)x \\ u^{(1)} = u - bu - \alpha u + \beta_1(k_1u + k_2v)x \\ y^{(1)} = y - by + \alpha u \\ v^{(1)} = v - bv \end{cases} \tag{13}$$

Using $v^{(1)} = v - bv = v$ we have $v = 0$. In the equation

$$u^{(1)} = u - bu - \alpha u + \beta_1 k_1 ux = u$$

we assume that $u > 0$ (otherwise, clearly that λ_1 is unique) and so we have $x = \frac{b+\alpha}{\beta_1 k_1}$. Substituting $x = \frac{b+\alpha}{\beta_1 k_1}$ into

$$x^{(1)} = x + b - bx - \beta_1 k_1 ux = x$$

it follows up

$$u = \frac{b - bx}{\beta_1 k_1 x} = \frac{b(\beta_1 k_1 - b - \alpha)}{\beta_1 k_1 (b + \alpha)}.$$

Consequently, from $y^{(1)} = y - by + \alpha u = y$ we get

$$y = \frac{\alpha u}{b} = \frac{\alpha(\beta_1 k_1 - b - \alpha)}{\beta_1 k_1 (b + \alpha)}$$

and the condition $\beta_1 k_1 > b + \alpha$ should be required which is easy to verify. Hence, we have an additional fixed point λ_{16} to λ_1.

5. If $\alpha b \beta_1 \beta_2 k_1 > 0$ then the operator can have the fixed point λ_{17} which all coordinates are positive. First, using $x^{(1)} = x$ we have $x = \frac{b}{b+\beta_1 A}$, and so by $u^{(1)} = u$ we get

$$u = \frac{\beta_1 A x}{b + \alpha} = \frac{b\beta_1 A}{(b + \beta_1 A)(b + \alpha)}.$$

Consequently, from the equation $y^{(1)} = y$ we have

$$y = \frac{\alpha u}{b + \beta_2 A} = \frac{\alpha b \beta_1 A}{(b + \beta_1 A)(b + \beta_2 A)(b + \alpha)},$$

and by $v^{(1)} = v$ it follows up

$$v = \frac{\beta_2 A y}{b} = \frac{\alpha \beta_1 \beta_2 A^2}{(b + \beta_1 A)(b + \beta_2 A)(b + \alpha)}.$$

In addition, using $A(u, v) = k_1 u + k_2 v$ one has the quadratical equation (9). Thus, the proof of Proposition is completed.

□

Note that the set of positive solutions of (9) is non-empty when $\beta_1 k_1 \geq b + \alpha$ (see the statements before Conjecture 2). For example, $\alpha = 0.3$, $b = 0.2$, $\beta_1 = 0.6$, $\beta_2 = 0.4$, $k_1 = k_2 = 1$. Then the equation (9) has the form

$$30A^2 - 5A - 1 = 0$$

and the positive solution is $A = \frac{5+\sqrt{145}}{60} \approx 0.284$.

4.2 Type of fixed points

Definition 2 ([1]). A fixed point p for $F:\mathbb{R}^m \to \mathbb{R}^m$ is called *hyperbolic* if the Jacobian matrix $\mathbf{J} = \mathbf{J}_F$ of the map F at the point p has no eigenvalues on the unit circle.

There are three types of hyperbolic fixed points:

1. p is an attracting fixed point if all of the eigenvalues of $\mathbf{J}(p)$ are less than one in absolute value.
2. p is an repelling fixed point if all of the eigenvalues of $\mathbf{J}(p)$ are greater than one in absolute value.
3. p is a saddle point otherwise.

Proposition 3. *Let λ_1 be the fixed point of the operator V. Then*

$$\lambda_1 = \begin{cases} \text{nonhyperbolic}, & \text{if } b=0 \text{ or } \beta_1 k_1 = b+\alpha \\ \text{attractive}, & \text{if } b>0 \text{ and } \beta_1 k_1 < b+\alpha \\ \text{saddle}, & \text{if } b>0 \text{ and } \beta_1 k_1 > b+\alpha \end{cases}$$

Proof. The Jacobian of the operator (6) is as follows:

$$J = \begin{bmatrix} 1-b-\beta_1 A & -\beta_1 k_1 x & 0 & -\beta_1 k_2 x \\ \beta_1 A & 1-b-\alpha+\beta_1 k_1 x & 0 & \beta_1 k_2 x \\ 0 & \alpha-\beta_2 k_1 y & 1-b-\beta_2 A & -\beta_2 k_2 y \\ 0 & \beta_2 k_1 y & \beta_2 A & 1-b+\beta_2 k_2 y \end{bmatrix}$$

Then at the fixed point λ_1 the Jacobian is

$$J(\lambda_1) = \begin{bmatrix} 1-b & -\beta_1 k_1 & 0 & -\beta_1 k_2 \\ 0 & 1-b-\alpha+\beta_1 k_1 & 0 & \beta_1 k_2 \\ 0 & \alpha & 1-b & 0 \\ 0 & 0 & 0 & 1-b \end{bmatrix}$$

and the eigenvalues of this matrix are $\mu_1 = 1-b$, $\mu_2 = 1-b-\alpha+\beta_1 k_1$ which are easy to verify. By the conditions (8) one has $\mu_1 \geq 0, \mu_2 \geq 0$. It is easy to see that, if $b=0$ or $\beta_1 k_1 = b+\alpha$ then $\mu_1 = 1$ or $\mu_2 = 1$ respectively. Moreover, by definition and conditions (8) it can be shown easily that if $b>0$, $\beta_1 k_1 < b+\alpha$ then the fixed point λ_1 is an attracting, otherwise, saddle fixed point. ☐

Proposition 4. *Let λ_2 be the fixed point of the operator V. Then*

$$\lambda_2 = \begin{cases} \text{nonhyperbolic}, & \text{if } b=0 \text{ or } b+\beta_1 k_1 = 2 \\ \text{attractive}, & \text{if } b>0 \text{ and } b+\beta_1 k_1 < 2 \\ \text{saddle}, & \text{otherwise} \end{cases}$$

Proof. At the fixed point $\lambda_2 = (0, 1, 0, 0)$ the Jacobian is as follows:

$$J(\lambda_2) = \begin{bmatrix} 1-b-\beta_1 k_1 & 0 & 0 & 0 \\ \beta_1 k_1 & 1-b-\alpha & 0 & 0 \\ 0 & \alpha & 1-b-\beta_2 k_1 & 0 \\ 0 & 0 & \beta_2 k_1 & 1-b \end{bmatrix}$$

Clearly, that the eigenvalues are $\mu_1 = 1-b-\beta_1 k_1$, $\mu_2 = 1-b-\alpha$, $\mu_3 = 1-b-\beta_2 k_1$ and $\mu_4 = 1-b$. In conditions (8) there involved the inequalities $|\alpha - b - \beta_2 k_1| \leq 1$ and $b+\alpha \leq 1$. It is easy to verify that if $b > 0$ then $|\mu_2| < 1, |\mu_4| < 1$ and from $b+\alpha \leq 1$ implies $b + \beta_2 k_1 < 2$, i.e., $|\mu_3| < 1$. Moreover, if $b + \beta_1 k_1 < 2$ then $|\mu_1| < 1$, and so λ_2 is an attractive fixed point. If $b = 0$ (resp. $b + \beta_1 k_1 = 2$) then $\mu_4 = 1$ (resp. $\mu_1 = 1$) and so λ_2 is nonhyperbolic, if $b > 0$ and $b + \beta_1 k_1 > 2$ then $|\mu_1| > 1$, $|\mu_2| < 1$, $|\mu_3| < 1$, $|\mu_4| < 1$, so λ_2 is a saddle fixed point. □

Proposition 5. *Let λ_3 be the fixed point of the operator V. Then*

$$\lambda_3 = \begin{cases} \textit{nonhyperbolic}, & \textit{if } b = 0 \textit{ or } b = \beta_2 k_2 \\ \textit{attractive}, & \textit{if } b > 0 \textit{ and } \beta_2 k_2 < b \\ \textit{saddle}, & \textit{if } b > 0 \textit{ and } \beta_2 k_2 > b \end{cases}$$

Proof. At the fixed point $\lambda_3 = (0, 0, 1, 0)$ the Jacobian is as follows

$$J(\lambda_3) = \begin{bmatrix} 1-b & 0 & 0 & 0 \\ 0 & 1-b-\alpha & 0 & 0 \\ 0 & \alpha-\beta_2 k_1 & 1-b & -\beta_2 k_2 \\ 0 & \beta_2 k_1 & 0 & 1-b+\beta_2 k_2 \end{bmatrix}$$

The eigenvalues are $\mu_1 = 1-b$, $\mu_2 = 1-b-\alpha$, $\mu_3 = 1-b+\beta_2 k_2$. Here also by the conditions (8) the proof is dealt similarly. □

Proposition 6. *Let λ_4 be the fixed point of the operator V. Then*

$$\lambda_4 = \begin{cases} \textit{nonhyperbolic}, & \textit{if } b = 0 \textit{ or } b + \beta_1 k_2 = 2 \\ \textit{attractive}, & \textit{if } b > 0 \textit{ and } b + \beta_1 k_2 < 2 \\ \textit{saddle}, & \textit{otherwise} \end{cases}$$

Proof. At the fixed point $\lambda_4 = (0, 0, 0, 1)$ the Jacobian is

$$J(\lambda_4) = \begin{bmatrix} 1-b-\beta_1 k_2 & 0 & 0 & 0 \\ \beta_1 k_2 & 1-b-\alpha & 0 & 0 \\ 0 & \alpha & 1-b-\beta_2 k_2 & 0 \\ 0 & 0 & \beta_2 k_2 & 1-b \end{bmatrix}$$

It is clear that, the eigenvalues are $\mu_1 = 1-b-\beta_1 k_2$, $\mu_2 = 1-b-\alpha$, $\mu_3 = 1-b-\beta_2 k_2$ and $\mu_4 = 1-b$. Based on conditions (8) and by the form of eigenvalues it is east to verify the proof of Proposition. □

Remark 2. We have studied the types of fixed points λ_1, λ_2, λ_3, λ_4 and the types of other fixed points can be investigated with too many calculations.

5 The limit points of trajectories

In this section we study the limit behavior of trajectories of initial point $\lambda^{(0)} \in S^3$ under operator (6), i.e the sequence $V^n(\lambda^{(0)})$, $n \geq 1$. Note that since V is a continuous operator, its trajectories have as a limit some fixed points obtained in Proposition 2.

5.1 Case no susceptibility of persons ($\beta_1 = \beta_2 = 0$).

We study here the case where in the model there is no susceptibility of persons.

Proposition 7. *For an initial point $\lambda^0 = (x^0, u^0, y^0, v^0) \in S^3$ (except fixed points) the trajectory (under action of operator (6)) has the following limit*

$$\lim_{n\to\infty} V^{(n)}(\lambda^0) = \begin{cases} \lambda^0 & \text{if } \alpha = b = 0 \\ (x^0, 0, 1 - x^0 - v^0, v^0) & \text{if } b = 0, \alpha > 0 \\ \lambda_1 & \text{if } b > 0 \end{cases}$$

Proof. If $\beta_1 = \beta_2 = 0$ then the operator (6) has the following form:

$$V : \begin{cases} x^{(1)} = x + b - bx \\ u^{(1)} = u(1 - b - \alpha) \\ y^{(1)} = y - by + \alpha u \\ v^{(1)} = v(1 - b) \end{cases} \tag{14}$$

Evidently, that $b = \alpha = 0$ then every point is fixed point.

Assume that $b = 0, \alpha > 0$ then by (14) we get $x^{(n)} = x^0, v^{(n)} = v^0$ and $u^{(n)} = u^0(1 - \alpha)^n \to 0$ as $n \to \infty$. Moreover, $y^{(1)} = y + \alpha u \geq y$, so the limit

$$\lim_{n\to\infty} y^{(n)} = \bar{y} = 1 - x^0 - v^0$$

exists. Consequently, it follows

$$\lim_{n\to\infty} V^{(n)} = \bar{\lambda} = (x^0, 0, 1 - x^0 - v^0, v^0).$$

Suppose $b > 0$ then the sequences $u^{(n)}$, $v^{(n)}$ have zero limits. From the first equation of (14) one has

$$x^{(n+1)} = (1 - b)^n x + b \sum_{k=0}^{n} (1 - b)^k \tag{15}$$

and using $0 < b \leq 1$ we have

$$\lim_{n\to\infty} x^{(n+1)} = x \lim_{n\to\infty} (1 - b)^n + b \sum_{k=0}^{\infty} (1 - b)^k = 1 \quad \text{for any} \quad \lambda^0 \in S^3. \tag{16}$$

From the relation $x^{(n)} + u^{(n)} + y^{(n)} + v^{(n)} = 1$ follows the assertion of Proposition. $\square$

5.2 Case no susceptibility of persons in S ($\beta_1 = 0, \beta_2 > 0$).

Proposition 8. *For an initial point $\lambda^0 = \left(x^0, u^0, y^0, v^0\right) \in S^3$ (except fixed points) the trajectory (under action of the operator (6)) has the following limit*

$$\lim_{n\to\infty} V^{(n)}(\lambda^0) = \begin{cases} \lambda^0 & \text{if } \alpha = b = 0, k_1u^0 + k_2v^0 = 0 \\ (x^0, 1-x^0, 0, 0) & \text{if } \alpha = b = 0, k_1u^0 + k_2v^0 > 0, y^0 = v^0 = 0 \\ (x^0, u^0, 0, 1-x^0-u^0) & \text{if } \alpha = b = 0, k_1u^0 + k_2v^0 > 0, y^0 + v^0 > 0 \\ \lambda_1 & \text{if } b > 0 \\ (x^0, 0, \bar{y}, 1-x^0-\bar{y}) & \text{if } b = 0, \alpha > 0, k_2 = 0, \text{ where } \bar{y} = \bar{y}(\lambda^0) \\ (x^0, 0, 1-x^0, 0) & \text{if } b = 0, \alpha > 0, k_2 > 0, k_1u^0 + k_2v^0 = 0 \\ (x^0, 0, 0, 1-x^0) & \text{if } b = 0, \alpha > 0, k_2 > 0, k_1u^0 + k_2v^0 > 0 \end{cases}$$

Proof. If $\beta_1 = 0$ then the operator (6) has the following representation:

$$V: \begin{cases} x^{(1)} = x + b(1-x) \\ u^{(1)} = u(1-b-\alpha) \\ y^{(1)} = y - by + \alpha u - \beta_2(k_1u + k_2v)y \\ v^{(1)} = v - bv + \beta_2(k_1u + k_2v)y \end{cases} \tag{17}$$

Case: $b = \alpha = 0$**.** In this case it is easy to see that every of the sequences $x^{(n)}$, $u^{(n)}$, $y^{(n)}$, $v^{(n)}$ has a limit. Denote by $\bar{y}, \bar{v}$ the limits of $y^{(n)}$ and $v^{(n)}$ respectively.

If

$$A(u^0, v^0) = k_1u^0 + k_2v^0 = 0$$

then from (17) it follows

$$A(u^{(n)}, v^{(n)}) = k_1u^{(n)} + k_2v^{(n)} = k_1u^0 + k_2v^0 = 0.$$

Therefore, $V^{(n)}(\lambda^0) = \lambda^0$, $n = 1, 2, 3, \dots$ and for all $\lambda^0 \in S^3$.

If $k_1u^0 + k_2v^0 > 0$ then we consider all possible subcases:

- If $y^0 = v^0 = 0$ then $y^{(n)} = v^{(n)} = 0$, so $\lim_{n\to\infty} V^{(n)} = (x^0, 1-x^0, 0, 0)$.
- Let be $y^0 + v^0 > 0$. Since $k_1u^{(n)} + k_2v^{(n)} \geq 0$ it follows that the sequence $v^{(n)}$ is a non decreasing. Using this one has

$$k_1u^{(n)} + k_2v^{(n)} = k_1u^0 + k_2\bar{v} \geq k_1u^0 + k_2v^0 > 0$$

and

$$\lim_{n\to\infty}(k_1u^{(n)} + k_2v^{(n)}) = k_1u^0 + k_2\bar{v} > 0. \tag{18}$$

Then from the third equation of (17) we have

$$\lim_{n\to\infty} y^{(n+1)} = \lim_{n\to\infty} y^{(n)} - \beta_2 \lim_{n\to\infty}(k_1u^{(n)} + k_2v^{(n)})y^{(n)}$$

$$\Rightarrow \beta_2(k_1u^0 + k_2\bar{v})\bar{y} = 0 \Rightarrow \bar{y} = 0.$$

Thus, a trajectory of the operator (17) converges to $(x^0, u^0, 0, 1-x^0-u^0)$.

Case: $b > 0$**.** In this case by the result (16) we have $x^{(n)} \to 1$ as $n \to \infty$. So all the other sequences have zero limit. Thus, limit of the considering operator is $\lambda_1 = (1, 0, 0, 0)$.

Case: $b = 0$**,** $\alpha > 0$**,** $k_2 = 0$**.** Then $x^{(n)} = x^0$, $u^{(n)} = u^0(1-\alpha)^n \to 0$ as $n \to \infty$, and $v^{(1)} = v + \beta_2 k_1 uy \geq v$, i.e., the sequence $v^{(n)}$ has limit, so $y^{(n)}$ also has limit. If we denote $\bar{y}$ the limit of $y^{(n)}$, of course, it depends on initial point λ^0 and consequently, the limit of $v^{(n)}$ is $1 - x^0 - \bar{y}$.

Case: $b = 0$**,** $\alpha > 0$**,** $k_2 > 0$**.** Here also, as previous case, $x^{(n)} = x^0$, $u^{(n)} = u^0(1-\alpha)^n \to 0$ and the sequences $y^{(n)}, v^{(n)}$ have limits.

If $k_1u^0 + k_2v^0 = 0$ then we can consider two possible subcases:

- If $k_1 = v^0 = 0$ then $v^{(n)} = 0$ and $A(u^{(n)}, v^{(n)}) = 0$, so

$$\lim_{n\to\infty} y^{(n+1)} = \lim_{n\to\infty} y^{(n)} + \alpha \lim_{n\to\infty} u^{(n)} = \bar{y} = 1 - x^0.$$

- If $u^0 = v^0 = 0$ then $u^{(n)} = v^{(n)} = 0$, so $\bar{y} = 1 - x^0$.

If $k_1u^0 + k_2v^0 > 0$ then we consider the following subcases:

- If $y^0 = v^0 = 0$ then $k_1u^0 > 0$, so $y^{(1)} = \alpha u^0 > 0$ and from this

$$v^{(2)} = \beta_2 k_1 \alpha(1-\alpha)(u^0)^2 > 0.$$

- If $y^0 > 0, v^0 = 0$ then $k_1u^0 > 0$, so $v^{(1)} = \beta_2 k_1 u^0 y^0 > 0$.
- If $v^0 > 0$ then $v^{(1)} > 0$. Here also by taking a limit of $v^{(n+1)}$ we have $\beta_2 k_2 \bar{v}\bar{y} = 0$. i.e., $\bar{y} = 0$, so $\bar{v} = 1 - x^0$.

The proof of the Proposition is completed. □

5.3 Case no birth (death) rate and recovery rate ($b = \alpha = 0$).

Proposition 9. *For an initial point $\lambda^0 = \left(x^0, u^0, y^0, v^0\right) \in S^3$ (except fixed points) the trajectory (under action of the operator (6)) has the following limit*

$$\lim_{n\to\infty} V^{(n)}(\lambda^0) = \begin{cases} \lambda^0 & \text{if } k_1u^0 + k_2v^0 = 0 \\ (0, 1 - y^0 - v^0, y^0, v^0) & \text{if } \beta_1 > 0,\ \beta_2 = 0,\ k_1u^0 + k_2v^0 > 0 \\ (0, u^0, 0, 1 - u^0) & \text{if } \beta_1 > 0,\ \beta_2 > 0,\ k_1u^0 + k_2v^0 > 0 \end{cases}$$

We note that the case $\beta_1 = 0$, $\beta_2 > 0$ is not considered, because, it coincides with some cases of Proposition 8.

Proof. If $b = 0, \alpha = 0$ then the operator (6) is

$$V : \begin{cases} x^{(1)} = x - \beta_1(k_1u + k_2v)x \\ u^{(1)} = u + \beta_1(k_1u + k_2v)x \\ y^{(1)} = y - \beta_2(k_1u + k_2v)y \\ v^{(1)} = v + \beta_2(k_1u + k_2v)y \end{cases} \tag{19}$$

From (19) one can check that the sequences $x^{(n)}$, $u^{(n)}$, $y^{(n)}$, $v^{(n)}$ are monotone, so they have limits and let $\bar{x}$, $\bar{u}$, $\bar{v}$ be the limits of $x^{(n)}$, $u^{(n)}$, $v^{(n)}$ respectively. The case $k_1u^0 + k_2v^0 = 0$ is clear.

If $\beta_1 > 0$, $\beta_2 = 0$, $k_1u^0 + k_2v^0 > 0$ then from $k_1u^{(n)} + k_2v^{(n)} \geq 0$ we have that $u^{(n)}$ non decreasing sequence and as (18), we have that

$$\lim_{n\to\infty}(k_1u^{(n)} + k_2v^{(n)}) = k_1\bar{u} + k_2v^0 > 0\,.$$

Therefore,

$$\lim_{n\to\infty} x^{(n+1)} = \lim_{n\to\infty} x^{(n)} - \beta_1 \lim_{n\to\infty}(k_1u^{(n)} + k_2v^{(n)})x^{(n)}$$

and it follows $\beta_1(k_1\bar{u} + k_2v^0)\bar{x} = 0$, i.e., $\bar{x} = 0$, so $\bar{u} = 1 - y^0 - v^0$.

The case $\beta_1 > 0$, $\beta_2 > 0$, $k_1u^0 + k_2v^0 > 0$ using

$$\lim_{n\to\infty}(k_1u^{(n)} + k_2v^{(n)}) = k_1\bar{u} + k_2\bar{v} \geq k_1u^0 + k_2v^0 > 0\,,$$

can be considered similarly. The Proposition is proved. □

5.4 Case no recovery rate and infectivity of persons in I_1 $(\alpha = 0, k_2 = 0)$.

Proposition 10. *For an initial point* $\lambda^0 = \left(x^0, u^0, y^0, v^0\right) \in S^3$ *(except fixed points) the trajectory (under action of the operator* (6)*) has the following limit*

$$\lim_{n\to\infty} V^{(n)}(\lambda^0) = \begin{cases} \lambda_1 & \text{if } \beta_1k_1 \leq b \text{ or } u^0 = 0 \\ \lambda_{15} & \text{if } \beta_1k_1 > b \text{ and } u^0 > 0 \end{cases}$$

Proof. Here we consider the case $b > 0$, otherwise it coincides with previous Proposition cases. If $\alpha = k_2 = 0$ then the operator (6) has the form

$$V: \begin{cases} x^{(1)} = x + b - bx - \beta_1k_1ux \\ u^{(1)} = u - bu + \beta_1k_1ux \\ y^{(1)} = y - by - \beta_2k_1uy \\ v^{(1)} = v - bv + \beta_2k_1uy \end{cases} \tag{20}$$

From the third equation of (20) one has

$$y^{(1)} = y - by - \beta_2k_1uy \leq y(1-b),$$

i.e., $y^{(n)} \leq y^0(1-b)^n$, so $y^{(n)} \to 0$ as $n \to \infty$. Moreover, from the sum of last two equations of (20) we get

$$\lim_{n\to\infty}(y^{(n+1)} + v^{(n+1)}) = (1-b)\lim_{n\to\infty}(y^{(n)} + v^{(n)}) = (y^0 + v^0)\lim_{n\to\infty}(1-b)^n = 0\,.$$

Hence, the sequence $v^{(n)}$ also converges to zero.

- If $\beta_1k_1 = 0$ then $u^{(n)} = u^0(1-b)^n$ has zero limit. Since the sequences $y^{(n)}$ and $v^{(n)}$ also converge to zero, it follows that the sequence $x^{(n)}$ has limit one. Thus, the limit of the operator V is λ_1.

- If $0 < \beta_1 k_1 \le b$ then from

$$u^{(1)} = u - (b - \beta_1 k_1 x)u \le u$$

we get that the sequence $u^{(n)}$ has limit. We assume that $\lim\limits_{n\to\infty} u^{(n)} = \bar{u} > 0$, then from

$$u^{(n+1)} = u^{(n)} - (b - \beta_1 k_1 x^{(n)})u^{(n)}$$

we have $\lim\limits_{n\to\infty} x^{(n)} = \dfrac{b}{\beta_1 k_1}$, and

$$\bar{u} = 1 - \frac{b}{\beta_1 k_1} = \frac{\beta_1 k_1 - b}{\beta_1 k_1}.$$

The condition $\beta_1 k_1 \le b$ contradicts to assumption $\bar{u} > 0$, so $\bar{u} = 0$. Hence, limit point of the operator is λ_1.

- If $\beta_1 k_1 > b$ and $u^0 > 0$ then from the first two equations of the operator (20) we formulate a new operator:

$$W : \begin{cases} x^{(1)} = x + b - bx - \beta_1 k_1 ux \\ u^{(1)} = u - bu + \beta_1 k_1 ux \end{cases} \tag{21}$$

Here we normalize the operator (21) as following:

$$W_0 : \begin{cases} x^{(1)} = \frac{x+b-bx-\beta_1 k_1 ux}{x+u+b-b(x+u)} \\ u^{(1)} = \frac{u-bu+\beta_1 k_1 ux}{x+u+b-b(x+u)} \end{cases} \tag{22}$$

For this operator $x^{(n)} + u^{(n)} = 1$, $n \ge 1$, so from $x^{(1)} + u^{(1)} = 1$ we have

$$\begin{cases} x^{(2)} = x^{(1)} + b - bx^{(1)} - \beta_1 k_1 u^{(1)} x^{(1)} \\ u^{(2)} = u^{(1)} - bu^{(1)} + \beta_1 k_1 u^{(1)} x^{(1)} \end{cases}$$

Thus, operators W and W_0 have same dynamics. From the first equation of the last system we obtain

$$x^{(2)} = (1 - b - \beta_1 k_1)x^{(1)} + \beta_1 k_1 (x^{(1)})^2 + b\,.$$

If we denote $x^{(1)} = x$, $x^{(2)} = f_{b,\beta_1 k_1}(x)$ then we get

$$f_{b,\beta_1 k_1}(x) = b + (1 - b - \beta_1 k_1)x + \beta_1 k_1 x^2\,.$$

Definition 3 ([1, p. 47]). Let $f\colon A \to A$ and $g\colon B \to B$ be two maps. f and g are said to be topologically conjugate if there exists a homeomorphism $h\colon A \to B$ such that, $h \circ f = g \circ h$. The homeomorphism h is called a topological conjugacy.

Let $F_\mu(x) = \mu x(1 - x)$ be quadratic family (discussed in [1]) and

$$f_{b,\beta_1 k_1}(x) = b + (1 - b - \beta_1 k_1)x + \beta_1 k_1 x^2\,.$$

Lemma 1. *Two maps $F_\mu(x)$ and $f_{b,\beta_1 k_1}(x)$ are topologically conjugate for $\mu = \beta_1 k_1 - b + 1$.*

Proof. We take the linear map (as in [6]) $h(x) = px + q$ and by Definition 3 we should have $h(F_\mu(x)) = f_{b,\beta_1 k_1}(h(x))$, i.e.,

$$p\mu x(1-x) + q = b + (px+q)(1-b-\beta_1 k_1) + \beta_1 k_1 (px+q)^2$$

from this identity we get

$$\begin{cases} -p\mu = \beta_1 k_1 p^2 \\ p\mu = p(1-b-\beta_1 k_1) + 2pq\beta_1 k_1 \\ q = q(1-b-\beta_1 k_1) + \beta_1 k_1 q^2 + b \end{cases} \Rightarrow \begin{cases} p = -\frac{\mu}{\beta_1 k_1} \\ q = \frac{\mu - 1 + b + \beta_1 k_1}{2\beta_1 k_1} \\ \beta_1 k_1 q^2 - (b + \beta_1 k_1) q + b = 0 \end{cases}$$

The roots of the equation

$$\beta_1 k_1 q^2 - (b + \beta_1 k_1) q + b = 0$$

are $q = 1$, $q = \frac{b}{\beta_1 k_1}$. If we choose $q = 1$ then by $q = \frac{\mu - 1 + b + \beta_1 k_1}{2\beta_1 k_1}$ we have $\mu = \beta_1 k_1 - b + 1$. Then the homeomorphism is $h(x) = \frac{b - \beta_1 k_1 - 1}{\beta_1 k_1} x + 1$. Moreover, since $b < \beta_1 k_1 \leq 2$ we have $1 < \mu < 3$. □

The importance of this Lemma is that if two maps are topologically conjugate then they have essentially the same dynamics (see [1, p. 53]). The operator

$$f_{b,\beta_1 k_1}(x) = b + (1 - b - \beta_1 k_1)x + \beta_1 k_1 x^2$$

has two fixed points $p_1 = 1$ and $p_2 = \frac{b}{\beta_1 k_1}$. In addition,

$$f'_{b,\beta_1 k_1}(x) = 1 - b - \beta_1 k_1 + 2\beta_1 k_1 x\,,$$

from this and $\beta_1 k_1 > b$ it obtains that the fixed point $p_1 = 1$ is repelling, $p_2 = \frac{b}{\beta_1 k_1}$ is attractive. Moreover, for any initial point $x \in (0,1)$ and for $\mu \in (1,3)$ the trajectory of the operator $F_\mu(x)$ converges to the attractive fixed point (see [1, p. 32]). Thus, for the case $\beta_1 k_1 > b$ the limit point of the operator (20) is λ_{15}. □

5.5 Case no only susceptibility of persons in S_1 ($\beta_2 = 0, \beta_1 > 0$).

Proposition 11. *For an initial point $\lambda^0 = \left(x^0, u^0, y^0, v^0\right) \in S^3$ (except fixed points) the trajectory (under action of the operator (6)) has the following limit*

$$\lim_{n\to\infty} V^{(n)}(\lambda^0) = \begin{cases} (x^0, 0, 1 - x^0 - v^0, v^0) & \text{if } b = 0,\ \alpha > 0 \text{ and } k_1 u^0 + k_2 v^0 = 0 \\ (0, 0, 1 - v^0, v^0) & \text{if } b = 0,\ \alpha > 0,\ k_2 v^0 > 0 \\ (\bar{x}, 0, 1 - \bar{x} - v^0, v^0) & \text{if } b = k_2 v^0 = 0,\ k_1 u^0 > 0,\ \alpha > 0 \\ \lambda_1 & \text{if } b\alpha > 0,\ k_1 u^0 + k_2 v^0 = 0 \\ \lambda_1 & \text{if } b\alpha > 0,\ k_2 v^0 = 0,\ \beta_1 k_1 \leq b + \alpha \end{cases}$$

where $\bar{x} = \bar{x}(\lambda^0)$

Proof. Here we consider the case $\beta_1 > 0$, otherwise, this proposition is same with Proposition 7. We note that the case $b = \alpha = 0$ is considered in Proposition 9. If $\beta_2 = 0$ then the operator (6) has the form

$$V : \begin{cases} x^{(1)} = x + b - bx - \beta_1(k_1 u + k_2 v)x \\ u^{(1)} = u - bu - \alpha u + \beta_1(k_1 u + k_2 v)x \\ y^{(1)} = y - by + \alpha u \\ v^{(1)} = v(1 - b) \end{cases} \tag{23}$$

Case: $b = 0, \alpha > 0, k_1 u^0 + k_2 v^0 = 0$**.** From $A(u^0, v^0) = k_1 u^0 + k_2 v^0 = 0$ we have the following simple cases:

- If $k_1 = k_2 = 0$ then $x^{(n)} = x^0, v^{(n)} = v^0$ and $u^{(n)} = u^0(1-\alpha)^n \to 0$, as $n \to \infty$.
- If $k_1 = v^0 = 0$ then $v^{(n)} = 0$, $x^{(n)} = x^0$, so $u^{(n)} = u^0(1-\alpha)^n \to 0$ and $y^{(n)} \to 1 - x^0$ as $n \to \infty$.
- If $u^0 = k_2 = 0$, then $u^{(n)} = 0, v^{(n)} = v^0$, so $x^{(n)} = x^0, y^{(n)} = y^0 = 1 - x^0$ for all n.
- If $u^0 = v^0 = 0$, then $u^{(n)} = 0, v^{(n)} = 0$, so $x^{(n)} = x^0, y^{(n)} = y^0 = 1 - x^0$ for all n.

Case: $b = 0, \alpha > 0, k_2 v^0 > 0$**.** Then $v^{(n)} = v^0$ and $x^{(1)} = x - \beta_1(k_1 u + k_2 v)x \le x$, $y^{(1)} = y + \alpha u \ge y$, i.e., the sequences $x^{(n)}$, $y^{(n)}$, $v^{(n)}$ have limits, so $u^{(n)}$ also has limit. From the equation $y^{(n+1)} = y^{(n)} + \alpha u^{(n)}$ we get limit and by $\alpha > 0$ it obtains that $u^{(n)}$ converges to zero. Moreover, by the second equation of the system (23),

$$u^{(n+1)} = (1-\alpha)u^{(n)} + \beta_1(k_1 u^{(n)} + k_2 v^{(n)})x^{(n)},$$

if we take a limit then we have $0 = \beta_1 k_2 v^0 \bar{x}$, from this and $k_2 v^0 > 0$ we have $\bar{x} = 0$, where $\bar{x}$ is a limit of the sequence $x^{(n)}$.

Case: $b = k_2 v^0 = 0,\ k_1 u^0 > 0,\ \alpha > 0$**.** Here also as previous case, all sequences have limits and $v^{(n)} = v^0$. From $y^{(n+1)} = y^{(n)} + \alpha u^{(n)}$ we get limit and by $\alpha > 0$ it obtains that $u^{(n)}$ converges to zero. But, the limit $\lim_{n\to\infty} x^{(n)} = \bar{x}$ depends on initial conditions x^0, u^0.

Case: $b > 0,\ \alpha > 0,\ k_1 u^0 + k_2 v^0 = 0$**.** We have the following subcases:

- If $k_1 = k_2 = 0$ then $x^{(1)} = x + (1-x)b \ge x$ and by the results (16) we have that the sequence $x^{(n)}$ has limit 1.
- If $k_1 = v^0 = 0$ then $v^{(n)} = 0$, $u^{(n)} = u^0(1-b-\alpha)^n \to 0$ and as previous case $x^{(n)} \to 1, n \to \infty$.
- If $u^0 = k_2 = 0$ then $u^{(n)} = 0, y^{(n)} = y^0(1-b)^n \to 0$ and from $v^{(n)} \to 0$ implies $x^{(n)} \to 1$ as $n \to \infty$.
- If $u^0 = v^0 = 0$ then $u^{(n)} = 0, v^{(n)} = 0$, for any $n = 1, 2, 3, \ldots$ and $y^{(n)} = y^0(1-b)^n \to 0$ as $n \to \infty$. So, it follows $x^{(n)} \to 1$ as $n \to \infty$.

Case: $b > 0$**,** $\alpha > 0$**,** $k_2 v^0 = 0$**,** $\beta_1 k_1 \leq b + \alpha$**.** From $k_2 v^0 = 0$ we have $k_2 v^{(n)} = 0$ and

$$u^{(1)} = u - (b + \alpha - \beta_1 k_1 x)u \leq u,$$

i.e., the sequence $u^{(n)}$ has limit. Let us show the existence of the limit of $y^{(n)}$.

Lemma 2. *If* $\beta_1 k_1 \leq b + \alpha$ *then the set*

$$M = \{(u; y) \in S^1 : by - \alpha u \leq 0\}$$

is an invariant with respect to operator (23).

Proof. Let $(u, y) \in M$, i.e., $by - \alpha u \geq 0$. We check the condition $by^{(1)} - \alpha u^{(1)} \geq 0$:

$$\begin{aligned} by^{(1)} - \alpha u^{(1)} &= b(y - by + \alpha u) - \alpha(u - bu - \alpha u + \beta_1 k_1 ux) \\ &= by - \alpha u + b\alpha u - b^2 y + b\alpha u + \alpha^2 u - \alpha\beta_1 k_1 ux \\ &= by - \alpha u - b(by - \alpha u) + \alpha u(b + \alpha - \beta_1 k_1 x) \\ &= (by - \alpha u)(1 - b) + \alpha u(b + \alpha - \beta_1 k_1 x) \geq 0. \end{aligned}$$

Thus, the Lemma is proved. □

By the Lemma (2) we show the existence of limit $y^{(n)}$. Assume that $(u^0, y^0) \in M$ then $y^{(n)}$ is decreasing for any n. Let be $(u^0, y^0) \notin M$ then there are two possible cases:

1. If after some finite step k, $(u^{(k)}, y^{(k)}) \in M$ then $(u^{(n)}, y^{(n)}) \in M$ for all $n > k$, so $y^{(n)}$ is decreasing for all $n > k$.
2. If $(u^{(n)}, y^{(n)}) \notin M$ for any $n \in N$ then $by^{(n)} - \alpha u^{(n)} > 0$, $\forall n \in N$, and so the sequence $y^{(n)}$ is increasing.

Therefore, the sequences $u^{(n)}$ and $y^{(n)}$ have limits and from $v^{(n)} \to 0$ we get that all sequences have limits. Since limit point must be fixed point and for the case $\beta_1 k_1 \leq b + \alpha$ the fixed point λ_1 is unique.

Thus, the proof of the Proposition is completed. □

For the case $b\alpha > 0$, $k_2 v^0 > 0$ numerical calculations suggest the following conjecture. (Fig. 1, Fig 2)

Conjecture 1. If $\beta_2 = 0$ then for an initial point $\lambda^0 = \left(x^0, u^0, y^0, v^0\right) \in S^3$ (except fixed points) the trajectory has the following limit

$$\lim_{n\to\infty} V^{(n)}(\lambda^0) = \begin{cases} \lambda_1 & \text{if } \beta_1 k_1 \leq b + \alpha,\ b\alpha > 0 \text{ and } k_2 v^0 > 0 \\ \lambda_{16} & \text{if } u^0 + v^0 > 0 \text{ and } \beta_1 k_1 > b + \alpha,\ b\alpha > 0 \end{cases}$$

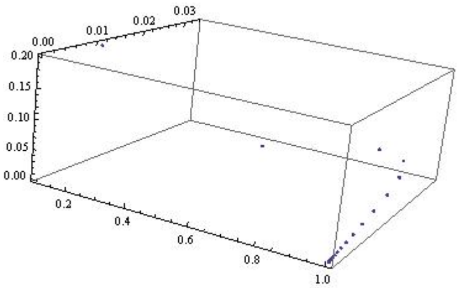

Figure 1: $\alpha = 0.2$, $b = 0.6$, $\beta_1 = 0.5$, $\beta_2 = 0$, $k_1 = 1$, $k_2 = 0.3$, $x^0 = 0.1$, $u^0 = 0.01$, $y^0 = 0.2$, $\lim\limits_{n\to\infty} V^{(n)}(\lambda^0) = \lambda_1$.

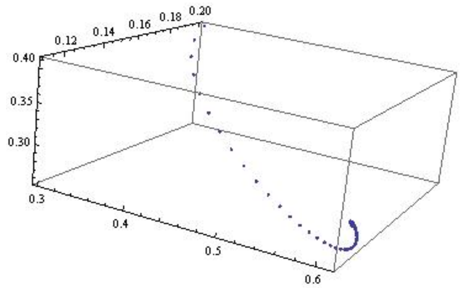

Figure 2: $\alpha = 0.2$, $b = 0.1$, $\beta_1 = 0.5$, $\beta_2 = 0$, $k_1 = 1$, $k_2 = 0.3$, $x^0 = 0.3$, $u^0 = 0.2$, $y^0 = 0.4$, $\lim\limits_{n\to\infty} V^{(n)}(\lambda^0) = \lambda_{10}$.

Let $\alpha b \beta_1 \beta_2 k_1 > 0$ and assume that all sequences have limits. First, we denote by $f(x)$, $g(x)$:

$$f(x) = b + \beta_1 x, \qquad g(x) = \frac{b\beta_1 k_1}{b+\alpha} + \frac{\alpha\beta_1\beta_2 k_2 x}{(b+\beta_2 x)(b+\alpha)}\,. \tag{24}$$

Then the roots of the equation (9) are roots of the equation $f(x) = g(x)$. We consider graphical solutions of this equation.

Case: $\beta_1 k_1 > b + \alpha$**.** In this case $\frac{b\beta_1 k_1}{b+\alpha} > b$ and the graphic of the function $g(x)$ has horizontal asymptote $y = \frac{\beta_1(bk_1+\alpha k_2)}{b+\alpha} = const$, so the equation $f(x) = g(x)$ has unique positive solution (Fig. 3).

Case: $\beta_1 k_1 < b + \alpha$**.** In this case, slope of the $f(x)$ is $Tan\varphi = \beta_1$ and slope of a tangent at the point $x = 0$ of $g(x)$ is $Tan\psi = \frac{\alpha\beta_1\beta_2 k_2}{b(b+\alpha)}$. It is clear that, if $Tan\varphi \geq Tan\psi$, i.e., $\beta_1 \geq \frac{\alpha\beta_1\beta_2 k_2}{b(b+\alpha)}$ or $b(b+\alpha) \geq \alpha\beta_2 k_2$ then $f(x) = g(x)$ does not have positive solution (Fig.4).

Case: $\beta_1 k_1 = b + \alpha$**.** In this case the equation $f(x) = g(x)$ has solution $x = 0$, i.e., operator (6) has unique fixed point λ_1.

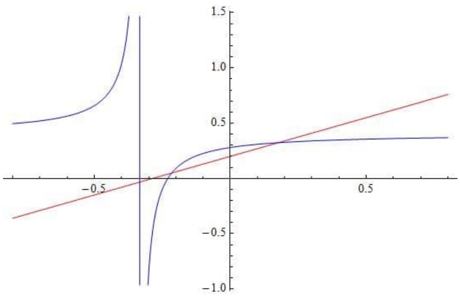

Figure 3: $\beta_1 k_1 > b + \alpha$

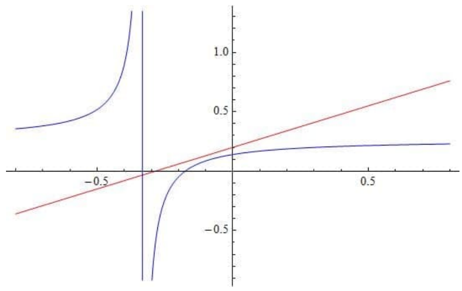

Figure 4: $\beta_1 k_1 < b+\alpha$, $b(b+\alpha) > \alpha\beta_2 k_2$

These conclusions together with numerical calculations for the operator V (6) lead the following conjecture.

Conjecture 2. If $\alpha b \beta_1 \beta_2 k_1 > 0$ then for an initial point $\lambda^0 = \left(x^0, u^0, y^0, v^0\right) \in S^3$ (except fixed points) the trajectory has the following limit

$$\lim_{n\to\infty} V^{(n)}(\lambda^0) = \begin{cases} \lambda_1 & \text{if } u^0 = v^0 = 0 \text{ or } \beta_1 k_1 \leq b+\alpha,\, b(b+\alpha) \geq \alpha\beta_2 k_2 \\ \lambda_{17} & \text{if } u^0 + v^0 > 0 \text{ and } \beta_1 k_1 > b+\alpha \end{cases}$$

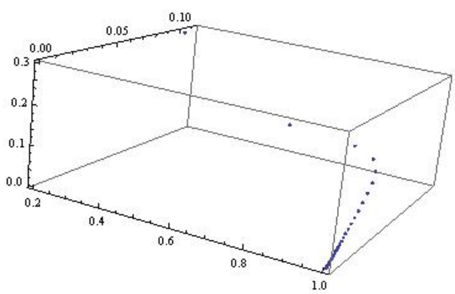

Figure 5: $\alpha = 0.1$, $b = 0.6$, $\beta_1 = 0.5$, $\beta_2 = 0.01$, $k_1 = 1.2$, $k_2 = 1.1$, $x^0 = 0.2$, $u^0 = 0.1$, $y^0 = 0.3$, $\lim_{n\to\infty} V^{(n)}(\lambda^0) = \lambda_1$.

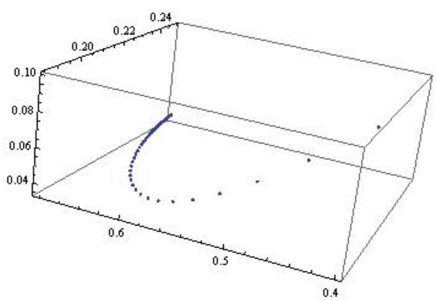

Figure 6: $\alpha = 0.01$, $b = 0.1$, $\beta_1 = 0.8$, $\beta_2 = 0.2$, $k_1 = 0.5$, $k_2 = 1.2$, $x^0 = 0.2$, $u^0 = 0.4$, $y^0 = 0.1$, $\lim_{n\to\infty} V^{(n)}(\lambda^0) = \lambda_{11}$.

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