\(\alpha\)-modules and generalized submodules

Rafiquddin, Ayazul Hasan, Mohammad Fareed Ahmad

Abstract. A QTAG-module \(M\) is an \(\alpha\)-module, where \(\alpha\) is a limit ordinal, if \(M/H_\beta(M)\) is totally projective for every ordinal \(\beta < \alpha\). In the present paper \(\alpha\)-modules are studied with the help of \(\alpha\)-pure submodules, \(\alpha\)-basic submodules, and \(\alpha\)-large submodules. It is found that an \(\alpha\)-closed \(\alpha\)-module is an \(\alpha\)-injective. For any ordinal \(\omega \leq \alpha \leq \omega_1\) we prove that an \(\alpha\)-large submodule \(L\) of an \(\omega_1\)-module \(M\) is summable if and only if \(M\) is summable.

1 Introduction and Preliminary Terminology

Let \(R\) be any ring. A module \(M\) in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length, it is called a uniserial module. An element \(x \in M\) is uniform, if \(xR\) is a non-zero uniform (hence uniserial) module, and for any \(R\)-module \(M\) with a unique decomposition series, \(d(M)\) denotes its decomposition length.

Modules are the natural generalizations of abelian groups. The results for abelian groups can be generalized for modules after imposing some conditions on modules/rings. In 1976 Singh [15] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

1. Every finitely generated submodule of any homomorphic image of \(M\) is a direct sum of uniserial modules.

2. Given any two uniserial submodules \(U\) and \(V\) of a homomorphic image of \(M\), for any submodule \(W\) of \(U\), any non-zero homomorphism \(f: W \to V\) can be

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extended to a homomorphism \( g: U \to V \), provided the composition length
\( d(U/W) \leq d(V/f(W)) \).

Later on Benabdallah, Singh, Khan etc. contributed a lot to the study of TAG-modules [2], [17]. In 1987 Singh made an improvement and studied the modules satisfying only the condition 1 and called them QTAG-modules. The study of QTAG-modules and their structure began with work of Singh in [16]. This work, executed by many authors, clearly parallels the earlier work on torsion abelian groups. They studied different notions and structures on QTAG-modules and developed the theory of these modules by introducing different notions and characterizing different submodules of QTAG-modules. Yet there is much to explore.

Throughout this paper, all the rings are associative with unity \((1 \neq 0)\) and modules \( M \) are unital QTAG-modules. For a uniform element \( x \in M \), \( e(x) = d(xR) \) and

\[
H_M(x) = \sup \left\{ d\left( \frac{yR}{xR} \right) \mid y \in M, \ x \in yR \text{ and } y \text{ uniform} \right\}
\]

are the exponent and height of \( x \) in \( M \), respectively. \( H_k(M) \) denotes the submodule of \( M \) generated by the elements of height at least \( k \) and \( H^k(M) \) is the submodule of \( M \) generated by the elements of exponents at most \( k \). Let us denote by \( M^{1} \), the submodule of \( M \), containing elements of infinite height. The module \( M \) is \( h \)-divisible if \( M = M^1 = \bigcap_{k=0}^{\infty} H_k(M) \). The module \( M \) is \( h \)-reduced if it does not contain any \( h \)-divisible submodule. In other words, it is free from the elements of infinite height. The module \( M \) is said to be bounded, if there exists an integer \( n \) such that \( H_M(x) \leq n \) for every uniform element \( x \in M \).

The sum of all simple submodules of \( M \) is called the socle of \( M \), denoted by \( \text{Soc}(M) \) and a submodule \( S \) of \( \text{Soc}(M) \) is called a subsocle of \( M \). The cardinality of the minimal generating set of \( M \) is denoted by \( g(M) \). For all ordinals \( \alpha \), \( f_M(\alpha) \) is the \( \alpha \)th Ulm invariant of \( M \) and it is equal to \( g\left( \text{Soc}(H_0(M))/\text{Soc}(H_{\alpha+1}(M)) \right) \).

A submodule \( N \) of \( M \) is \( h \)-pure in \( M \) if \( N \cap H_k(M) = H_k(N) \), for every integer \( k \geq 0 \). For an ordinal \( \alpha \), a submodule \( N \subseteq M \) is an \( \alpha \)-high submodule of \( M \) if \( N \) is maximal among the submodules of \( M \) that intersect \( H_\alpha(M) \) trivially.

For an ordinal \( \alpha \), a submodule \( N \) of \( M \) is said to be an \( \alpha \)-pure, if \( H_\beta(M) \cap N = H_\beta(N) \) for all \( \beta \leq \alpha \) and a submodule \( N \) of \( M \) is said to be isotype in \( M \), if it is \( \alpha \)-pure for every ordinal \( \alpha \) [13]. For a submodule \( N \subseteq M \), the valuation of \( N \) induced by height in \( M \) is defined by \( v(x) = H_M(x) \), the height of \( x \) in \( M \), for all \( x \in N \) and \( N = K \oplus L \) is a valued direct sum if \( v(k + \ell) = \min\{v(k), v(\ell)\} \) for all \( k \in K \) and \( \ell \in L \) [5].

A submodule \( B \subseteq M \) is a basic submodule [9] of \( M \), if \( B \) is \( h \)-pure in \( M \), \( B = \oplus B_i \), where each \( B_i \) is the direct sum of uniserial modules of length \( i \) and \( M/B \) is \( h \)-divisible. A fully invariant submodule \( L \subseteq M \) is large [1], if \( L + B = M \), for every basic submodule \( B \) in \( M \).

Imitating [11], the submodules \( H_k(M), k \geq 0 \) form a neighborhood system of zero, thus a topology known as \( h \)-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of \( N \subseteq M \) is defined as
\[ \mathcal{N} = \bigcap_{k=0}^{\infty} (N + H_k(M)). \]
Therefore the submodule \( N \subseteq M \) is closed with respect to \( h \)-topology if \( \overline{N} = N \).

An \( h \)-reduced QTAG-module \( M \) is summable [14] if \( \text{Soc}(M) = \bigoplus_{\beta<\alpha} S_\beta \), where \( S_\beta \) is the set of all elements of \( H_\beta(M) \) which are not in \( H_{\beta+1}(M) \), where \( \alpha \) is the length of \( M \). Moreover, \( M \) is called totally projective [10], if
\[ H_\alpha(\text{Ext}(M/H_\beta(M), M')) = 0 \]
for all ordinal \( \alpha \) and QTAG-modules \( M' \).

It is interesting to note that almost all the results which hold for TAG-modules are also valid for QTAG-modules [13]. Our notations and terminology generally agree with those in [3] and [4].

2 \textbf{\( \alpha \)-modules and \( \alpha \)-basic submodules}

Mehdi et al. [12] defined \( \alpha \)-modules and introduced some new concepts for these modules. The same type of study was continued in [6] and a number of results have been obtained in terms of generalized submodules. Here we also continue the similar study of \( \alpha \)-modules that depend on the notions of summability, purity, basic submodules, projectivity and injectivity. For facilitating the exposition and for the convenience of the readers, we recall the definition of \( \alpha \)-modules.

\textbf{Definition 1.} Let \( \alpha \) denote the class of all QTAG-modules \( M \) such that \( M/H_\beta(M) \) is totally projective for all ordinals \( \beta < \alpha \), a limit ordinal. These modules are called \( \alpha \)-modules.

To develop the study, we need to prove some results, and we start with the following.

\textbf{Proposition 1.} If \( N \) is an \( \alpha \)-pure submodule of an \( \alpha \)-module \( M \), then \( N \) is itself an \( \alpha \)-module.

\textit{Proof.} We actually only need that \( N \cap H_\gamma(N) = H_\gamma(N) \) for all \( \gamma < \alpha \). For then it is a simple calculation to show that \( N + H_\beta(M)/H_\beta(M) \) is isotype in \( M/H_\beta(M) \) for each \( \beta < \alpha \). And therefore, \( N + H_\beta(M)/H_\beta(M) \cong N/H_\beta(N) \) is totally projective for all \( \beta < \alpha \). \( \square \)

As generalized the notion of a basic submodule in [12], by defining \( B \) to be an \( \alpha \)-basic submodule of an \( \alpha \)-module \( M \) if \( B \) is totally projective of length at most \( \alpha \), \( B \) is \( \alpha \)-pure submodule of \( M \), and \( M/B \) is \( h \)-divisible.

In order to establish the existence of \( \alpha \)-basic submodules we require the following notion for technical convenience.

\textbf{Definition 2.} Let \( \alpha \) be a limit ordinal and \( M \) a QTAG-module. An \( \alpha \)-high tower of \( M \) is a well-ordered ascending chain \( \{M_\beta\}_{\beta<\alpha} \) of submodules of \( M \) such that, for each \( \beta \), \( M_\beta \) is a \( \beta \)-high submodule of \( M \).
Now we need to prove the following lemma.

**Lemma 1.** Let $\alpha$ be a limit ordinal and $\{M_\beta\}_{\beta<\alpha}$ an $\alpha$-high tower of a QTAG-module $M$. If each $M_\beta$ is summable, then $N = \bigcup_{\beta<\alpha} M_\beta$ is summable.

**Proof.** As $\alpha$ is a limit ordinal, we may choose a strictly increasing sequence $\beta_1 < \beta_2 < \cdots < \beta_n < \cdots$ of ordinals having $\alpha$ as its limit. Then $N = \bigcup_{n<\omega} M_{\beta_n}$. Set $T_0 = \text{Soc}(M_{\beta_1})$ and, for $n > 1$, let $T_n$ be such that

$$\text{Soc}(H_{\beta_n}(M)) = T_n \oplus \text{Soc}(H_{\beta_{n+1}}(M))$$

with $T_n \subseteq M_{\beta_{n+1}}$. Then we have a direct-sum decomposition $\text{Soc}(N) = \bigoplus_{n<\omega} T_n$ which is normal in the sense that

$$H_M(t_1 + \cdots + t_n) = \min[H_M(t_1), \ldots, H_M(t_n)]$$

provided $t_i \in T_i$ for $i = 1, \ldots, n$. Now each $M_\beta$ is isotype, summable, and of countable length. Therefore, each subsocle of $M_\beta$ is a summable subsocle of $M$. In particular, each $T_n$ is a summable subsocle of $M$. Since the decomposition $\text{Soc}(N) = \bigoplus_{n<\omega} T_n$ is normal, it follows that $\text{Soc}(N)$ is a summable subsocle of $M$. Since each $M_\beta$ is isotype, $N$ is itself an isotype submodule of $M$ and consequently $N$ is summable. $\square$

We continue the study with the following corollary.

**Corollary 1.** Let $\alpha$ be a limit ordinal and $\{M_\beta\}_{\beta<\alpha}$ an $\alpha$-high tower of a QTAG-module $M$, where each $M_\beta$ is totally projective, then $N = \bigcup_{\beta<\alpha} M_\beta$ is totally projective of length at most $\alpha$.

**Proof.** As noted above, $N$ is an isotype submodule of $M$ and clearly $N$ has a length at most $\alpha$. Thus $M_\beta$ is also a $\beta$-high submodule of $N$ for each $\beta < \alpha$. Since $N$ is summable by Lemma 1 implies that $N$ is totally projective. $\square$

Now we prove the following.

**Theorem 1.** Let $M$ be a QTAG-module. Then $M$ contains an $\alpha$-basic submodule if and only if $M$ is an $\alpha$-module.

**Proof.** If $B$ is an $\alpha$-pure submodule of $M$ and if $M/B$ is h-divisible, then it follows that $M/H_\beta(M) \cong B/H_\beta(B)$ for all $\beta < \alpha$. Consequently, only $\alpha$-modules can have $\alpha$-basic submodules (see [12]). Suppose now that $M$ is an $\alpha$-module and select an $\alpha$-high tower $\{M_\beta\}_{\beta<\alpha}$. Now

$$M_\beta \cong M_\beta + H_\beta(M)/H_\beta(M),$$

and since $M_\beta$ is isotype in $M$, $M_\beta + H_\beta(M)/H_\beta(M)$ is isotype in $M/H_\beta(M)$. By Corollary 1, $B = \bigcup_{\beta<\alpha} M_\beta$ is totally projective. It is easily seen that

$$\text{Soc}(M) \subseteq \text{Soc}(B) + H_\beta(M)$$
for each $\beta < \alpha$, and therefore $B$ is $\alpha$-pure in $M$. Moreover, $B \cap H_1(M) = H_1(B)$ and

$$\text{Soc}(M) \subseteq \text{Soc}(B) + H_\beta(M)$$

for $\beta < \omega$ imply that $M/B$ is $h$-divisible. Thus, $B$ is the required $\alpha$-basic submodule of $M$. \qed

Lemma 2. Suppose $N$ is an isotype submodule of a QTAG-module $M$ and that \( \{N_\beta\}_{\beta<\alpha} \) is an $\alpha$-high tower of $N$, then there exists an $\alpha$-high tower $\{M_\beta\}_{\beta<\alpha}$ of $M$ such that, for each $\beta$, $N_\beta \subseteq M_\beta$ and $N_\beta = N \cap M_\beta$.

Proof. Let us first note that $N_\beta = N \cap M_\beta$ is a consequence of $N_\beta \subseteq M_\beta$. Indeed, $N_\beta \subseteq M_\beta$ implies $N_\beta \subseteq N \cap M_\beta$ and

$$(N \cap M_\beta) \cap H_\beta(N) = (N \cap M_\beta) \cap H_\beta(M) = 0.$$ 

The maximality of a $\beta$-high submodule then yields the equality. Assume now that $\beta < \alpha$ and that for each $\gamma < \beta$ we have a $\gamma$-high submodule $M_\gamma$ of $M$ such that $N_\gamma \subseteq M_\gamma$ and $M_\eta \subseteq M_\gamma$ for all $\eta < \gamma$. In order to be able to choose the desired $M_\beta$, it suffices to show that

$$(N_\beta + \bigcup_{\gamma<\beta} M_\gamma) \cap \text{Soc}(H_\beta(M)) = 0.$$ 

Suppose $x + y \in \text{Soc}(H_\beta(M))$ where $x \in N_\beta$ and $y \in M_\gamma$ for some $\gamma < \beta$. Then

$$H(x') = -H(y') \in H_1(M) \cap N \cap M_\gamma = H_1(M) \cap N_\gamma = H_1(N_\gamma),$$ 

where $d(\frac{x}{x'}R) = d(\frac{y}{y'}R) = 1$, and hence there is $u \in N_\gamma$ such that

$$x - u \in \text{Soc}(N) = \text{Soc}(N_\gamma) \oplus \text{Soc}(H_\gamma(N)).$$ 

Thus we can write $x = u + v + z$ where $v \in \text{Soc}(N_\gamma)$ and $z \in \text{Soc}(H_\gamma(N))$. Then

$$u + v + y = x + y - z \in H_\gamma(M) \cap M_\gamma = 0 \quad \text{and} \quad x + y = z \in N.$$ 

Therefore $y \in N \cap M_\gamma = N_\gamma \subseteq N_\beta$ and, consequently,

$$x + y \in N_\beta \cap H_\beta(M) = N_\beta \cap H_\beta(N) = 0$$

as desired. \qed

Lemma 3. Let $M$ be a totally projective QTAG-module such that $M = \bigcup_{\beta<\alpha} M_\beta$ where $\{M_\beta\}_{\beta<\alpha}$ is an $\alpha$-high tower. If $N$ is an $\alpha$-pure submodule of $M$ such that for each $\beta$, $N \cap M_\beta$ is a $\beta$-high submodule of $N$, then $N$ is a direct summand of $M$.

Proof. We need only show that $M/N$ is totally projective having length at most $\alpha$. Since $N \cap M_\beta$ is $(\beta + 1)$-pure in $N$ and $N$ is $\alpha$-pure in $M$, $N \cap M_\beta$ is $(\beta + 1)$-pure in $M$ and, a fortiori, $(\beta + 1)$-pure in $M_\beta$. Since $M_\beta$ is totally projective, $M_\beta$ is
\(\beta\)-projective. Therefore, there is direct decomposition \(M_\beta = (N \cap M_\beta) \oplus K_\beta\) for each \(\beta < \alpha\). Now

\[
\frac{M}{N} = \bigcup_{\beta < \alpha} M_\beta + \frac{N}{N} \quad \text{and} \quad M_\beta + \frac{N}{N} \cong M_\beta/(M_\beta \cap N) \cong K_\beta
\]

is totally projective for each \(\beta\). By Corollary 1, it is enough to show that \(M_\beta + \frac{N}{N}\)

is a \(\beta\)-high submodule of \(\frac{M}{N}\) whenever \(\omega \leq \beta < \alpha\). Since \(N\) is \(\alpha\)-pure in \(M\), we have

\[
\text{Soc}(H_\beta(M/N)) = \text{Soc}(H_\beta(M)) + \frac{N}{N}
\]

for \(\beta < \alpha\) and it then easily follows that

\[
\text{Soc}(\frac{M}{N}) = \text{Soc}(M_\beta + \frac{N}{N}) \oplus \text{Soc}(H_\beta(M/N))
\]

Because of this direct decomposition, it is enough to show that \(M_\beta + \frac{N}{N}\) is an \(h\)-pure submodule of \(\frac{M}{N}\) for \(\beta \geq \omega\).

Now

\[
\text{Soc}(M_\beta + N) = \text{Soc}(K_\beta \oplus N)
\]

\[
= \text{Soc}(K_\beta) \oplus \text{Soc}(N)
\]

\[
= \text{Soc}(K_\beta) \oplus \text{Soc}(N \cap M_\beta) \oplus \text{Soc}(H_\beta(N))
\]

\[
= \text{Soc}(M_\beta) \oplus \text{Soc}(H_\beta(N)).
\]

If \(\beta \geq \omega\) and if \(x \in \text{Soc}(M_\beta + N)\), then we can write \(x = y + z\) where \(y \in \text{Soc}(M_\beta)\) and \(z \in \text{Soc}(H_\beta(N)) \subseteq H_\omega(N)\). If \(x\) has finite height in \(M\), then this height is just the height of \(y\) in \(M\) (= height of \(y\) in \(M_\beta\)) and thus just the height of \(x = y + z\) in \(M_\beta + N\). On the other hand, if \(x\) has infinite height in \(M\), then \(y\) has infinite height in \(M_\beta\) and \(x = y + z\) has infinite height in \(M_\beta + N\), it follows that \(M_\beta + N\) is an \(h\)-pure submodule of \(M\). Thus \(M_\beta + \frac{N}{N}\) is \(h\)-pure in \(\frac{M}{N}\). \(\square\)

**Proposition 2.** Let \(N\) be an \(\alpha\)-pure submodule of an \(\alpha\)-module \(M\) such that \(N\) is totally projective of length at most \(\alpha\). Then there exists a submodule \(K\) of \(M\) such that \(N \oplus K\) is an \(\alpha\)-basic submodule of \(M\).

**Proof.** Since \(N\) is totally projective of length \(\leq \alpha\), \(N\) is the union of an \(\alpha\)-high tower \(\{N_\beta\}_{\beta < \alpha}\) of itself. By Lemma 2, there exists an \(\alpha\)-high tower \(\{M_\beta\}_{\beta < \alpha}\) of \(M\) such that \(N_\beta = N \cap M_\beta\) for each \(\beta\). Let \(B = \bigcup_{\beta < \alpha} M_\beta\). By the proof of Theorem 1, \(B\) is an \(\alpha\)-basic submodule of \(M\). But \(\{M_\beta\}_{\beta < \alpha}\) is also an \(\alpha\)-high tower of \(B\), and by Lemma 3 we have the required direct decomposition \(B = N \oplus K\). \(\square\)

Now we prove the following result.

**Theorem 2.** If \(N\) is an \(\alpha\)-pure submodule of an \(\alpha\)-module \(M\), then \(\frac{M}{N}\) is an \(\alpha\)-module.

**Proof.** Let \(B\) be an \(\alpha\)-basic submodule of \(N\) and choose \(K\) such that \(B \oplus K\) is an \(\alpha\)-basic submodule of \(M\). Now if \(x \in \text{Soc}(N \cap K)\), we can write for each \(\beta < \alpha\), \(x = y_\beta + z_\beta\), where \(y_\beta \in \text{Soc}(N)\) and \(z_\beta \in H_\beta(N)\). Thus

\[-y_\beta + x \in H_\beta(B \oplus K) = H_\beta(B) \oplus H_\beta(K)\]
\[ x \in \bigcap_{\beta < \alpha} H_\beta(K) = H_\alpha(K) = 0. \]

We then have a direct decomposition \( N \oplus K \). If \( H_1(a') \in N \oplus K \), then \( H_1(a') = y + H_1(b') + c \), where \( d(\frac{aR}{aR}) = d(\frac{bR}{bR}) = 1 \), \( y \in B \), \( b \in N \) and \( c \in K \). Since \( H_1(M) \cap (B \oplus K) = H_1(B \oplus K) \), we conclude that \( H_1(M) \cap (N \oplus K) = H_1(N \oplus K) \).

Now \( \text{Soc}(M) \subseteq \text{Soc}(B \oplus K) + H_\beta(M) \subseteq \text{Soc}(N \oplus K) + H_\beta(M) \) for all \( \beta < \alpha \), and therefore \( N \oplus K \) is an \( \alpha \)-pure submodule of \( M \). Consequently, \( N \oplus K/N \) is \( \alpha \)-pure in \( M/N \). Also \( N \oplus K/N \cong K \) and \( (M/N)/(N \oplus K/N) \cong (M/B \oplus K)/[(N \oplus K)/(B \oplus K)] \) is \( h \)-divisible. We have constructed an \( \alpha \)-basic submodule of \( M/N \) and we conclude that \( M/N \) is indeed an \( \alpha \)-module. \( \square \)

As a consequence of the above theorem, we have the following striking analog of a familiar property of \( h \)-pure submodules.

**Corollary 2.** Let \( N \) be a submodule of an \( \alpha \)-module \( M \). Then \( N \) is an \( \alpha \)-pure submodule of \( M \) if and only if \( N + H_\beta(M)/H_\beta(M) \) is a direct summand of \( M/H_\beta(M) \) for all \( \beta < \alpha \).

**Proof.** \( N + H_\beta(M)/H_\beta(M) \) being a direct summand of \( M/H_\beta(M) \) implies that \( N + H_\beta(M)/H_\beta(M) \) is \( \beta \)-pure in \( M/H_\beta(M) \), which is equivalent to \( N \) being \( \beta \)-pure in \( M \). Since \( \alpha \) is a limit ordinal, \( N \) is \( \alpha \)-pure in \( M \) if and only if \( N \) is \( \beta \)-pure in \( M \) for all \( \beta < \alpha \).

Conversely, assume that \( N \) is \( \alpha \)-pure in \( M \). Then \( M/N \) is an \( \alpha \)-module and therefore, for \( \beta < \alpha \),

\[
(M/N)/H_\beta(M/N) = (M/N)/(H_\beta(M) + N/N) \
\cong (M/H_\beta(M))/(N + H_\beta(M)/H_\beta(M))
\]

is totally projective of length at most \( \beta \). Since \( N + H_\beta(M)/H_\beta(M) \) is \( \beta \)-pure in \( M/H_\beta(M) \), \( N + H_\beta(M)/H_\beta(M) \) is a direct summand of \( M/H_\beta(M) \). \( \square \)

**Proposition 3.** If \( N \) is an \( \alpha \)-pure submodule of an \( \alpha \)-module \( M \), and if \( H_\beta(N) \) is a direct summand of \( H_\beta(M) \) for some \( \beta < \alpha \), then \( N \) is a direct summand of \( M \).

**Proof.** Assuming the conditions of the Theorem 2, we have for some \( \beta < \alpha \):
(i) \((M/N)/H_\beta(M/N)\) is totally projective;
(ii) \(N \cap H_\beta(M) = H_\beta(N)\);
(iii) \(N + H_\beta(M)/H_\beta(M)\) is a direct summand of \(M/H_\beta(M)\); and
(iv) \(H_\beta(M) = H_\beta(N) \oplus K\).

It follows that \(M = N \oplus L\) where \(L \supseteq K\). \(\square\)

As a corollary, we have the following generalization of the well-known fact that bounded \(h\)-pure submodules are direct summands.

**Corollary 3.** If \(N\) is an \(\alpha\)-pure submodule of an \(\alpha\)-module \(M\) and if \(H_\beta(N) = 0\) for some \(\beta < \alpha\), then \(N\) is a direct summand of \(M\).

As defined in [10], a QTAG-module \(M\) is fully transitive if for every pair of uniform elements \(x, y \in M\), \(H_M(x_i) \leq H_M(y_i)\) for all \(i \geq 0\) implies that there exists an endomorphism of \(M\) that maps \(x\) onto \(y\). Here \(d(xR_i R) = d(yR_i R) = i\).

The next corollary tells us that \(\alpha\)-modules of length \(\alpha\) are fully transitive (see [6]). This, of course, is merely a reflection of the fact that modules of length \(\leq \alpha\) behave in the \(\alpha\) context exactly as modules without elements of infinite height in the classical situations.

**Corollary 4.** If \(M\) is an \(\alpha\)-module of length \(\alpha\), then every finite subset of \(M\) is contained in a countably generated direct summand.

**Proof.** Let \(S\) be a finite subset of \(M\). Then \(S \subseteq T\) for some countably generated, \(\alpha\)-pure submodule \(T\) of \(M\). We may assume that \(T\) has length \(\alpha\). Then \(T\) is a direct sum of modules of length less than \(\alpha\). Consequently, \(T\) is contained in a direct summand \(K\) of \(T\) having length less than \(\alpha\). By the preceding corollary, \(K\) is a direct summand of \(M\). \(\square\)

For a limit ordinal \(\alpha\), an \(\alpha\)-module \(M\) is an \(\alpha\)-projective if
\[ H_\alpha(\text{Ext}(M, M')) = 0 \]
for all \(\alpha\)-modules \(M'\), that is, there exists a submodule \(N\) bounded by \(\alpha\) such that \(M/N\) is totally projective, and an \(\alpha\)-module \(M\) is an \(\alpha\)-injective if
\[ H_\alpha(\text{Ext}(M', M)) = 0 \]
for all \(\alpha\)-modules \(M'\), that is, it is a direct summand of every \(\alpha\)-module in which it occurs as an \(\alpha\)-pure submodule.

To characterize the \(\alpha\)-injective modules we must generalize the notion of a closed module. Mimicking [12], for any QTAG-module \(M\), the submodules \(\{H_k(M)\}_k\), \(k = 0, 1, 2, \ldots, \infty\) from a neighborhood system of zero, giving rise to \(h\)-topology. If \(k\) is replaced by an arbitrary limit ordinal less than or equal to \(\alpha\), then \(h\)-topology may be extended to \(\alpha\)-topology, and all the definitions and results which hold for \(h\)-topology may be extended for \(\alpha\)-topology. In \(\alpha\)-topology, for any submodule \(N\) of \(M\), the closure of \(N\) as \(\bigcap_{\beta<\alpha}(N + H_\beta(M))\) denoted by \(\overline{N}\).
**Definition 3.** We call a QTAG-module an $\alpha$-closed module if it is the maximal closed submodule of its closure in the $\alpha$-topology.

With the help of the above discussion, we are able to infer the following.

**Proposition 4.** Let $M$ be an $\alpha$-closed $\alpha$-module. Then $M$ is an $\alpha$-injective.

**Proof.** We first show that $H_\alpha(\text{Ext}(T, M)) = 0$ for all $\alpha$-modules $T$. Assume that $M$ is an $\alpha$-pure submodule of $M'$ with $M'/M \cong T$ for all $\alpha$-modules $M'$. Since $\alpha$ is a limit ordinal, it follows that $M' = H_\beta(M') + M$ for all $\beta < \alpha$. Therefore, if $y \in M'$, we can find for each $\beta < \alpha$ a $x_\beta \in M$ such that $y - x_\beta \in H_\beta(M')$. Moreover, we can assume that the exponent of $x_\beta$ does not exceed that of $y$. Indeed, if $y$ has exponent $n$, then

$$H_n(x'_\beta) \in H_{\beta+n}(M') \cap M = H_{\beta+n}(M),$$

where $d(x_\beta R)$ = $n$ and $H_n(x'_\beta) = H_n(z'_\beta)$, where $d(x_\beta R) = d(z_\beta R)$ = $n$ for some $z_\beta \in H_\beta(M)$. Then $x_\beta = x_\beta - z_\beta$ has an exponent at most $n$ and $y - x_\beta \in H_\beta(M')$. But $\{x_\beta : \beta < \alpha\}$ is a chain in $M$ with elements uniformly bounded in exponent and, therefore, converges to some $x \in M$. Hence

$$y - x \in \bigcap_{\beta < \alpha} H_\beta(M') = H_\alpha(M').$$

We conclude that $M' = M \oplus H_\alpha(M')$.

Now let $M'$ be an arbitrary $\alpha$-module and let $B$ be an $\alpha$-basic submodule of $M'$. We then have the exact sequence

$$H_\alpha(\text{Ext}(M'/B, M)) \longrightarrow H_\alpha(\text{Ext}(M', M)) \longrightarrow H_\alpha(\text{Ext}(B, M)).$$

The left-hand term of the above sequence vanishes since $M'/B$ is isomorphic to a direct sum of copies of $T$ and the right-hand term vanishes since $B$ is an $\alpha$-projective. Thus, $H_\alpha(\text{Ext}(M', M)) = 0$ and we conclude that $M$ is an $\alpha$-injective.

We can now show that there are enough $\alpha$-injective modules and that an $\alpha$-injective module is the sum of an $\alpha$-closed module and an $h$-divisible module.

**Theorem 3.** Let $M$ be an $\alpha$-module. Then $M$ is an $\alpha$-pure submodule of an $\alpha$-injective module and $M$ is an $\alpha$-injective module if and only if $M$ is the direct sum of an $h$-divisible module and an $\alpha$-closed $\alpha$-module.

**Proof.** It is evident from Proposition 4 that the direct sum of an $h$-divisible module and an $\alpha$-closed $\alpha$-module is necessarily an $\alpha$-injective. Next, we need the observation that every $\alpha$-module $M$ of length at most $\alpha$ can be imbedded as an $\alpha$-pure submodule of an $\alpha$-closed module $T_M(\alpha)$ such that $T_M(\alpha)/M$ is $h$-divisible. Indeed, $T_M(\alpha)$ may be taken as the maximal closed submodule of the closure of $M$. 


in the $\alpha$-topology. It follows, by the same reasoning as in the proof of Theorem 1, that

$$T_M(\alpha)/H_\beta(T_M(\alpha)) \cong M/H_\beta(M)$$

for all $\beta < \alpha$, and therefore that $T_M(\alpha)$ is an $\alpha$-module.

Now let $M$ be an arbitrary $\alpha$-module. Let $D$ be a minimal $h$-divisible module containing $H_\alpha(M)$. Take $P$ to be the amalgamated sum of $M$ and $D$ over $H_\alpha(M)$. Then $P = M' \oplus D$ where $M' \cong M/H_\alpha(M)$ and $M' \cap M$ is an $\alpha$-high submodule of $M$. Also, $P/M$ is $h$-divisible and

$$\text{Soc}(P) \subseteq \text{Soc}(M) + H_\beta(P)$$

for all $\beta < \alpha$. It follows that $M$ is an $\alpha$-pure submodule of $P$. By the transitivity of $\alpha$-purity, $M$ is an $\alpha$-pure in the $\alpha$-injective $T_{M'}(\alpha) \oplus D$.

Finally, assume that $M$ is itself an $\alpha$-injective and that we have it imbedded, as above, as an $\alpha$-pure submodule of $\overline{P} = T_{M'}(\alpha) \oplus D$. Since $M$ is an $\alpha$-injective, $\overline{P} = M \oplus Q$ where $Q \cong \overline{P}/M$ is obviously $h$-divisible, since both $P/M$ and $\overline{P}/P$ are $h$-divisible. But then $Q \subseteq D$, and since $\text{Soc}(D) \subseteq H_\alpha(M)$, we conclude that $Q = 0$ and $M = T_{M'}(\alpha) \oplus D$. \hfill \Box

Now we are in a position to prove the following result.

**Theorem 4.** If $M$ and $M'$ are $\alpha$-closed $\alpha$-modules with the same Ulm invariants, then $M \cong M'$.

**Proof.** Take $B$ and $B'$ to be $\alpha$-basic submodules of $M$ and $M'$, respectively. It is easily seen that $B$ and $B'$ have the same Ulm invariants as $M$ and $M'$. Therefore, there is an isomorphism $f$ of $B$ onto $B'$. Since $B$ is an $\alpha$-pure submodule of $M$, we have the exact sequence

$$\text{Hom}(M, M') \rightarrow \text{Hom}(B, M') \rightarrow H_\alpha(\text{Ext}(M/B, M')) = 0$$

Thus, there is a homomorphism $f' : M \rightarrow M'$ that extends $f$. Let $x \in \text{Ker} f'$ and assume that $x \neq 0$. Then $x$ has some height $\beta < \alpha$ and we can write $x = y + z$ where $y \in B$ and $z \in H_{\beta+1}(M)$. But then $x$ has height $\beta$ and $f(y) = f'(y) = -f'(z)$ has height at least $\beta + 1$. This, however, is a contradiction, since $f$ is an isomorphism of $B$ onto $B'$ and $B'$ is an isotype submodule of $M'$. We conclude that $\text{Ker} f' = 0$. Then

$$f'(M)/B' = f'(M)/f'(B) \cong M/B$$

is $h$-divisible. Hence $f'(M)/B'$ is a direct summand of $M'/B'$, and since $B'$ is an $\alpha$-pure submodule of $M'$, it follows that $f'(M)$ is an $\alpha$-pure submodule of $M'$. Since $f'(M) \cong M$ is an $\alpha$-injective, we have a direct decomposition $M' = f'(M)) \oplus L$ where $L \cong M'/f'(M)$ is $h$-divisible. But $M'$ is $h$-reduced and therefore $L = 0$ and $f'(M) = M'$, that is, $f'$ is an isomorphism of $M$ onto $M'$. \hfill \Box
3 $\alpha$-large submodules of summable modules

If $F$ is a fully invariant submodule of the $\alpha$-module $M$, then $F$ is called an $\alpha$-large submodule of $M$ if $F = B + F$ for all $\alpha$-basic submodules $B$ of $M$. This generalization of the concept of large submodule is studied in [12]. It is well-known that $H_\beta(M)$ is always an $\alpha$-large submodule of $M$ provided that $\beta < \alpha$ as well as $H_\beta(M) \subseteq L$ whenever $L$ is an $\alpha$-large submodule in $M$.

Likewise, It was proved that an $h$-reduced QTAG-module $M$ of length $\geq \alpha$ contains a proper $\alpha$-basic submodule $B$ if and only if $M$ is an $\alpha$-module where $\alpha$ is cofinal with $\omega$. Since $\omega_1$, the first uncountable limit ordinal, is not cofinal with $\omega$, some additional clarifications are necessary. In fact, $B$ is an $\omega_1$-basic submodule of $M$ only when $B = 1$ or $B = M$, and so $L$ is an $\omega_1$-large submodule of $M$ uniquely when $L = M$ and either $L = 1$ or $L \neq 1$ and it can take different forms; for instance $L = H_\beta(M)$ where $\beta < \text{length of } M \leq \omega_1$.

In [12] it was seen that the properties of $\alpha$-large submodules for $\alpha > \omega$ are not preserved in general by these of the QTAG-module and conversely; for instance the direct sum of countably generated modules. However, this is not the case for totally projective modules.

**Theorem 5 ([6], [12]).** Let $L$ be an $\alpha$-large submodule of the QTAG-module $M$. Then $L$ is totally projective if and only if $M$ is totally projective.

The main goal of this section is to strengthen the above assertion to a class of modules, called summable modules. It is evident that direct sum of countably generated modules are themselves summable. In [8] it was constructed a summable $\omega_1$-module need not be a direct sum of countably generated modules. So, the investigation of the discussed above theme for $\alpha$-large submodules of summable $\omega_1$-modules will be of interest.

Now we have accumulated all the machinery necessary to prove the following.

**Theorem 6.** Suppose that $M$ is an $\omega_1$-module with an $\alpha$-large submodule $L$ for some ordinal $\alpha$ such that $\omega \leq \alpha \leq \omega_1$. Then $M$ is summable if and only if $L$ is summable.

**Proof.** “$\Rightarrow$”. In virtue of [12] there is a countable limit ordinal $\tau \leq \alpha$ such that $H_\tau(M) = H_\omega(L)$. Moreover, $L/H_\omega(L) = L/H_\tau(M)$ is an $\alpha$-large submodule of $M/H_\tau(M)$, where the latter quotient is totally projective by assumption. Therefore, Theorem 5 applies to deduce that $L/H_\omega(L)$ is totally projective, in fact, a direct sum of uniserial modules. That is why, some high submodule $N$ of $L$ is a direct sum of countably generated modules. Indeed, what suffices to show is that $N/H_\omega(L)$ is a direct sum uniserial modules because $H_\omega(L)$ is bounded. In order to do that, we observe that

$$(N + H_\omega(L))/H_\omega(L) \subseteq L/H_\omega(L)$$

is also a direct sum of uniserial modules as a submodule. But $N$ is isotype in $L$, whence

$$(N + H_\omega(L))/H_\omega(L) \cong N/(N \cap H_\omega(L)) = N/H_\omega(L)$$
which substantiates our claim. On the other hand, \( M \) being summable yields that \( H_\omega(L) = H_\tau(M) \) is summable, and we are done.

\[ \leftarrow \] Same as above, \( H_\tau(M) = H_\omega(L) \) for some countable limit ordinal \( \tau \leq \alpha \).

But \( L \) being summable implies that \( H_\omega(L) = H_\tau(M) \) is summable. Likewise, \( M/H_\tau(M) \) is totally projective of countable length, hence a direct sum of countably generated modules. Let \( N \) be a \( \tau \)-high submodule of \( M \). Since \( H_\tau(N) \) is high in \( H_\tau(M) \) one may write \( H_\tau(M) = H_\tau(N) \oplus T \) for some submodule \( T \), whence \( H_\tau(M) = T \). Moreover,

\[
\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(H_\tau(M)) = \text{Soc}(N) \oplus \text{Soc}(T).
\]

In fact

\[
\text{Soc}(N) \cap \text{Soc}(T) \subseteq N \cap T = N \cap (H_\tau(M) \cap T) = (N \cap H_\tau(M)) \cap T = H_\tau(N) \cap T = 0
\]
because \( N \) is isotype in \( M \). Consequently, there is a valuated direct sum

\[
\text{Soc}(H_\tau(M)) = H_\tau(N) \oplus \text{Soc}(T).
\]

Even more,

\[
\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T)
\]
is a valuated direct sum, where \( T \) is a valuated submodule of \( H_\tau(M) \) with \( \text{Soc}(T) = \text{Soc}(H_\tau(M)) \). Indeed, if \( x \in \text{Soc}(M) \) then

\[
x \in \text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T).
\]

Since \( \text{Soc}(T) = \text{Soc}(H_\tau(M)) \) and \( N \) is \( h \)-pure in \( M \). It easily follows that

\[
x \in N + H_\tau(M) + M = N \oplus T \oplus +M
\]
because \( H_\tau(M) = H_\tau(K) \oplus T \), and by induction the desired decomposition now follows.

If \( y \in \text{Soc}(N) \) and \( z \in \text{Soc}(T) \) then

\[
H_M(y + z) = \min\{H_M(y), H_M(z)\}
\]
since either

\[
H_M(y) < \alpha \leq H_M(z) \quad \text{or} \quad H_\tau(M) = H_\tau(N) \oplus T
\]
when \( H_M(y) \geq \alpha \). Therefore, \( H_\tau(M) \) is summable if and only if \( T \) has this property. Next, observe that

\[
N \cong N/\{0\} = N/H_\tau(N) = N/(N \cap H_\tau(M)) \cong (N + H_\tau(M))/H_\tau(M),
\]
where the last quotient is obviously isotype in \( M/H_\tau(M) \), and thus it is a direct sum of countably generated modules as well. It follows that \( N \) is a direct sum of countably generated modules. Furthermore, both \( T \) and \( N \) are summable. But \( \text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T) \) is a valuated direct sum and from this, our assertion follows directly by the definition of summability.
We close the paper with a problem as follows:

**Problem 1.** Does it follow that if both $H_\beta(M)$ and $M/H_\beta(M)$ are $\sigma$-summable modules (see [14]) for some ordinal $\beta$, then $M$ is $\sigma$-summable?

For summable modules we refer to [7]. Notice also that it can be obtained some results in this aspect under certain limitations on $\beta$ which depends on $n < \omega$.

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