Communications in Mathematics 27 (2019) 13–26
DOI: 10.2478/cm-2019-0002

α-modules and generalized submodules

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Abstract. A QTAG-module M is an α-module, where α is a limit ordinal, if $M/H_\beta(M)$ is totally projective for every ordinal $\beta < \alpha$. In the present paper α-modules are studied with the help of α-pure submodules, α-basic submodules, and α-large submodules. It is found that an α-closed α-module is an α-injective. For any ordinal $\omega \leq \alpha \leq \omega_1$ we prove that an α-large submodule L of an ω_1-module M is summable if and only if M is summable.

1 Introduction and Preliminary Terminology

Let R be any ring. A module M in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length, it is called a uniserial module. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module, and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

Modules are the natural generalizations of abelian groups. The results for abelian groups can be generalized for modules after imposing some conditions on modules/rings. In 1976 Singh [15] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

1. Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

2. Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f\colon W \to V$ can be

2010 MSC: 16K20
Key words: α-modules, α-pure submodules, α-basic submodules, α-large submodules.
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extended to a homomorphism $g\colon U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

Later on Benabdallah, Singh, Khan etc. contributed a lot to the study of TAG-modules [2], [17]. In 1987 Singh made an improvement and studied the modules satisfying only the condition 1 and called them QTAG-modules. The study of QTAG-modules and their structure began with work of Singh in [16]. This work, executed by many authors, clearly parallels the earlier work on torsion abelian groups. They studied different notions and structures on QTAG-modules and developed the theory of these modules by introducing different notions and characterizing different submodules of QTAG-modules. Yet there is much to explore.

Throughout this paper, all the rings are associative with unity $(1 \neq 0)$ and modules M are unital QTAG-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and

$$H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \,\middle|\, y \in M,\ x \in yR \text{ and } y \text{ uniform} \right\}$$

are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$. The module M is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded, if there exists an integer n such that $H_M(x) \leq n$ for every uniform element $x \in M$.

The sum of all simple submodules of M is called the socle of M, denoted by $\operatorname{Soc}(M)$ and a submodule S of $\operatorname{Soc}(M)$ is called a subsocle of M. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th} Ulm invariant of M and it is equal to $g\Big(\operatorname{Soc}\big(H_\alpha(M)\big)/\operatorname{Soc}\big(H_{\alpha+1}(M)\big)\Big)$.

A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For an ordinal α, a submodule $N \subseteq M$ is an α-high submodule of M if N is maximal among the submodules of M that intersect $H_\alpha(M)$ trivially.

For an ordinal α, a submodule N of M is said to be an α-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \alpha$ and a submodule N of M is said to be isotype in M, if it is α-pure for every ordinal α [13]. For a submodule $N \subseteq M$, the valuation of N induced by height in M is defined by $v(x) = H_M(x)$, the height of x in M, for all $x \in N$ and $N = K \oplus L$ is a valuated direct sum if $v(k + \ell) = \min\{v(k), v(\ell)\}$ for all $k \in K$ and $\ell \in L$ [5].

A submodule $B \subseteq M$ is a basic submodule [9] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A fully invariant submodule $L \subseteq M$ is large [1], if $L + B = M$, for every basic submodule B in M.

Imitating [11], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as

$\overline{N} = \bigcap_{k=0}^{\infty}(N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$.

An h-reduced QTAG-module M is summable [14] if $\mathrm{Soc}(M) = \bigoplus_{\beta<\alpha} S_\beta$, where S_β is the set of all elements of $H_\beta(M)$ which are not in $H_{\beta+1}(M)$, where α is the length of M. Moreover, M is called totally projective [10], if

$$H_\alpha(\mathrm{Ext}(M/H_\alpha(M), M')) = 0$$

for all ordinal α and QTAG-modules M'.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for QTAG-modules [13]. Our notations and terminology generally agree with those in [3] and [4].

2 α-modules and α-basic submodules

Mehdi et al. [12] defined α-modules and introduced some new concepts for these modules. The same type of study was continued in [6] and a number of results have been obtained in terms of generalized submodules. Here we also continue the similar study of α-modules that depend on the notions of summability, purity, basic submodules, projectivity and injectivity. For facilitating the exposition and for the convenience of the readers, we recall the definition of α-modules.

Definition 1. Let α denote the class of all QTAG-modules M such that $M/H_\beta(M)$ is totally projective for all ordinals $\beta < \alpha$, a limit ordinal. These modules are called α-modules.

To develop the study, we need to prove some results, and we start with the following.

Proposition 1. *If N is an α-pure submodule of an α-module M, then N is itself an α-module.*

Proof. We actually only need that $N \cap H_\gamma(M) = H_\gamma(N)$ for all $\gamma < \alpha$. For then it is a simple calculation to show that $N + H_\beta(M)/H_\beta(M)$ is isotype in $M/H_\beta(M)$ for each $\beta < \alpha$. And therefore, $N + H_\beta(M)/H_\beta(M) \cong N/H_\beta(N)$ is totally projective for all $\beta < \alpha$. □

As generalized the notion of a basic submodule in [12], by defining B to be an α-basic submodule of an α-module M if B is totally projective of length at most α, B is α-pure submodule of M, and M/B is h-divisible.

In order to establish the existence of α-basic submodules we require the following notion for technical convenience.

Definition 2. Let α be a limit ordinal and M a QTAG-module. An α-high tower of M is a well-ordered ascending chain $\{M_\beta\}_{\beta<\alpha}$ of submodules of M such that, for each β, M_β is a β-high submodule of M.

Now we need to prove the following lemma.

Lemma 1. *Let α be a limit ordinal and $\{M_\beta\}_{\beta<\alpha}$ an α-high tower of a QTAG--module M. If each M_β is summable, then $N = \bigcup_{\beta<\alpha} M_\beta$ is summable.*

Proof. As α is a limit ordinal, we may choose a strictly increasing sequence $\beta_1 < \beta_2 < \cdots < \beta_n < \ldots$ of ordinals having α as its limit. Then $N = \bigcup_{n<\omega} M_{\beta_n}$. Set $T_0 = \mathrm{Soc}(M_{\beta_1})$ and, for $n > 1$, let T_n be such that

$$\mathrm{Soc}(H_{\beta_n}(M)) = T_n \oplus \mathrm{Soc}(H_{\beta_{n+1}}(M))$$

with $T_n \subseteq M_{\beta_{n+1}}$. Then we have a direct-sum decomposition $\mathrm{Soc}(N) = \bigoplus_{n<\omega} T_n$ which is normal in the sense that

$$H_M(t_1 + \cdots + t_n) = \min[H_M(t_1), \ldots, H_M(t_n)]$$

provided $t_i \in T_i$ for $i = 1, \ldots, n$. Now each M_β is isotype, summable, and of countable length. Therefore, each subsocle of M_β is a summable subsocle of M. In particular, each T_n is a summable subsocle of M. Since the decomposition $\mathrm{Soc}(N) = \bigoplus_{n<\omega} T_n$ is normal, it follows that $\mathrm{Soc}(N)$ is a summable subsocle of M. Since each M_β is isotype, N is itself an isotype submodule of M and consequently N is summable. □

We continue the study with the following corollary.

Corollary 1. *Let α be a limit ordinal and $\{M_\beta\}_{\beta<\alpha}$ an α-high tower of a QTAG-module M, where each M_β is totally projective, then $N = \bigcup_{\beta<\alpha} M_\beta$ is totally projective of length at most α.*

Proof. As noted above, N is an isotype submodule of M and clearly N has a length at most α. Thus M_β is also a β-high submodule of N for each $\beta < \alpha$. Since N is summable by Lemma 1 implies that N is totally projective. □

Now we prove the following.

Theorem 1. *Let M be a QTAG-module. Then M contains an α-basic submodule if and only if M is an α-module.*

Proof. If B is an α-pure submodule of M and if M/B is h-divisible, then it follows that $M/H_\beta(M) \cong B/H_\beta(B)$ for all $\beta < \alpha$. Consequently, only α-modules can have α-basic submodules (see [12]). Suppose now that M is an α-module and select an α-high tower $\{M_\beta\}_{\beta<\alpha}$. Now

$$M_\beta \cong M_\beta + H_\beta(M)/H_\beta(M)\,,$$

and since M_β is isotype in M, $M_\beta + H_\beta(M)/H_\beta(M)$ is isotype in $M/H_\beta(M)$. By Corollary 1, $B = \bigcup_{\beta<\alpha} M_\beta$ is totally projective. It is easily seen that

$$\mathrm{Soc}(M) \subseteq \mathrm{Soc}(B) + H_\beta(M)$$

for each $\beta < \alpha$, and therefore B is α-pure in M. Moreover, $B \cap H_1(M) = H_1(B)$ and

$$\mathrm{Soc}(M) \subseteq \mathrm{Soc}(B) + H_\beta(M)$$

for $\beta < \omega$ imply that M/B is h-divisible. Thus, B is the required α-basic submodule of M. □

Lemma 2. *Suppose N is an isotype submodule of a QTAG-module M and that $\{N_\beta\}_{\beta<\alpha}$ is an α-high tower of N, then there exists an α-high tower $\{M_\beta\}_{\beta<\alpha}$ of M such that, for each β, $N_\beta \subseteq M_\beta$ and $N_\beta = N \cap M_\beta$.*

Proof. Let us first note that $N_\beta = N \cap M_\beta$ is a consequence of $N_\beta \subseteq M_\beta$. Indeed, $N_\beta \subseteq M_\beta$ implies $N_\beta \subseteq N \cap M_\beta$ and

$$(N \cap M_\beta) \cap H_\beta(N) = (N \cap M_\beta) \cap H_\beta(M) = 0\,.$$

The maximality of a β-high submodule then yields the equality. Assume now that $\beta < \alpha$ and that for each $\gamma < \beta$ we have a γ-high submodule M_γ of M such that $N_\gamma \subseteq M_\gamma$ and $M_\eta \subseteq M_\gamma$ for all $\eta < \gamma$. In order to be able to choose the desired M_β, it suffices to show that

$$(N_\beta + \bigcup_{\gamma<\beta} M_\gamma) \cap \mathrm{Soc}(H_\beta(M)) = 0\,.$$

Suppose $x + y \in \mathrm{Soc}(H_\beta(M))$ where $x \in N_\beta$ and $y \in M_\gamma$ for some $\gamma < \beta$. Then

$$H(x') = -H(y') \in H_1(M) \cap N \cap M_\gamma = H_1(M) \cap N_\gamma = H_1(N_\gamma)\,,$$

where $d\big(\frac{xR}{x'R}\big) = d\big(\frac{yR}{y'R}\big) = 1$, and hence there is $u \in N_\gamma$ such that

$$x - u \in \mathrm{Soc}(N) = \mathrm{Soc}(N_\gamma) \oplus \mathrm{Soc}(H_\gamma(N))\,.$$

Thus we can write $x = u + v + z$ where $v \in \mathrm{Soc}(N_\gamma)$ and $z \in \mathrm{Soc}(H_\gamma(N))$. Then

$$u + v + y = x + y - z \in H_\gamma(M) \cap M_\gamma = 0 \quad \text{and} \quad x + y = z \in N\,.$$

Therefore $y \in N \cap M_\gamma = N_\gamma \subseteq N_\beta$ and, consequently,

$$x + y \in N_\beta \cap H_\beta(M) = N_\beta \cap H_\beta(N) = 0$$

as desired. □

Lemma 3. *Let M be a totally projective QTAG-module such that $M = \bigcup_{\beta<\alpha} M_\beta$ where $\{M_\beta\}_{\beta<\alpha}$ is an α-high tower. If N is an α-pure submodule of M such that for each β, $N \cap M_\beta$ is a β-high submodule of N, then N is a direct summand of M.*

Proof. We need only show that M/N is totally projective having length at most α. Since $N \cap M_\beta$ is $(\beta+1)$-pure in N and N is α-pure in M, $N \cap M_\beta$ is $(\beta+1)$-pure in M and, a fortiori, $(\beta+1)$-pure in M_β. Since M_β is totally projective, M_β is

β-projective. Therefore, there is direct decomposition $M_\beta = (N \cap M_\beta) \oplus K_\beta$ for each $\beta < \alpha$. Now

$$M/N = \bigcup_{\beta<\alpha} M_\beta + N/N \quad \text{and} \quad M_\beta + N/N \cong M_\beta/(M_\beta \cap N) \cong K_\beta$$

is totally projective for each β. By Corollary 1, it is enough to show that $M_\beta + N/N$ is a β-high submodule of M/N whenever $\omega \leq \beta < \alpha$. Since N is α-pure in M, we have

$$\mathrm{Soc}(H_\beta(M/N)) = \mathrm{Soc}(H_\beta(M)) + N/N$$

for $\beta < \alpha$ and it then easily follows that

$$\mathrm{Soc}(M/N) = \mathrm{Soc}(M_\beta + N/N) \oplus \mathrm{Soc}(H_\beta(M/N))\,.$$

Because of this direct decomposition, it is enough to show that $M_\beta + N/N$ is an h-pure submodule of M/N for $\beta \geq \omega$.

Now

$$\begin{aligned}
\mathrm{Soc}(M_\beta + N) &= \mathrm{Soc}(K_\beta \oplus N) \\
&= \mathrm{Soc}(K_\beta) \oplus \mathrm{Soc}(N) \\
&= \mathrm{Soc}(K_\beta) \oplus \mathrm{Soc}(N \cap M_\beta) \oplus \mathrm{Soc}(H_\beta(N)) \\
&= \mathrm{Soc}(M_\beta) \oplus \mathrm{Soc}(H_\beta(N)).
\end{aligned}$$

If $\beta \geq \omega$ and if $x \in \mathrm{Soc}(M_\beta + N)$, then we can write $x = y + z$ where $y \in \mathrm{Soc}(M_\beta)$ and $z \in \mathrm{Soc}(H_\beta(N)) \subseteq H_\omega(N)$. If x has finite height in M, then this height is just the height of y in M (= height of y in M_β) and thus just the height of $x = y + z$ in $M_\beta + N$. On the other hand, if x has infinite height in M, then y has infinite height in M_β and $x = y + z$ has infinite height in $M_\beta + N$, it follows that $M_\beta + N$ is an h-pure submodule of M. Thus $M_\beta + N/N$ is h-pure in M/N. □

Proposition 2. *Let N be an α-pure submodule of an α-module M such that N is totally projective of length at most α. Then there exists a submodule K of M such that $N \oplus K$ is an α-basic submodule of M.*

Proof. Since N is totally projective of length $\leq \alpha$, N is the union of an α-high tower $\{N_\beta\}_{\beta<\alpha}$ of itself. By Lemma 2, there exists an α-high tower $\{M_\beta\}_{\beta<\alpha}$ of M such that $N_\beta = N \cap M_\beta$ for each β. Let $B = \bigcup_{\beta<\alpha} M_\beta$. By the proof of Theorem 1, B is an α-basic submodule of M. But $\{M_\beta\}_{\beta<\alpha}$ is also an α-high tower of B, and by Lemma 3 we have the required direct decomposition $B = N \oplus K$. □

Now we prove the following result.

Theorem 2. *If N is an α-pure submodule of an α-module M, then M/N is an α-module.*

Proof. Let B be an α-basic submodule of N and choose K such that $B \oplus K$ is an α-basic submodule of M. Now if $x \in \mathrm{Soc}(N \cap K)$, we can write for each $\beta < \alpha$, $x = y_\beta + z_\beta$, where $y_\beta \in \mathrm{Soc}(N)$ and $z_\beta \in H_\beta(N)$. Thus

$$-y_\beta + x \in H_\beta(B \oplus K) = H_\beta(B) \oplus H_\beta(K)$$

and

$$x \in \bigcap_{\beta<\alpha} H_\beta(K) = H_\alpha(K) = 0\,.$$

We then have a direct decomposition $N \oplus K$. If $H_1(a') \in N \oplus K$, then $H_1(a') = y + H_1(b') + c$, where $d\bigl(\frac{aR}{a'R}\bigr) = d\bigl(\frac{bR}{b'R}\bigr) = 1$, $y \in B$, $b \in N$ and $c \in K$. Since

$$H_1(M) \cap (B \oplus K) = H_1(B \oplus K)\,,$$

we conclude that

$$H_1(M) \cap (N \oplus K) = H_1(N \oplus K)\,.$$

Now

$$\mathrm{Soc}(M) \subseteq \mathrm{Soc}(B \oplus K) + H_\beta(M) \subseteq \mathrm{Soc}(N \oplus K) + H_\beta(M)$$

for all $\beta < \alpha$, and therefore $N \oplus K$ is an α-pure submodule of M. Consequently, $N \oplus K/N$ is α-pure in M/N. Also

$$N \oplus K/N \cong K$$

and

$$(M/N)/(N \oplus K/N) \cong (M/B \oplus K)/[(N \oplus K)/(B \oplus K)]$$

is h-divisible. We have constructed an α-basic submodule of M/N and we conclude that M/N is indeed an α-module. □

As a consequence of the above theorem, we have the following striking analog of a familiar property of h-pure submodules.

Corollary 2. *Let N be a submodule of an α-module M. Then N is an α-pure submodule of M if and only if $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$ for all $\beta < \alpha$.*

Proof. $N + H_\beta(M)/H_\beta(M)$ being a direct summand of $M/H_\beta(M)$ implies that $N + H_\beta(M)/H_\beta(M)$ is β-pure in $M/H_\beta(M)$, which is equivalent to N being β-pure in M. Since α is a limit ordinal, N is α-pure in M if and only if N is β-pure in M for all $\beta < \alpha$.

Conversely, assume that N is α-pure in M. Then M/N is an α-module and therefore, for $\beta < \alpha$,

$$\begin{aligned}(M/N)/H_\beta(M/N) &= (M/N)/(H_\beta(M) + N/N)\\ &\cong (M/H_\beta(M))/(N + H_\beta(M)/H_\beta(M))\end{aligned}$$

is totally projective of length at most β. Since $N + H_\beta(M)/H_\beta(M)$ is β-pure in $M/H_\beta(M)$, $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$. □

Proposition 3. *If N is an α-pure submodule of an α-module M, and if $H_\beta(N)$ is a direct summand of $H_\beta(M)$ for some $\beta < \alpha$, then N is a direct summand of M.*

Proof. Assuming the conditions of the Theorem 2, we have for some $\beta < \alpha$:

(i) $(M/N)/H_\beta(M/N)$ is totally projective;

(ii) $N \cap H_\beta(M) = H_\beta(N)$;

(iii) $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$; and

(iv) $H_\beta(M) = H_\beta(N) \oplus K$.

It follows that $M = N \oplus L$ where $L \supseteq K$. □

As a corollary, we have the following generalization of the well-known fact that bounded h-pure submodules are direct summands.

Corollary 3. *If N is an α-pure submodule of an α-module M and if $H_\beta(N) = 0$ for some $\beta < \alpha$, then N is a direct summand of M.*

As defined in [10], a QTAG-module M is fully transitive if for every pair of uniform elements $x,\ y \in M,\ H_M(x_i) \leq H_M(y_i)$ for all $i \geq 0$ implies that there exists an endomorphism of M that maps x onto y. Here $d\big(\frac{xR}{x_iR}\big) = d\big(\frac{yR}{y_iR}\big) = i$.

The next corollary tells us that α-modules of length α are fully transitive (see [6]). This, of course, is merely a reflection of the fact that modules of length $\leq \alpha$ behave in the α context exactly as modules without elements of infinite height in the classical situations.

Corollary 4. *If M is an α-module of length α, then every finite subset of M is contained in a countably generated direct summand.*

Proof. Let S be a finite subset of M. Then $S \subseteq T$ for some countably generated, α-pure submodule T of M. We may assume that T has length α. Then T is a direct sum of modules of length less than α. Consequently, T is contained in a direct summand K of T having length less than α. By the preceding corollary, K is a direct summand of M. □

For a limit ordinal α, an α-module M is an α-projective if

$$H_\alpha(\mathrm{Ext}(M, M')) = 0$$

for all α-modules M', that is, there exists a submodule N bounded by α such that M/N is totally projective, and an α-module M is an α-injective if

$$H_\alpha(\mathrm{Ext}(M', M)) = 0$$

for all α-modules M', that is, it is a direct summand of every α-module in which it occurs as an α-pure submodule.

To characterize the α-injective modules we must generalize the notion of a closed module. Mimicking [12], for any QTAG-module M, the submodules $\{H_k(M)\}_k$, $k = 0, 1, 2, \ldots, \infty$ from a neighborhood system of zero, giving rise to h-topology. If k is replaced by an arbitrary limit ordinal less than or equal to α, then h-topology may be extended to α-topology, and all the definitions and results which hold for h-topology may be extended for α-topology. In α-topology, for any submodule N of M, the closure of N as $\bigcap_{\beta<\alpha}(N + H_\beta(M))$ denoted by $\overline{N}$.

Definition 3. We call a QTAG-module an α-closed module if it is the maximal closed submodule of its closure in the α-topology.

With the help of the above discussion, we are able to infer the following.

Proposition 4. *Let M be an α-closed α-module. Then M is an α-injective.*

Proof. We first show that $H_\alpha(\mathrm{Ext}(T, M)) = 0$ for all α-modules T. Assume that M is an α-pure submodule of M' with $M'/M \cong T$ for all α-modules M'. Since α is a limit ordinal, it follows that $M' = H_\beta(M') + M$ for all $\beta < \alpha$. Therefore, if $y \in M'$, we can find for each $\beta < \alpha$ a $x_\beta \in M$ such that $y - x_\beta \in H_\beta(M')$. Moreover, we can assume that the exponent of x_β does not exceed that of y. Indeed, if y has exponent n, then

$$H_n(x'_\beta) \in H_{\beta+n}(M') \cap M = H_{\beta+n}(M)\,,$$

where $d\big(\frac{x_\beta R}{x'_\beta R}\big) = n$ and $H_n(x'_\beta) = H_n(z'_\beta)$, where $d\big(\frac{x_\beta R}{x'_\beta R}\big) = d\big(\frac{z_\beta R}{z'_\beta R}\big) = n$ for some $z_\beta \in H_\beta(M)$. Then $\overline{x}_\beta = x_\beta - z_\beta$ has an exponent at most n and $y - \overline{x}_\beta \in H_\beta(M')$. But $\{x_\beta : \beta < \alpha\}$ is a chain in M with elements uniformly bounded in exponent and, therefore, converges to some $x \in M$. Hence

$$y - x \in \bigcap_{\beta<\alpha} H_\beta(M') = H_\alpha(M')\,.$$

We conclude that $M' = M \oplus H_\alpha(M')$.

Now let M' be an arbitrary α-module and let B be an α-basic submodule of M'. We then have the exact sequence

$$H_\alpha(\mathrm{Ext}(M'/B, M)) \longrightarrow H_\alpha(\mathrm{Ext}(M', M)) \longrightarrow H_\alpha(\mathrm{Ext}(B, M)).$$

The left-hand term of the above sequence vanishes since M'/B is isomorphic to a direct sum of copies of T and the right-hand term vanishes since B is an α-projective. Thus, $H_\alpha(\mathrm{Ext}(M', M)) = 0$ and we conclude that M is an α-injective.

□

We can now show that there are enough α-injective modules and that an α-injective module is the sum of an α-closed module and an h-divisible module.

Theorem 3. *Let M be an α-module. Then M is an α-pure submodule of an α-injective module and M is an α-injective module if and only if M is the direct sum of an h-divisible module and an α-closed α-module.*

Proof. It is evident from Proposition 4 that the direct sum of an h-divisible module and an α-closed α-module is necessarily an α-injective. Next, we need the observation that every α-module M of length at most α can be imbedded as an α-pure submodule of an α-closed module $T_M(\alpha)$ such that $T_M(\alpha)/M$ is h-divisible. Indeed, $T_M(\alpha)$ may be taken as the maximal closed submodule of the closure of M

in the α-topology. It follows, by the same reasoning as in the proof of Theorem 1, that

$$T_M(\alpha)/H_\beta(T_M(\alpha)) \cong M/H_\beta(M)$$

for all $\beta < \alpha$, and therefore that $T_M(\alpha)$ is an α-module.

Now let M be an arbitrary α-module. Let D be a minimal h-divisible module containing $H_\alpha(M)$. Take P to be the amalgamated sum of M and D over $H_\alpha(M)$. Then $P = M' \oplus D$ where $M' \cong M/H_\alpha(M)$ and $M' \cap M$ is an α-high submodule of M. Also, P/M is h-divisible and

$$\mathrm{Soc}(P) \subseteq \mathrm{Soc}(M) + H_\beta(P)$$

for all $\beta < \alpha$. It follows that M is an α-pure submodule of P. By the transitivity of α-purity, M is an α-pure in the α-injective $T_{M'}(\alpha) \oplus D$.

Finally, assume that M is itself an α-injective and that we have it imbedded, as above, as an α-pure submodule of $\overline{P} = T_{M'}(\alpha) \oplus D$. Since M is an α-injective, $\overline{P} = M \oplus Q$ where $Q \cong \overline{P}/M$ is obviously h-divisible, since both P/M and $\overline{P}/P$ are h-divisible. But then $Q \subseteq D$, and since $\mathrm{Soc}(D) \subseteq H_\alpha(M)$, we conclude that $Q = 0$ and $M = T_{M'}(\alpha) \oplus D$. □

Now we are in a position to prove the following result.

Theorem 4. *If M and M' are α-closed α-modules with the same Ulm invariants, then $M \cong M'$.*

Proof. Take B and B' to be α-basic submodules of M and M', respectively. It is easily seen that B and B' have the same Ulm invariants as M and M'. Therefore, there is an isomorphism f of B onto B'. Since B is an α-pure submodule of M, we have the exact sequence

$$\mathrm{Hom}(M, M') \to \mathrm{Hom}(B, M') \to H_\alpha(\mathrm{Ext}(M/B, M')) = 0$$

Thus, there is a homomorphism $f' \colon M \to M'$ that extends f. Let $x \in \mathrm{Ker} f'$ and assume that $x \neq 0$. Then x has some height $\beta < \alpha$ and we can write $x = y + z$ where $y \in B$ and $z \in H_{\beta+1}(M)$. But then x has height β and $f(y) = f'(y) = -f'(z)$ has height at least $\beta + 1$. This, however, is a contradiction, since f is an isomorphism of B onto B' and B' is an isotype submodule of M'. We conclude that $\mathrm{Ker} f' = 0$. Then

$$f'(M)/B' = f'(M)/f'(B) \cong M/B$$

is h-divisible. Hence $f'(M)/B'$ is a direct summand of M'/B', and since B' is an α-pure submodule of M', it follows that $f'(M)$ is an α-pure submodule of M'. Since $f'(M) \cong M$ is an α-injective, we have a direct decomposition $M' = f'(M)) \oplus L$ where $L \cong M'/f'(M)$ is h-divisible. But M' is h-reduced and therefore $L = 0$ and $f'(M) = M'$, that is, f' is an isomorphism of M onto M'. □

3 α-large submodules of summable modules

If F is a fully invariant submodule of the α-module M, then F is called an α-large submodule of M if $M = B + F$ for all α-basic submodules B of M. This generalization of the concept of large submodule is studied in [12]. It is well-known that $H_\beta(M)$ is always an α-large submodule of M provided that $\beta < \alpha$ as well as $H_\beta(M) \subseteq L$ whenever L is an α-large submodule in M.

Likewise, It was proved that an h-reduced QTAG-module M of length $\geq \alpha$ contains a proper α-basic submodule B if and only if M is an α-module where α is cofinal with ω. Since ω_1, the first uncountable limit ordinal, is not cofinal with ω, some additional clarifications are necessary. In fact, B is an ω_1-basic submodule of M only when $B = 1$ or $B = M$, and so L is an ω_1-large submodule of M uniquely when $L = M$ and either $L = 1$ or $L \neq 1$ and it can take different forms; for instance $L = H_\beta(M)$ where $\beta <$ length of $M \leq \omega_1$.

In [12] it was seen that the properties of α-large submodules for $\alpha > \omega$ are not preserved in general by these of the QTAG-module and conversely; for instance the direct sum of countably generated modules. However, this is not the case for totally projective modules.

Theorem 5 ([6], [12]). *Let L be an α-large submodule of the QTAG-module M. Then L is totally projective if and only if M is totally projective.*

The main goal of this section is to strengthen the above assertion to a class of modules, called summable modules. It is evident that direct sum of countably generated modules are themselves summable. In [8] it was constructed a summable ω_1-module need not be a direct sum of countably generated modules. So, the investigation of the discussed above theme for α-large submodules of summable ω_1-modules will be of interest.

Now we have accumulated all the machinery necessary to prove the following.

Theorem 6. *Suppose that M is an ω_1-module with an α-large submodule L for some ordinal α such that $\omega \leq \alpha \leq \omega_1$. Then M is summable if and only if L is summable.*

Proof. " $\Rightarrow$ ". In virtue of [12] there is a countable limit ordinal $\tau \leq \alpha$ such that $H_\tau(M) = H_\omega(L)$. Moreover, $L/H_\omega(L) = L/H_\tau(M)$ is an α-large submodule of $M/H_\tau(M)$, where the latter quotient is totally projective by assumption. Therefore, Theorem 5 applies to deduce that $L/H_\omega(L)$ is totally projective, in fact, a direct sum of uniserial modules. That is why, some high submodule N of L is a direct sum of countably generated modules. Indeed, what suffices to show is that $N/H_\omega(N)$ is a direct sum uniserial modules because $H_\omega(N)$ is bounded. In order to do that, we observe that

$$(N + H_\omega(L))/H_\omega(L) \subseteq L/H_\omega(L)$$

is also a direct sum of uniserial modules as a submodule. But N is isotype in L, whence

$$(N + H_\omega(L))/H_\omega(L) \cong N/(N \cap H_\omega(L)) = N/H_\omega(L)$$

which substantiates our claim. On the other hand, M being summable yields that $H_\omega(L) = H_\tau(M)$ is summable, and we are done.

" $\Leftarrow$ ". Same as above, $H_\tau(M) = H_\omega(L)$ for some countable limit ordinal $\tau \leq \alpha$. But L being summable implies that $H_\omega(L) = H_\tau(M)$ is summable. Likewise, $M/H_\tau(M)$ is totally projective of countable length, hence a direct sum of countably generated modules. Let N be a τ-high submodule of M. Since $H_\tau(N)$ is high in $H_\tau(M)$ one may write $H_\tau(M) = H_\tau(N) \oplus T$ for some submodule T, whence $H_\tau(M) = T$. Moreover,

$$\mathrm{Soc}(M) = \mathrm{Soc}(N) \oplus \mathrm{Soc}(H_\tau(M)) = \mathrm{Soc}(N) \oplus \mathrm{Soc}(T)\,.$$

In fact

$$\begin{aligned}\mathrm{Soc}(N) \cap \mathrm{Soc}(T) \subseteq N \cap T = N \cap (H_\tau(M) \cap T)& \\ = (N \cap H_\tau(M)) \cap T = H_\tau(N) \cap T = 0&\end{aligned}$$

because N is isotype in M. Consequently, there is a valuated direct sum

$$\mathrm{Soc}(H_\tau(M)) = H_\tau(N) \oplus \mathrm{Soc}(T)\,.$$

Even more,

$$\mathrm{Soc}(M) = \mathrm{Soc}(N) \oplus \mathrm{Soc}(T)$$

is a valuated direct sum, where T is a valuated submodule of $H_\tau(M)$ with $\mathrm{Soc}(T) = \mathrm{Soc}(H_\tau(M))$. Indeed, if $x \in \mathrm{Soc}(M)$ then

$$x \in \mathrm{Soc}(M) = \mathrm{Soc}(N) \oplus \mathrm{Soc}(T)\,.$$

Since $\mathrm{Soc}(T) = \mathrm{Soc}(H_\tau(M))$ and N is h-pure in M. It easily follows that

$$x \in N + H_\tau(M) + M = N \oplus T \oplus +M$$

because $H_\tau(M) = H_\tau(K) \oplus T$, and by induction the desired decomposition now follows.

If $y \in \mathrm{Soc}(N)$ and $z \in \mathrm{Soc}(T)$ then

$$H_M(y+z) = \min\{H_M(y), H_M(z)\}$$

since either

$$H_M(y) < \alpha \leq H_M(z) \quad \text{or} \quad H_\tau(M) = H_\tau(N) \oplus T$$

when $H_M(y) \geq \alpha$. Therefore, $H_\tau(M)$ is summable if and only if T has this property.

Next, observe that

$$N \cong N/\{0\} = N/H_\tau(N) = N/(N \cap H_\tau(M)) \cong (N + H_\tau(M))/H_\tau(M)\,,$$

where the last quotient is obviously isotype in $M/H_\tau(M)$, and thus it is a direct sum of countably generated modules as well. It follows that N is a direct sum of countably generated modules. Furthermore, both T and N are summable. But $\mathrm{Soc}(M) = \mathrm{Soc}(N) \oplus \mathrm{Soc}(T)$ is a valuated direct sum and from this, our assertion follows directly by the definition of summability. □

We close the paper with a problem as follows:

Problem 1. Does it follow that if both $H_\beta(M)$ and $M/H_\beta(M)$ are σ-summable modules (see [14]) for some ordinal β, then M is σ-summable?

For summable modules we refer to [7]. Notice also that it can be obtained some results in this aspect under certain limitations on β which depends on $n < \omega$.

Acknowledgement

The authors are extremely thankful to the referee for the expert suggestions which improved the presentation of the paper, and to the editor, for his valuable editorial work.

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Received: 2 May, 2018
Accepted for publication: 16 January, 2019
Communicated by: Eric Swartz