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Non-split supermanifolds associated with the cotangent bundle

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Abstract. Here, I study the problem of classification of non-split supermanifolds having as retract the split supermanifold (M, Ω), where Ω is the sheaf of holomorphic forms on a given complex manifold M of dimension > 1. I propose a general construction associating with any d-closed $(1,1)$-form ω on M a supermanifold with retract (M, Ω) which is non-split whenever the Dolbeault class of ω is non-zero. In particular, this gives a non-empty family of non-split supermanifolds for any flag manifold $M \neq \mathbb{CP}^1$. In the case where M is an irreducible compact Hermitian symmetric space, I get a complete classification of non-split supermanifolds with retract (M, Ω). For each of these supermanifolds, the 0- and 1-cohomology with values in the tangent sheaf are calculated. As an example, I study the Π-symmetric super-Grassmannians introduced by Yu. Manin.

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Editor's note The work on the problem of deformations of analytical supermanifolds, especially the study of non-splitness, was dormant for ca 40 years after the discovery of the first non-split example by Green. Apart from Onishchik and his students nobody studied this problem. The problem drew new attention of both mathematicians and theoretical physicists after Donagi and Witten showed that the moduli space of super Riemann surfaces is not split and how this fact affects working with modules of string theory, see [9*].

This work by Onishchik was preprinted in 1997 as Prépubl. Univ. Poitiers Départ de Math. N. 109. It was found very helpful several times since, see [52*] – [56*] and [5*].

A.L.Onishchik used to tell me that he understood the meaning of the "non-split" supermanifold having learned the papers by Green [17] and Vaintrob [50*], [51*]. Here, I updated the references; my additions are marked with a $*$. I also considerably edited English, but each time I read this text after a break I see something to be corrected, so I am afraid there still is something left. I also changed the outdated or *ad hoc* notation of several supergroups and superalgebras using the currently used notation and inserted a couple of clarifying parenthetical remarks (marked by *D.L.*).

For basics of supermanifold theory, I recommend [8*] and [29*] which still contain many results, notions and open problems not covered in other sources; see also comments in [34*, Section 4.8]. I also recommend the wonderful introduction into the theory of schemes and ringed spaces [31*], and the definition and calculation of curvature tensors of almost complex supermanifolds, and real-complex supermanifolds endowed with a non-integrable distribution (see [6*]), examples of such supermanifolds are all superstrings usually considered in the works of physicists and most of the super Grassmannians.

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In what follows, "I" means "Onishchik". *D.Leites.*

1 Introduction

One of the most important features of the theory of complex analytic supermanifolds is the existence of non-split supermanifolds. The simplest example is the superquadric $\mathcal{Q}^{1|2}$ in the projective superplane $\mathcal{CP}^{2|2}$, see Example 2.8 below; it is of dimension $1|2$ and has as its base the projective line $\mathbb{CP}^1$. This superquadric belongs to one of four series of homogeneous complex supermanifolds constructed by Yu. Manin [30] — the flag supermanifolds; as a rule, they are non-split.

With any supermanifold a split one, called its *retract*, is associated. In this paper, I study non-split supermanifolds with retract (M,Ω), where Ω is the sheaf of holomorphic forms on a complex manifold M. I present a construction assigning to any d-closed (1,1)-form ω on M a supermanifold with retract (M,Ω); this supermanifold is non-split whenever ω has a non-zero Dolbeault cohomology class. In particular, for any compact Kähler manifold M, we obtain a family of supermanifolds with retract (M,Ω) parametrized by $H^{1,1}(M,\mathbb{C})$, all members of which are non-split, except the one corresponding to 0. This family is non-empty, e.g., when M is a flag manifold.

The next problem is the classification of *all* non-split supermanifolds with retract (M,Ω), where M is a flag manifold. I solve it in the case where M is an irreducible Hermitian symmetric space. In this case, the family mentioned above contains precisely one non-split supermanifold. I prove that this is the only non-split supermanifold with retract (M,Ω) if one excludes the case of the Grassmannians $M=\mathrm{Gr}^n_s$, where $2\le s\le n-2$, while the non-split supermanifolds for such a Grassmannian form an 1-parameter family. The proof is based on certain general results concerning classification of supermanifolds with a given retract. We also calculate the 0- and 1-cohomology of the tangent sheaf for all the supermanifolds associated with the cotangent bundle over a compact irreducible Hermitian symmetric space.

The well known examples of supermanifolds studied here are the supermanifolds of Π-symmetric flags that form one of Manin's series mentioned above. If M is a symmetric space, then $M=\mathrm{Gr}^n_s$, and we have the Π-symmetric super-Grassmannians $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$. As a corollary, we calculate the 0- and 1-cohomology of their tangent sheaf. Note that this special question initiated the study exposed in this paper. For the three other series of super-Grassmannians this calculation was performed in [37], [43], [44].

2 Generalities about superalgebras and supermanifolds

2.1 Algebraic background To fix the notation, we give here some definitions.

Let $\mathbb{Z}_2$ denote not the ring of 2-adic integers but $\mathbb{Z}/2\mathbb{Z}=\{\bar 0,\bar 1\}$. A *vector superspace* is any $\mathbb{Z}_2$-graded vector space V. In this paper, the ground field is the field of complex numbers $\mathbb{C}$, By definition, we have

$$V=V_{\bar 0}\oplus V_{\bar 1},$$

where $V_{\bar 0}$, $V_{\bar 1}$ are vector subspaces called the *even part* and the *odd part* of V, respectively. The *non-zero* elements of these subspaces are said to be *even* or *odd*, respectively, and we define the *parity function* by setting for *non-zero* elements

$$p(x)=\begin{cases}\bar 0 \text{ if } x\in V_{\bar 0}\\ \bar 1 \text{ if } x\in V_{\bar 1}.\end{cases}$$

We write $\dim V=n|m$, where $\dim V_{\bar 0}=n,\ \dim V_{\bar 1}=m$; this is the *superdimension* of a vector superspace V. A standard example of a vector superspace of dimension $n|m$ is $\mathbb{C}^{n|m}=\mathbb{C}^n\oplus\Pi\mathbb{C}^m$, where Π is the change of parity functor.

A *superalgebra* is a $\mathbb{Z}_2$-graded algebra over $\mathbb{C}$, i.e., a vector superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ endowed with a multiplication $(a, b) \mapsto ab$ satisfying the following condition

$$A_i A_j \subset A_{i+j} \text{ for any } i, j \in \mathbb{Z}_2.$$

A *morphism* $f : A \longrightarrow B$ of superalgebras is, by definition, a *parity preserving* homomorphism of algebras, i.e., satisfying $f(A_i) \subset B_i$ for any $i \in \mathbb{Z}_2$. If A and B are superalgebras with units, we also assume that $f(1) = 1$.

Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a graded (i.e., a $\mathbb{Z}$-graded) algebra. This means that

$$A_i A_j \subset A_{i+j} \text{ for any } i, j \in \mathbb{Z}.$$

Setting

$$A_{\bar{0}} = \bigoplus_{i\in\mathbb{Z}} A_{2i},\ A_{\bar{1}} = \bigoplus_{i\in\mathbb{Z}} A_{2i+1},$$

we, clearly, endow A with a parity ($\mathbb{Z}_2$-grading), turning it into a superalgebra. In this case, we say that the $\mathbb{Z}$-grading and the $\mathbb{Z}_2$-grading in A are *compatible*.

2.1 Example (The exterior a.k.a. Grassmann algebra). Let E denote a complex vector space of dimension m and let $\bigwedge E$ be the exterior (or Grassmann) algebra over E. Then, we have the natural $\mathbb{Z}$-grading

$$\bigwedge E = \bigoplus_{p=0}^{n} \bigwedge^p E$$

making $\bigwedge E$ a graded algebra. Using the above procedure, we can regard $\bigwedge E$ as a superalgebra. Note that setting $p(x) = \bar{1}$ for any $x \in E$ we endow $\bigwedge E$ with a (natural) parity. Any basis $\xi_1, \ldots, \xi_m$ of E is a set of (odd) generators of $\bigwedge E$, and we often write

$$\bigwedge E = \bigwedge_{\mathbb{C}}(\xi_1, \ldots, \xi_m).$$

Many formulas of Linear Algebra are superized by means of the Sign Rule "**if something of parity a is moved past something of parity b, the sign $(-1)^{ab}$ accrues; formulas defined on homogeneous elements are extended to all elements via linearity**". Here are examples.

A superalgebra A is called *supercommutative* if

$$ba = (-1)^{p(a)p(b)} ab$$

for any even or odd $a, b \in A$. The associativity of A is meant in the usual sense. Clearly, $\bigwedge E$ from Example 2.1 is an associative supercommutative superalgebra with unit.

Let V and W be two vector superspaces. Then, the tensor product $U = V \otimes W$ becomes a superspace if we endow it with the following $\mathbb{Z}_2$-grading:

$$U_{\bar{0}} = V_{\bar{0}} \otimes W_{\bar{0}} \oplus V_{\bar{1}} \otimes W_{\bar{1}}, \quad U_{\bar{1}} = V_{\bar{0}} \otimes W_{\bar{1}} \oplus V_{\bar{1}} \otimes W_{\bar{0}},$$

For any two superalgebras A and B, let us endow the superspace $A \otimes B$ with the multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{p(b_1)p(a_2)}(a_1a_2) \otimes (b_1b_2), \quad a_i \in A,\ b_i \in B.$$

Then, $A \otimes B$ is a superalgebra (the *tensor product* of A and B). The tensor product of two associative (supercommutative) superalgebras is associative (respectively, supercommutative).

Let V and W be two vector superspaces. Then, the vector space $L(V, W)$ of all linear mappings $V \longrightarrow W$ is endowed with the following $\mathbb{Z}_2$-grading:

$$L(V, W)_k = \{f \in L(V, W) \mid f(V_i) \subset W_{i+k},\ i \in \mathbb{Z}_2\},\ k \in \mathbb{Z}_2.$$

Thus, a non-zero $f \in L(V, W)$ is even (odd) if it preserves (respectively, changes) the parity. For example, any morphism of superalgebras is, by definition, even.

Regarding $\mathbb{C}$ as $\mathbb{C}^{1|0}$, we get a natural $\mathbb{Z}_2$-grading in the dual vector space $V^* = L(V, \mathbb{C})$ of a superspace V. Clearly,

$$(V^*)_{\bar{0}} = \{f \in V^* \mid f(V_{\bar{1}}) = 0\} \text{ and } (V^*)_{\bar{1}} = \{f \in V^* \mid f(V_{\bar{0}}) = 0\}$$

are identified with $(V_{\bar{0}})^*$ and $(V_{\bar{1}})^*$, respectively. For another vector superspace W, the superspace $V^* \otimes W$ is identified with $L(V, W)$, as usual.

2.2 Example (Associative superalgebra of supermatrices). Let V be a vector superspace. Then, $L(V) = L(V, V)$, the associative algebra with unit of all linear transformations of V is a superalgebra if we endow it with the above $\mathbb{Z}_2$-grading.

A corresponding example can be constructed by means of matrices. Let $\mathbb{M}_{n+m}(\mathbb{C})$ denote the (associative, with unit) algebra of $(n+m) \times (n+m)$ matrices over $\mathbb{C}$. Endow this algebra with a $\mathbb{Z}_2$-grading in the following way. Write a matrix $X \in \mathbb{M}_{n+m}(\mathbb{C})$ in the form

$$X = \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix},$$

where X_{00} fills the first n rows and n columns. Then, $p(X) = \bar{0}$ whenever $X_{01} = 0,\ X_{10} = 0$, and $p(X) = \bar{1}$ whenever $X_{00} = 0,\ X_{11} = 0$. Clearly, this $\mathbb{Z}_2$-grading endows $\mathbb{M}_{n+m}(\mathbb{C})$ with a superalgebra structure. We denote it by $\mathbb{M}_{n|m}(\mathbb{C})$.

Let V be a vector superspace of dimension $n|m$. Choosing in V a basis

$$e_1, \ldots, e_n, f_1, \ldots, f_m \text{ with } e_i \in V_{\bar{0}},\ f_j \in V_{\bar{1}},$$

we get a natural isomorphism of superalgebras $L(V) \simeq \mathbb{M}_{n|m}(\mathbb{C})$.

Let $\mathfrak{g}$ be a superalgebra and let us agree to denote the multiplication in $\mathfrak{g}$ by $[-,-]$ and to call it the *bracket*. We say that $\mathfrak{g}$ is a *Lie superalgebra* if the following conditions are satisfied for any $x, y, z \in \mathfrak{g}$:

$$[y, x] = -(-1)^{p(x)p(y)}[x, y],$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]].$$

A graded algebra $\mathfrak{g}$ is called a *graded Lie superalgebra* if $\mathfrak{g}$ is a Lie superalgebra being provided with a $\mathbb{Z}_2$-grading (parity). Observe that these two gradings are not necessarily compatible.

2.3 Example (Lie superalgebra of supermatrices). As in the classical case, there is a functor $\mathfrak{L}$ converting associative superalgebras into Lie ones. If A is an associative superalgebra, then $\mathfrak{L}(A)$ is the vector superspace A endowed with the bracket

$$[a,b] = ab - (-1)^{p(a)p(b)}ba) \quad \text{for any } a,b \in A.$$

Let us note some special cases.

If A is supercommutative, then $\mathfrak{L}(A)$ is a Lie superalgebra with zero bracket called *commutative* Lie superalgebra.

For any vector superspace V, the superalgebra $L(V)$ from Example 2.1 gives the general linear Lie superalgebra $\mathfrak{gl}(V) = \mathfrak{L}(L(V))$.

By the same example, we get the Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C}) = \mathfrak{L}(\mathbb{M}_{n|m}(\mathbb{C}))$.

2.4 Example (Lie superalgebra of superderivations). Let A be an arbitrary superalgebra. A linear transformation $v \in \mathfrak{gl}(A)$ is called a *derivation* of A if

$$v(ab) = v(a)b + (-1)^{p(v)p(a)}av(b) \quad \text{for any } a,b \in A.$$

Denote

$$\mathfrak{der}\, A := (\mathfrak{der}\, A)_{\bar{0}} \oplus (\mathfrak{der}\, A)_{\bar{1}},$$

where $(\mathfrak{der}\, A)_i \subset \mathfrak{gl}(A)_i$, $i \in \mathbb{Z}_2$, is the vector space of even or odd derivations of A. One checks easily that $\mathfrak{der}\, A$ is a subalgebra of $\mathfrak{gl}(A)$ and, hence, a Lie superalgebra (the *superalgebra of derivations* of A).

Let $\mathfrak{g}$ be a Lie superalgebra. For any $x \in L$, define the *adjoint operator* $\mathrm{ad}_x \in L(\mathfrak{g})$ by setting

$$\mathrm{ad}_x\,(y) := [x,y] \quad \text{for any} \quad y \in \mathfrak{g}.$$

A straightforward verification shows that $\mathrm{ad}_x \in \mathfrak{der}\,\mathfrak{g}$ for any $x \in \mathfrak{g}$ and $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{der}\,\mathfrak{g}$ is a homomorphism of Lie superalgebras.

Clearly, $\mathfrak{g}_{\bar{0}}$ is a usual Lie algebra. If $x \in \mathfrak{g}_{\bar{0}}$, then ad_x is an even derivation. Restricting it onto $\mathfrak{g}_p$, where $p = \bar{0}, \bar{1}$, we get two linear representations ad_p of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}_p$, where $p = \bar{0}, \bar{1}$.

Similarly, let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a graded Lie superalgebra. Then, $\mathfrak{g}_0$ is a Lie algebra, and the restriction $\mathrm{ad}\,|_{\mathfrak{g}_0}$ determines a representation ad_p of $\mathfrak{g}_0$ in $\mathfrak{g}_p$ for any $p \in \mathbb{Z}$.

If A is an associative and supercommutative superalgebra, then $\mathfrak{der}\, A$ is naturally an A-module according to the rule

$$(au)(b) = au(b) \quad \text{for any} \quad u \in \mathfrak{der}\, A, \ a,b \in A.$$

The same definitions apply to sheaves of graded algebras on a topological space M. If, in particular, $\mathcal{A}$ is a sheaf of associative supercommutative graded algebras, then the sheaf

$\mathcal{D}er\mathcal{A}$ of derivations of $\mathcal{A}$ is defined, which is a sheaf of Lie superalgebras and a sheaf of $\mathcal{A}$-modules on M.

Let us consider the case where A is a graded algebra and regard it as a superalgebra with respect to the compatible $\mathbb{Z}_2$-grading. Then, $\mathfrak{der}\, A$ has a natural structure of the graded Lie superalgebra. More precisely, $\mathfrak{der}\, A = \sum_{p\in\mathbb{Z}}(\mathfrak{der}\, A)_p$, where

$$(\mathfrak{der}\, A)_p = \{\delta \in \mathfrak{der}\, A \mid \delta(A_q) \subset A_{q+p} \quad \text{for any } q \in \mathbb{Z}\}.$$

One can always define the *grading derivation* $\varepsilon \in (\mathfrak{der}\, A)_0$ given by the formula

$$\varepsilon(f) = pf \text{ for any} f \in A_p \text{ and } p \geq 0. \tag{2.1}$$

One easily checks that

$$[\varepsilon, v] = pv \text{ for any } v \in (\mathfrak{der}\, A)_p,\ p \in \mathbb{Z}. \tag{2.2}$$

2.5 Example (Vectorial Lie superalgebras). Consider a complex vector space E of dimension m and its corresponding the Grassmann algebra $A = \bigwedge E$ (see Example 2.1). Denote $W(E) = \mathfrak{der}\, A$. These Lie superalgebras constitute one of the "Cartan type" series of simple (for $m \geq 2$) finite-dimensional vectorial Lie superalgebras (see [24]).

We need the well known description of derivations from $W(E)$ in terms of multilinear forms. Any $u \in W(E)_p$ is determined by its restriction onto $E = A_1$, which can be an arbitrary linear mapping $E \longrightarrow A_{p+1} = \bigwedge^{p+1} E$. Thus, $W(E)_p$ is isomorphic, as a vector space, to $L(E, \bigwedge^{p+1} E)$, which can be identified with $\bigwedge^{p+1} E \otimes E^*$. Let us denote by $i(\varphi) \in W(E)_p$ the derivation corresponding to a linear mapping $\varphi \in \bigwedge^{p+1} E \otimes E^*$.

The elements of the latter vector space can be regarded as vector-valued $(p+1)$-forms on E^*, i.e., as anti-symmetric $(p+1)$-linear mappings $(E^*)^{p+1} \longrightarrow E^*$. Regarding A as the set of all anti-symmetric multilinear forms on E^*, we have

$$\begin{aligned} &i(\varphi)(a)(x_1, \ldots, x_{p+q}) \\ &\quad = \tfrac{1}{(p+1)!(q-1)!} \sum_{\alpha\in S_{p+q}} (\operatorname{sgn}\alpha) a(\varphi(x_{\alpha_1}, \ldots, x_{\alpha_{p+1}}), x_{\alpha_{p+2}}, \ldots, x_{\alpha_{p+q}}) \text{ for the } x_k \in E^*. \end{aligned}$$

Denote by ξ_j for $j = 1, \ldots, m$ a basis of E, and by ξ_j^* for $j = 1, \ldots, m$ the dual basis of E^*. Clearly, the derivations $\frac{\partial}{\partial\xi_j} = i(\xi_j^*) \in W(E)_{-1}$ for $j = 1, \ldots, m$, constitute a basis of the A-module $W(E)$. It follows that the derivations

$$\xi_{i_1} \ldots \xi_{i_{p+1}} \frac{\partial}{\partial\xi_j} \text{ for } i_1 < \ldots < i_{p+1} \text{ and } j = 1, \ldots, m,$$

constitute a basis of $W(E)_p$ over $\mathbb{C}$. In particular, we see that $W(E)_p$ is non-zero only for $-1 \leq p \leq m$.

We also write

$$i(\varphi)(a) = a \mathbin{\overline{\wedge}} \varphi \quad \text{for any } a \in A \text{ and } \varphi \in A \otimes E^*.$$

A similar operation can be defined for two vector-valued forms of arbitrary degrees. Namely, let $\varphi \in A_p \otimes E^*$ and $\psi \in A_q \otimes E^*$ be given. Regarding these tensors as E^*-valued p- and q-forms on E^*, we define the form $\varphi \bar{\wedge} \psi \in A_{p+q-1} \otimes E^*$ by the formula

$$\begin{aligned}(\varphi \bar{\wedge} \psi)(x_1, \ldots, x_{p+q-1}) \\ = \frac{1}{(p-1)!q!} \sum_{\alpha \in S_{p+q-1}} (\operatorname{sgn} \alpha) \varphi(\psi(x_{\alpha_1}, \ldots, x_{\alpha_q}), x_{\alpha_{q+1}}, \ldots, x_{\alpha_{p+q-1}})\end{aligned} \tag{2.3}$$

for any $x_k \in E^*$. This operation can be used to express the bracket in $W(E)$. More precisely, define the bilinear operation $\{-,-\}$ on $A \otimes E^*$ by setting

$$\{\varphi, \psi\} = \psi \bar{\wedge} \varphi - (-1)^{(p-1)(q-1)} \varphi \bar{\wedge} \psi \tag{2.4}$$

for any $\varphi \in A_p \otimes E^*$ and $\psi \in A_q \otimes E^*$. Then,

$$i(\{\varphi, \psi\}) = [i(\varphi), i(\psi)].$$

In what follows, we will use the linear mapping $j : \bigwedge^p E \to L(E, \bigwedge^{p+1} E)$ given by the formula

$$j(\psi)(u) = \psi u \quad \text{for any} \quad u \in E. \tag{2.5}$$

It is injective whenever $p < m$. Clearly,

$$i(j(\psi)) = \psi \varepsilon, \quad \psi \in \bigwedge^p E. \tag{2.6}$$

Regarding $L(E, \bigwedge^{p+1} E)$ as $\bigwedge^{p+1} E \otimes E^*$, we easily see that

$$j(\psi) = \sum_{k=1}^{m} (\psi \xi_k) \otimes \xi_k^*. \tag{2.7}$$

Finally, regarding elements of $L(E, \bigwedge^{p+1} E)$ as vector-valued $(p+1)$-forms on E^*, we obtain

$$j(\psi)(x_1, \ldots, x_{p+1}) = p! \sum_{k=1}^{m} (-1)^{k-1} \psi(x_1, \ldots, \hat{x}_k, \ldots, x_{p+1}) x_k, \quad \text{where} \quad x_l \in E^*. \tag{2.8}$$

On the other hand, there is the contraction mapping $c : \bigwedge^{p+1} E \otimes E^* \longrightarrow \bigwedge^p E$ given by the formula

$$c(\varphi)(x_1, \ldots, x_p) = \sum_{k=1}^{m} \varphi(\xi_k^*, x_1, \ldots, x_p)(\xi_k), \quad \text{where} \quad x_l \in E^*.$$

An easy calculation shows that $cj = p!(m-p) \operatorname{id}$. It follows that

$$\bigwedge^{p+1} E \otimes E^* = \operatorname{Im} j \oplus \operatorname{Ker} c \quad \text{whenever } p < m. \tag{2.9}$$

2.7 Complex supermanifolds The word “supermanifold” will mean the same as in [2], [3*], [27], but the complex-analytic version of the theory will be considered (see [30]). Let us begin with a more general notion of the ringed space.

A $\mathbb{Z}_2$-*graded ringed space* is a pair $(M, \mathcal{O})$, where M is a topological space and $\mathcal{O}$ is a sheaf of associative unital supercommutative superalgebras on M. A *morphism* between two $\mathbb{Z}_2$-graded ringed spaces $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a pair (f, f^*), where $f : M \to N$ is a continuous mapping and $f^* : \mathcal{O}_N \to \mathcal{O}_M$ a morphism of sheaves of superalgebras. In particular, if $F = (f, f^*)$ is an automorphism of a ringed space $(M, \mathcal{O})$, then we can consider the mapping $f_* = (f^*)^{-1}$ instead of f^*; this is an automorphism of the sheaf $\mathcal{O}$ over M. The automorphisms of $(M, \mathcal{O})$ form the group $\mathrm{Aut}(M, \mathcal{O})$.

2.6 Example (Complex-analytic supermanifolds). On the space $\mathbb{C}^n$, consider the sheaf

$$\mathcal{F}_{n|m} := \bigwedge(\xi_1, \dots, \xi_m) \otimes \mathcal{F}_n = \bigwedge_{\mathcal{F}_n}(\xi_1, \dots, \xi_m),$$

where $\mathcal{F}_n$ is the sheaf of germs of holomorphic functions on $\mathbb{C}^n$. Here we assume that the functions from $\mathcal{F}_n$ are even, while ξ_j are odd. A *superdomain* in $\mathcal{C}^{n|m}$ is, by definition, a $\mathbb{Z}_2$-graded ringed space of the form $(U, \mathcal{F}_{n|m})$, where U is an open subset of $\mathbb{C}^n$.

A *complex-analytic supermanifold* of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a superdomain in $\mathcal{C}^{n|m}$. Thus, if $(M, \mathcal{O})$ is a supermanifold, then for any point $x_0 \in M$ there exist a neighborhood U of x_0 in M and an isomorphism of the ringed space $(U, \mathcal{O}|_U)$ onto a superdomain $(\widetilde{U}, \mathcal{F}_{n|m})$ in $\mathcal{C}^{n|m}$ called a *chart* on U. Let $x_1, \dots, x_n$ denote the standard coordinates in $\mathbb{C}^n$. Identifying $(U, \mathcal{O})$ with the superdomain by means of the chart, we get the elements x_i for $i = 1, \dots, n$, and ξ_j for $j = 1, \dots, m$ of $\mathcal{O}(U)$ called the *local coordinates* on U.

Let U (resp. V) be two open subsets of M admitting two charts with local coordinates for $i = 1, \dots, n$, and ξ_j for $j = 1, \dots, m$ (resp. y_i for $i = 1, \dots, n$ and η_j for $j = 1, \dots, m$). Then, in $U \cap V$ we can write

$$\begin{aligned} y_i &= \varphi_i(x_1, \dots, x_n, \xi_1, \dots, \xi_m), \quad \text{where} \quad i = 1, \dots, n; \\ \eta_j &= \psi_j(x_1, \dots, x_n, \xi_1, \dots, \xi_m), \quad \text{where} \quad j = 1, \dots, m, \end{aligned} \tag{2.10}$$

where φ_i, ψ_j are, respectively, even and odd sections of $\mathcal{F}_{n|m}$ called the *transition functions*. Similarly, there are transition functions realizing the inverse coordinate transformation. An *atlas* of $(M, \mathcal{O})$ is a cover of M by open subsets that admit certain charts; any atlas determines a supermanifold up to isomorphism.

Let $(M, \mathcal{O})$ be a supermanifold and

$$\mathcal{J} = (\mathcal{O}_{\bar{1}}) = \mathcal{O}_{\bar{1}} + (\mathcal{O}_{\bar{1}})^2 \tag{2.11}$$

the subsheaf of ideals of $\mathcal{O}$ generated by the subsheaf $\mathcal{O}_{\bar{1}}$ of odd elements. Manin (see [30]) denoted

$$\mathcal{O}_{\mathrm{rd}} := \mathcal{O}/\mathcal{J}. \text{ and } M_{\mathrm{rd}} := (M, \mathcal{O}_{\mathrm{rd}}).$$

So, M_{rd} is a usual complex analytic manifold of dimension n called the *odd reduction* of $(M, \mathcal{O})$, and we have a morphism

$$\mathrm{red} = (\mathrm{id}, p_0) : (M, \mathcal{O}_{\mathrm{rd}}) \longrightarrow (M, \mathcal{O}),$$

where $p_0 : \mathcal{O} \longrightarrow \mathcal{O}_{\mathrm{rd}}$ is the canonical projection. This morphism takes the odd local coordinates ξ_j to 0 and the even ones x_i to certain local coordinates $X_1, \dots, X_n$ on M_{rd}. Clearly, any chart of $(M, \mathcal{O})$ determines a chart on M, and, on the intersection of two charts, the transition functions transforming $X_i = p_0(x_i)$ into $Y_i = p_0(y_i)$ (see eq. (2.10)) have the form

$$Y_i = \varphi_i(X_1, \dots, X_n, 0, \dots, 0).$$

Though one should distinguish between the coordinates x_i of $(M, \mathcal{O})$ and the coordinates X_i on M, they often are denoted in the same way. In what follows, we usually denote the sheaf $\mathcal{O}_{\mathrm{rd}}$ by $\mathcal{F}$. The complex manifold $M_{\mathrm{rd}} = (M, \mathcal{F})$ will usually be denoted just by M.

Any morphism of supermanifolds

$$F = (f, f^*) : (M, \mathcal{O}_M) \longrightarrow (N, \mathcal{O}_N)$$

induces a morphism of manifolds $M \to N$. This just means that the mapping $f : M \to N$ is holomorphic. As a consequence, we get a canonical homomorphism of groups from $\mathrm{Aut}(M, \mathcal{O})$ to $\mathrm{Bih}\, M$, the group of all biholomorphic transformations of M.

Any superdomain is, clearly, a supermanifold. More complicated examples will be given below.

2.7 Example (Supermanifold (M, Ω)). Let M be a complex manifold of dimension n and $\Omega = \bigoplus_{p=0}^{n} \Omega^p$ be the sheaf of holomorphic exterior forms on M. Then, (M, Ω) is a supermanifold of dimension $n|n$. Indeed, let U be an open subset of M, where a chart with local coordinates $x_1, \dots, x_n$ is defined. Clearly, the sheaf $\Omega|_U$ can be identified with $\bigwedge_{\mathcal{F}_n}(dx_1, \dots, dx_m)$. Denoting $\xi_j := dx_j$, we see that the x_i, ξ_j are local coordinates for (M, Ω). If V is another open subset with local coordinates y_i and $\eta_j := dy_j$, then the transition functions in $U \cap V$ have the form

$$\begin{aligned} y_i &= \varphi_i(y_1, \dots, y_n), \quad i = 1, \dots, n, \\ \eta_j &= \sum_{k=1}^{n} \frac{\partial y_j}{\partial x_k} \xi_k, \quad j = 1, \dots, n, \end{aligned}$$

where φ_i are the usual transition functions for M.

The simplest class of supermanifolds are the so-called split ones. Let $(M, \mathcal{F})$ be a complex manifold and $\mathcal{E}$ a locally free analytic sheaf on it. Defining $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{E}$, we get a supermanifold $(M, \mathcal{O})$. A supermanifold is called *split* if it is isomorphic to a supermanifold of this form.

The structure sheaf $\mathcal{O}$ of a split supermanifold admits the $\mathbb{Z}$-grading $\mathcal{O} = \bigoplus_{p\geq 0} \mathcal{O}_p$, where

$$\mathcal{O}_p \simeq \bigwedge_{\mathcal{F}}^{p} \mathcal{E};$$

this $\mathbb{Z}$-grading on it is compatible with the $\mathbb{Z}_2$-grading. In what follows, we often omit the subscript $\mathcal{F}$ while denoting the exterior powers, the tensor products etc. of the sheaves of $\mathcal{F}$-modules.

Let U be a coordinate neighborhood in M, over which the sheaf $\mathcal{E}$ is free or, which is the same, the corresponding vector bundle $\mathbf{E}$ is trivial. Then, we can choose special local coordinates of $(M, \mathcal{O})$ in U; these are $x_i,\ \xi_j$, where $x_1, \dots, x_n$ are local coordinates of M, while $\xi_1, \dots, \xi_m$ is a basis of the free $\mathcal{F}_U$-module $\Gamma(U, \mathcal{E})$. These local coordinates will be called *splitting* ones. The transition functions between two systems of splitting coordinates (see eq. (2.10)) have the following special form:

$$\begin{aligned} y_i &= \varphi_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \\ \eta_j &= \sum_{j=1}^{m} \psi_{jk}(x_1, \dots, x_n)\xi_k, \quad j = 1, \dots, m, \end{aligned}$$

where $\varphi_i,\ \psi_{jk}$ are holomorphic functions in x_i, and the matrices (ψ_{jk}) are the transition functions of the vector bundle $\mathbf{E}$.

A classical example of a split supermanifold is (M, Ω) (see Example 2.7). The sheaf Ω corresponds to the cotangent bundle $\mathbf{E} = \mathbf{T}(M)^*$ over M. As splitting coordinates one can choose the x_i and $\xi_j := dx_j$, where x_i are local holomorphic coordinates on M.

Another important example is that of the *complex projective superspace*.

2.8 Example (Projective superspace $\mathcal{CP}^{n|m}$). Formally, a "point" of the projective superspace $\mathcal{CP}^{n|m}$ is determined by a row of "homogeneous coordinates"

$$(z_0\colon\ \dots\colon z_n\colon \zeta_1\colon\ \dots\colon \zeta_m),$$

where $p(z_i) = \bar{0}$ and $p(\zeta_j) = \bar{1}$ and $(z_0, \dots, z_n) \neq 0$, which is defined up to multiplication by a non-zero complex number. As M, we take the usual projective space $\mathcal{CP}^n$; its points are given by the homogeneous coordinates $(z_0 : \dots : z_n)$. As usual, consider the cover of M by the affine open sets $U_k = \{z_k \neq 0\}$ for any $k = 0, \dots, n$. In U_k, we can uniquely write the coordinate row (z, ζ) as

$$(x_1^{(k)}, \dots, x_k^{(k)}, 1, x_{k+1}^{(k)}, \dots, x_n^{(k)}, \xi_1^{(k)}, \dots, \xi_m^{(k)}),$$

where $x_i^{(k)},\ \xi_j^{(k)}$ are, by definition, the local coordinates of the supermanifold $\mathcal{CP}^{n|m}$ in U_k, expressed through homogeneous coordinates by

$$\begin{aligned} x_i^{(k)} &= \begin{cases} \frac{z_{i-1}}{z_k} & \text{for } 1 \leq i \leq k \\ \frac{z_i}{z_k} & \text{for } k+1 \leq i \leq n, \end{cases} \\ \xi_j^{(k)} &= \frac{\zeta_j}{z_k},\ 1 \leq j \leq m. \end{aligned}$$

One can easily write down the transition functions, showing that $\mathcal{CP}^{n|m}$ is a split supermanifold. The sheaf $\mathcal{E}$ is $\mathcal{F}(-1)^m$, where $\mathcal{F}(-1)$ is the invertible sheaf determined by a hyperplane of $\mathbb{CP}^n$.

2.11 Subsupermanifold, retract Let $(M, \mathcal{O}_M)$ be a supermanifold and $\mathcal{I}$ be a $\mathbb{Z}_2$-graded subsheaf of ideals of $\mathcal{O}_M$. Setting

$$N = \{x \in M \mid \pi(\varphi)(x) = 0 \quad \text{for all} \ \ \varphi \in (\mathcal{O}_M)_x\}, \quad \mathcal{O}_N = (\mathcal{O}_M/\mathcal{I})|_N,$$

we get the $\mathbb{Z}_2$-graded ringed space $(N, \mathcal{O}_N)$. If this space is a supermanifold, then it is called a *submanifold* of $(M, \mathcal{O}_M)$. If the sheaf of ideals $\mathcal{I}$ is generated, over an open set $U \subset M$, by its homogeneous sections $\varphi_1, \dots, \varphi_s$, then it is usual to say that the submanifold is determined in U by the equations $\varphi_i = 0$ for $i = 1, \dots, s$.

For example, the subsheaf $\mathcal{J}$, given by the formula eq. (2.11), determines the reduction M_{rd} of $(M, \mathcal{O}_M)$, which is thus a submanifold of $(M, \mathcal{O}_M)$. For other examples, see Subsections 2.9, 2.10.

There is a construction that to any supermanifold $(M, \mathcal{O})$ assigns a split one. Consider the filtration

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots \tag{2.12}$$

of $\mathcal{O}$ by the powers of the subsheaf of ideals $\mathcal{J}$ given by the formula (2.11). The associated graded sheaf

$$\operatorname{gr} \mathcal{O} = \bigoplus_{p \geq 0} \operatorname{gr}^p \mathcal{O},$$

where $\operatorname{gr}^p \mathcal{O} = \mathcal{J}^p/\mathcal{J}^{p+1}$, gives rise to the split supermanifold $(M, \operatorname{gr} \mathcal{O})$.

Indeed, $\operatorname{gr} \mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{F} = \operatorname{gr}^0 \mathcal{O} = \mathcal{O}_{\text{rd}}$ and $\mathcal{E} = \operatorname{gr}^1 \mathcal{O}$ is a locally free sheaf of $\mathcal{F}$-modules. Clearly, $(M, \mathcal{O})$ and $(M, \operatorname{gr} \mathcal{O})$ have the same dimension. The supermanifold $(M, \operatorname{gr} \mathcal{O})$ is called the *retract* of the supermanifold $(M, \mathcal{O})$.

A supermanifold is split if and only if it is isomorphic to its retract. If the x_i and ξ_j are arbitrary local coordinates of $(M, \mathcal{O})$ in a neighborhood $U \subset M$, then $X_i = x_i + \mathcal{J}^2$ and $\Xi_j = \xi_j + \mathcal{J}$ are splitting local coordinates of $(M, \operatorname{gr} \mathcal{O})$ in U, and one gets the transition functions between these splitting coordinates, if one takes the terms of degree 0 (respectively 1) in ξ_j in the transition functions φ_i (respectively ψ_j) for $(M, \mathcal{O})$ (see eq. (2.10)).

Thus, we see that with any supermanifold $(M, \mathcal{O})$ two objects of the classical complex analytic geometry are associated: the complex manifold $(M, \mathcal{F})$ and the holomorphic vector bundle $\mathbf{E}$ over $(M, \mathcal{F})$ corresponding to the sheaf $\mathcal{E}$. It turns out that $(M, \mathcal{O})$ is not, in general, determined by these two objects up to an isomorphism, since there exist non-split supermanifolds. For examples, see below.

To settle, if a given supermanifold $(M, \mathcal{O})$ is split, one can consider the following exact sequences of sheaves over M:

$$\begin{aligned} 0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O} \overset{p_0}{\to} \mathcal{F} \longrightarrow 0, \\ 0 \longrightarrow \mathcal{J}^2 \longrightarrow \mathcal{J} \overset{p_1}{\to} \mathcal{E} \longrightarrow 0. \end{aligned} \tag{2.13}$$

If the supermanifold $(M, \mathcal{O})$ is split, then both these exact sequences are split, i.e., there exist homomorphisms $q_0 : \mathcal{F} \longrightarrow \mathcal{O}$ and $q_1 : \mathcal{E} \longrightarrow \mathcal{J}$ such that $p_i q_i = \mathrm{id},\ i = 0, 1$. The obstructions to splitness lie in certain sheaf cohomology of M (see [30, Ch.4, Sect. 2], [2, Ch.4, Sects. 6, 7], [3*, Ch.3, Sects. 6, 7]).

2.12 Super-Grassmannians In this subsection, we will briefly consider certain examples of complex supermanifolds introduced by Yu. Manin in [30]. Actually, four series of compact complex supermanifolds corresponding to the following four series of classical complex Lie superalgebras, were constructed:

(1) $\mathfrak{gl}_{n|m}(\mathbb{C})$ — the general linear Lie superalgebra of the vector superspace $\mathbb{C}^{n|m}$;

(2) $\mathfrak{osp}_{n|m}(\mathbb{C})$ — the orthosymplectic Lie superalgebra that annihilates a non-degenerate even symmetric bilinear form in $\mathbb{C}^{n|m}$, m being even;

(3) $\mathfrak{pe}_{n|n}(\mathbb{C})$ — the linear Lie superalgebra that annihilates a non-degenerate odd symmetric bilinear form in $\mathbb{C}^{n|n}$ (Manin denoted $\mathfrak{pe}_{n|n}(\mathbb{C})$ by $\pi\mathfrak{sp}_n(\mathbb{C})$ in [30], see also [54*], but A. Weil's suggestion to call the odd non-degenerate bilinear form, and the Lie superalgebra/supergroup it preserves, *periplectic* took over and is now universally accepted together with the name *queer* for the following purely super analog of $\mathfrak{gl}_n$, *D.L.*);

(4) $\mathfrak{q}_n(\mathbb{C})$ — the linear Lie superalgebra that commutes with an odd involution in $\mathbb{C}^{n|n}$.

These supermanifolds are called the *flag supermanifolds* in case (1), the *supermanifolds of isotropic flags* in cases (2) and (3), and the *supermanifolds of* Π-*symmetric flags* in case (4). We will call them the *classical flag supermanifolds*. They are, in most cases, non-split.

Here we consider the classical flag supermanifolds under assumption that the flags have the minimal possible length; these are so-called *super-Grassmannians*. The super-Grassmannians are basic in Manin's constructions, because the flag supermanifolds are defined inductively as relative super-Grassmannians over the flag supermanifolds of lesser length.

As in Example 2.6, we denote by $e_1, \dots, e_n, f_1, \dots, f_m$ the standard basis of $\mathbb{C}^{n|m}$.

2.9 Example (The super-Grassmannian). The super-Grassmannian $\mathrm{Gr}^{n|m}_{k|l}$ of $(k|l)$-dimensional subspaces in $\mathcal{C}^{n|m}$ is a natural generalization of the projective superspace $\mathcal{CP}^{n|m} = \mathrm{Gr}^{n+1|m}_{1|0}$. Its structure is determined by the $(n+m) \times (k+l)$ coordinate matrix

$$Z = \begin{pmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z_{11} \end{pmatrix},$$

where Z_{00} and Z_{11} are $n \times k$- and $m \times l$-matrices, respectively, whose entries are even homogeneous coordinates, while Z_{01} and Z_{10} are $n \times l$- and $m \times k$-matrices, respectively, whose entries are odd ones. It is supposed that Z_{00} and Z_{11} are complex matrices of ranks k and l, respectively, so that each of them determines a point of the complex Grassmannian Gr^n_k or Gr^m_l, respectively.

Thus, we get an element x_0 of the manifold $M = \mathrm{Gr}_k^n \times \mathrm{Gr}_l^m$; this manifold is the reduction of the super-Grassmannian. The matrix Z is to be regarded up to the following equivalence:

$$Z \sim Z' \text{ if } Z' = ZQ, \text{ where } Q \text{ is an invertible } k \times l\text{-matrix.}$$

If we fix an invertible $k \times l$-submatrix of Z, then the remaining entries of Z give us the even and the odd local coordinates in a neighborhood of x_0. Using the equivalence, we can assume that the fixed submatrix is the unit matrix $I_{k|l} = \begin{pmatrix} I_k & 0 \\ 0 & I_l \end{pmatrix}$.

For example, choose

$$x_0 = \langle e_{n-k+1}, \dots, e_n, f_1, \dots, f_l \rangle = (\langle e_{n-k+1}, \dots, e_n \rangle, \langle f_1, \dots, f_l \rangle).$$

Then, the coordinate matrix can be written in the form

$$Z = \begin{pmatrix} X & \Xi \\ I_k & 0 \\ 0 & I_l \\ \mathrm{H} & Y \end{pmatrix}, \tag{2.14}$$

where

$$X = (x_{ij}),\ Y = (y_{\alpha p}),\ \Xi = (\xi_{ip}),\ \mathrm{H} = (\eta_{\alpha i}),$$
$$i = 1, \dots, n-k,\ j = 1, \dots, k,\ p = 1, \dots, l,\ \alpha = l+1, \dots, n.$$

Here x_{ij} and $y_{\alpha p}$ are even local coordinates satisfying $x_{ij}(x_0) = y_{\alpha p}(x_0) = 0$, while ξ_{ip} and $\eta_{\alpha i}$ are odd ones. In particular, we have

$$\dim \mathrm{Gr}_{k|l}^{n|m} = n(n-k) + m(m-l) \mid n(m-l) + m(n-k).$$

In the case where $0 < k < n$ and $0 < l < m$, the supermanifold $\mathrm{Gr}_{k|l}^{n|m}$ is non-split. The simplest non-split super-Grassmannian is $\mathrm{Gr}_{1|1}^{2|2}$ of dimension $2|2$.

2.10 Example (The isotropic super-Grassmannian. Superquadric). Let an even non-degenerate symmetric bilinear form b be given in $\mathcal{C}^{n|m}$. Then, it is possible to define the subsupermanifold $\mathrm{I\,Gr}_{k|l}^{n|m}$ of $\mathrm{Gr}_{k|l}^{n|m}$, consisting of subspaces that are (totally) isotropic with respect to b; this $\mathrm{I\,Gr}_{k|l}^{n|m}$ is called *isotropic super-Grassmannian*).

If n is odd, then we get the simplest non-split supermanifolds for

$$n = 3,\ k = 1,\ m = 2s \geq 2,\ l = 0.$$

The supermanifold $\mathcal{Q}^{1,m} = \mathrm{I\,Gr}_{1|0}^{3|m}$ is called the *superquadric* in the projective superplane $\mathcal{CP}^{2|m}$. In homogeneous coordinates (see Example 2.7), we can express the superquadric by the equation

$$z_0^2 - z_1 z_2 + \sum_{i=1}^{s} \zeta_i \zeta_{s+i} = 0.$$

The local coordinates on the superquadric are given by the formula

$$x = \frac{z_0}{z_1},\ \xi_j = \frac{\zeta_j}{z_1} \quad \text{for } z_1 \neq 0;$$
$$y = \frac{z_0}{z_2},\ \eta_j = \frac{\zeta_j}{z_2} \quad \text{for } z_2 \neq 0,$$

and the transition functions have the form

$$y = x^{-1}(1 + x^{-2}\sum_{i=1}^{s} \xi_i\xi_{s+i})^{-1},$$
$$\eta_j = x^{-2}(1 + x^{-2}\sum_{i=1}^{s} \xi_i\xi_{s+i})^{-1}\xi_j, \quad \text{where} \quad j = 1, \ldots, 2s.$$

Historically, this was (for $m = 2$) one of the first examples of non-split supermanifolds (see [17], [2], [30]).

2.11 Example (The odd isotropic super-Grassmannian). Quite similarly, an isotropic super-Grassmannian, associated with an odd non-degenerate anti-symmetric (or symmetric) bilinear form b is defined. In this case, $n = m$, and we denote by $\mathrm{I_{odd}\,Gr}_{k|l}^{n|n}$ the corresponding submanifold of $\mathrm{Gr}_{k|l}^{n|n}$.

2.12 Example (The Π-symmetric super-Grassmannian). Suppose that $m = n$ and that an odd involutive linear transformation Π of the vector superspace $\mathbb{C}^{n|n}$ is given. Then, we can define the submanifold $\Pi\,\mathrm{Gr}_{s|s}^{n|n}$ of $\mathrm{Gr}_{s|s}^{n|n}$ that consists of Π-invariant subspaces of dimension $s|s$ (the Π-*symmetric super-Grassmannian*). This super-Grassmannian is one of the main objects of our study, and therefore it will be considered in more details in Section 5. We only mention here that the retract of $\Pi\,\mathrm{Gr}_{s|s}^{n|n}$ is the supermanifold $(\mathrm{Gr}_s^n, \Omega)$ of Example 2.7.

3 Tangent sheaf and vector fields

3.1 Tangent space and tangent sheaf We retain the notation of Subsection 2.7. Let $(M, \mathcal{O})$ be a complex supermanifold. Fix a point $x \in M$. Using local coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_m$ in a neighborhood of x, we can identify the superalgebra $\mathcal{O}_x$ with $\mathbb{C}\{x_1, \ldots, x_n\} \otimes \bigwedge_{\mathbb{C}}(\xi_1, \ldots, \xi_m)$ for any $x \in M$. Notice that this is a local superalgebra whose unique maximal ideal is $\mathfrak{m}_x := (x_1, \ldots, x_n, \xi_1, \ldots, \xi_m)$. The vector superspace $T_x(M, \mathcal{O}) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the *tangent space* to $(M, \mathcal{O})$ at the point x. Since $\mathcal{F} = \mathcal{O}/\mathcal{J}$, we have the exact sequence

$$0 \longrightarrow \mathcal{J}_x \longrightarrow \mathfrak{m}_x \longrightarrow n_x \longrightarrow 0,$$

where $\mathfrak{n}_x$ is the maximal ideal of $\mathcal{F}_x$. This implies the following exact sequence:

$$0 \longrightarrow \mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x \longrightarrow \mathfrak{m}_x/(\mathfrak{m}_x)^2 \to \mathfrak{n}_x/(\mathfrak{n}_x)^2 \longrightarrow 0.$$

The fiber at x of the vector bundle $\mathbf{E}$ corresponding to $(M, \mathcal{O})$ is $\mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x = E_x$. Since $T_x(M) = (\mathfrak{n}_x/\mathfrak{n}_x^2)^*$ is the tangent vector space to M at x, we get the exact sequence

$$0 \longrightarrow T_x(M) \longrightarrow T_x(M, \mathcal{O}) \longrightarrow E_x^* \longrightarrow 0.$$

This gives the canonical identifications

$$T_x(M, \mathcal{O})_{\bar{0}} = T_x(M), \quad T_x(M, \mathcal{O})_{\bar{1}} = E_x^*.$$

The *tangent sheaf* of a supermanifold $(M, \mathcal{O})$ is by definition the sheaf $\mathcal{T} = \mathcal{D}er\, \mathcal{O}$ of derivations of the structure sheaf $\mathcal{O}$. Its stalk at $x \in M$ is the Lie superalgebra $\mathfrak{der}_{\mathbb{C}} \mathcal{O}_x$ of derivations of the superalgebra $\mathcal{O}_x$. Its sections are caled *holomorphic vector fields* on $(M, \mathcal{O})$. The vector superspace $\mathfrak{v}(M, \mathcal{O}) = \Gamma(M, \mathcal{T})$ of all holomorphic vector fields is finite-dimensional whenever M is compact. We regard it as a complex Lie superalgebra with the bracket

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX. \tag{3.15}$$

Fix a point $x \in M$. Any $\delta \in \mathfrak{der}\, \mathcal{O}_x$ is such that $\delta(\mathfrak{m}_x^2) \subset \mathfrak{m}_x$, and hence defines a linear mapping $\widetilde{\delta} : \mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \mathcal{O}_x/\mathfrak{m}_x = \mathbb{C}$ which is an element of $T_x(M, \mathcal{O})$. This permits us to define an even linear mapping $\mathrm{ev}_x : \mathfrak{v}(M, \mathcal{O}) \longrightarrow T_x(M, \mathcal{O})$ by

$$\mathrm{ev}_x(v) = \widetilde{v}_x.$$

We note that, in contrast with the non-super case, a vector field v is not, in general, uniquely determined by its values $\widetilde{v}_x$ at all $x \in M$.

Endow the tangent sheaf $\mathcal{T}$ with the following filtration:

$$\mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \ldots \supset \mathcal{T}_{(m)} \supset \mathcal{T}_{(m+1)} = 0, \tag{3.16}$$

where

$$\mathcal{T}_{(p)} = \{\delta \in \mathcal{T} \mid \delta(\mathcal{O}) \subset \mathcal{J}^p, \delta(\mathcal{J}) \subset \mathcal{J}^{p+1}\} \text{ for any } p \geq 0.$$

Thus, we have obtained a filtered sheaf of Lie superalgebras. Let $\mathrm{gr}\, \mathcal{T}$ denote the corresponding graded sheaf of algebras. Any $v \in \mathcal{T}_{(p)}$ maps $\mathcal{J}^q$ to $\mathcal{J}^{q+p}$, inducing a derivation from $(\mathcal{T}_{\mathrm{gr}})_p$, where $\mathcal{T}_{\mathrm{gr}} = \mathcal{D}er\ \mathrm{gr}\, \mathcal{O}$. As a result, we get a homomorphism $\sigma_p : \mathcal{T}_{(p)} \to (\mathcal{T}_{\mathrm{gr}})_p$. It is easy to check (see [35*]) that the following assertion is true.

3.1 Proposition (An exact sequence). *The following sequences of sheaves are exact:*

$$0 \longrightarrow \mathcal{T}_{(p+1)} \longrightarrow \mathcal{T}_{(p)} \stackrel{\sigma_p}{\to} (\mathcal{T}_{\mathrm{gr}})_p \longrightarrow 0, \quad \textit{where} \;\; p \geq -1.$$

The homomorphisms σ_p, where $p \geq -1$, determine an isomorphism of the sheaves of graded algebras $\mathrm{gr}\, \mathcal{T} \longrightarrow \mathcal{T}_{\mathrm{gr}} = \mathcal{D}er\ \mathrm{gr}\, \mathcal{O}$.

In what follows, we will use the cohomology groups $H^p(M, \mathcal{T})$ with values in the tangent sheaf. Recall that they are finite-dimensional vector spaces if M is compact. We have $H^0(M, \mathcal{T}) = \mathfrak{v}(M, \mathcal{O})$. Since $\mathcal{T}$ is a sheaf of Lie superalgebras, we can define the corresponding operation in

$$H^{\bullet}(M, \mathcal{T}) = \bigoplus_{p \geq 0} H^p(M, \mathcal{T}),$$

giving a graded algebra. This operation will be denoted by $[-,-]$; it coincides on $H^0(M, \mathcal{T})$ with the bracket defined above. The filtration (3.16) gives rise to a natural filtration in $H^{\bullet}(M, \mathcal{T})$, so we get a filtered algebra.

3.3 The tangent sheaf of the split supermanifold Here we make some remarks concerning vector fields on split supermanifolds. If $(M, \mathcal{O})$ is split, then $\mathcal{T}$ is a $\mathbb{Z}$-graded sheaf of Lie superalgebras, the grading being given by the formula

$$\mathcal{T} = \bigoplus_{p \geq -1} \mathcal{T}_p,$$

where

$$\mathcal{T}_p := \mathcal{D}er_p\mathcal{O} = \{\delta \in \mathcal{T} \mid \delta(\mathcal{O}_q) \subset \mathcal{O}_{q+p} \text{ for all } q \in \mathbb{Z}\}. \tag{3.17}$$

Hence, $\mathfrak{v}(M, \mathcal{O}) := \bigoplus_{p \geq -1} \mathfrak{v}(M, \mathcal{O})_p$ is a $\mathbb{Z}$-graded Lie superalgebra. Moreover, we get a grading in any cohomology $H^p(M, \mathcal{T})$, turning $H^{\bullet}(M, \mathcal{T})$ into a bigraded algebra. One easily verifies that the filtration (3.16) of $\mathcal{T}$ coincides with the filtration associated with the grading (3.17), so that

$$\mathcal{T}_{(p)} = \bigoplus_{r \geq p} \mathcal{T}_r.$$

Since $\mathcal{O} = \bigwedge \mathcal{E}$, where $\mathcal{E}$ is a locally free analytic sheaf on $M = (M, \mathcal{F})$, it follows that $\mathcal{T}$ can be regarded as an analytic sheaf on the complex manifold M. It was useful to interpret $\mathcal{T}$ directly in terms of the sheaf $\mathcal{E}$. A partial description of $\mathcal{T}_p$ for $p \geq -1$ is given by the following exact sequence of locally free analytic sheaves on M (see [35*]):

$$0 \longrightarrow \mathcal{E}^* \otimes \bigwedge^{p+1} \mathcal{E} \xrightarrow{i} \mathcal{T}_p \xrightarrow{\alpha} \Theta \otimes \bigwedge^{p} \mathcal{E} \longrightarrow 0, \tag{3.18}$$

where $\Theta = \mathcal{D}er\mathcal{F}$ is the tangent sheaf of the manifold M. The mapping α is the restriction of the derivation of degree p onto the subsheaf $\mathcal{F}$, while i identifies any sheaf homomorphism $\mathcal{E} \longrightarrow \bigwedge^{p+1} \mathcal{E}$ with a derivation of degree p that vanishes on $\mathcal{F}$. Clearly, $\operatorname{Im} i$ is the subsheaf of $\mathcal{T}$ consisting of $\mathcal{F}$-derivations; they act on the stalks of $\mathcal{O}$ as derivations of a Grassmann algebra (see Example 2.4).

In particular, in the case $p = -1$, we have an isomorphism

$$\mathcal{T}_{-1} \simeq \mathcal{H}om_{\mathcal{F}}(\mathcal{E}, \mathcal{F}) = \mathcal{E}^*, \tag{3.19}$$

and in the case $p = 0$, we have the exact sequence

$$0 \longrightarrow \mathcal{E}^* \otimes \mathcal{E} \xrightarrow{i} \mathcal{T}_0 \xrightarrow{\alpha} \Theta \longrightarrow 0. \tag{3.20}$$

Let $\mathbf{E}$ be the holomorphic vector bundle over M corresponding to the locally free sheaf $\mathcal{E}$. Clearly, $\mathcal{T}_0$ is the sheaf of infinitesimal automorphisms of $\mathbf{E}$ and $\mathcal{E}^* \otimes \mathcal{E} = \mathcal{E}nd\, \mathbf{E}$ is its subsheaf consisting of germs of endomorphisms preserving each fiber.

The first terms of the cohomology exact sequence, corresponding to the sequence (3.20), have the form

$$0 \longrightarrow \mathfrak{gl}(\mathbf{E}) \xrightarrow{i} \mathfrak{v}(M, \mathcal{O})_0 \xrightarrow{\alpha} \mathfrak{v}(M), \tag{3.21}$$

where $\mathfrak{gl}(\mathbf{E}) = \Gamma(M, \mathcal{E}^* \otimes \mathcal{E})$ is the Lie algebra of all endomorphisms of $\mathbf{E}$ (preserving the fibers) and $\mathfrak{v}(M) = \Gamma(M, \Theta)$ is the Lie algebra of holomorphic vector fields on M, whereas i and α are Lie algebra homomorphisms. If M is compact, then we have the corresponding exact sequence of complex Lie groups

$$e \longrightarrow \mathrm{GL}(\mathbf{E}) \longrightarrow \mathrm{Aut}\, \mathbf{E} \longrightarrow \mathrm{Bih}\, M, \tag{3.22}$$

where $\mathrm{Aut}\, \mathbf{E}$ is the group of automorphisms of $\mathbf{E}$ and $\mathrm{GL}(\mathbf{E})$ its normal subgroup that consists of automorphisms preserving the fibers (the *gauge group* of $\mathbf{E}$) (see [35*]). Note that $\mathrm{GL}(\mathbf{E})$ is the group of invertible elements of $\mathrm{Mat}(\mathbf{E})$ regarded as an associative algebra.

We remark that the Lie algebra $\mathfrak{gl}(\mathbf{E})$ is never zero (whenever $m > 0$). Indeed, it always contains the identity endomorphism ε. Regarded as a vector field, ε coincides with the grading derivation of the sheaf $\mathcal{O}$ acting by means of eq. (2.1). Clearly, in any splitting local coordinates x_i, ξ_j, it has the form

$$\varepsilon = \sum_{j=1}^{m} \xi_j \frac{\partial}{\partial \xi_j}. \tag{3.23}$$

3.4 The tangent sheaf of (M, Ω). The splitting mapping l. The Frölicher and Nijenhuis (FN) bracket Clearly, locally, the exact sequence (3.18) splits. But in the case where $\mathbf{E} = \mathbf{T}(M)^*$ (see Example 2.7), there is a canonical global splitting, discovered by Frölicher and Nijenhuis (see [12] and [25]; see also [20*]). In this case, we have $\mathcal{E} = \Omega^1$, and hence the sheaves $\bigwedge \mathcal{E} \otimes \Theta$ and $\bigwedge \mathcal{E} \otimes \mathcal{E}^*$ both coincide with the sheaf $\Omega \otimes \Theta$ of holomorphic vector-valued differential forms. Thus, the exact sequence (3.18) has the form

$$0 \longrightarrow \Omega^{p+1} \otimes \Theta \xrightarrow{i} \mathcal{T} \xrightarrow{\alpha} \Omega^p \otimes \Theta \longrightarrow 0. \tag{3.24}$$

The splitting mapping $l : \Omega \otimes \Theta \longrightarrow \mathcal{T}$ is defined by the formula

$$l(\varphi) = [i(\varphi), d], \tag{3.25}$$

where d is the exterior derivative which, clearly, is a section of $\mathcal{T}_1$. One verifies that $\alpha(l(\varphi)) = \varphi$, so that l is, in fact, a splitting of the sequence (3.24). It follows that there is the following decomposition into the direct sum of sheaves of vector spaces:

$$\mathcal{T} = i(\Omega \otimes \Theta) \oplus l(\Omega \otimes \Theta). \tag{3.26}$$

More precisely,

$$\mathcal{T}_p = i(\Omega_{p+1} \otimes \Theta) \oplus l(\Omega_p \otimes \Theta) \simeq (\Omega_{p+1} \otimes \Theta) \oplus (\Omega_p \otimes \Theta). \tag{3.27}$$

Note that for $p = 0$ the derivation $l(u),\ u \in \Theta$ is the classical Lie derivative along the vector field u.

We recall now the Lie bracket in $\mathcal{T}$. Clearly, $i(\Omega \otimes \Theta)$ is a sheaf of subalgebras of $\mathcal{T}$, and hence we get a bracket $\{-,-\}$ in the sheaf $\Omega \otimes \Theta$, which is often called the *algebraic* bracket and is given by the formula (2.4). In [12], another bracket $[-,-]$ was defined in $\Omega \otimes \Theta$, namely,

$$[\varphi, \psi] = \alpha([l(\varphi), l(\psi)]).$$

It is called the *FN-bracket*. Under this bracket and the grading

$$(\Omega \otimes \Theta)_p = \Omega_p \otimes \Theta,$$

the sheaf $\Omega \otimes \Theta$ is a sheaf of graded Lie superalgebras as well. We also have

$$\begin{aligned} &[l(\varphi), l(\psi)] = l([\varphi, \psi]), \\ &[i(\varphi), l(\psi)] = l(\varphi \mathbin{\bar\wedge} \psi) + (-1)^q i([\varphi, \psi]), \quad \varphi \in \Omega \otimes \Theta,\ \psi \in \Omega^q \otimes \Theta. \end{aligned} \tag{3.28}$$

Thus, l is a homomorphism of sheaves of graded Lie superalgebras, and the formula (3.26) describes a decomposition into the sum of sheaves of subalgebras (but not ideals!).

In particular, from the isomorphism (3.19) we get the isomorphisms

$$\begin{aligned} &i : \Theta \longrightarrow \mathcal{T}_{-1}, \\ &i : \mathfrak{v}(M) \longrightarrow \mathfrak{v}(M, \Omega)_{-1}, \end{aligned} \tag{3.29}$$

and from the sequence (3.21) we get the semi-direct decomposition of Lie algebras

$$\mathfrak{v}(M, \Omega)_0 = i(\mathfrak{gl}(\mathbf{T}(M)^*) \mathbin{⋑} l(\mathfrak{v}(M)). \tag{3.30}$$

For a compact M, we have the following global form of this decomposition:

$$\operatorname{Aut} \mathbf{T}(M)^* = \mathrm{GL}(\mathbf{T}(M)^*) \rtimes \operatorname{Bih} M, \tag{3.31}$$

where the group $\operatorname{Bih} M$ of biholomorphic transformations of M acts on differential forms in an obvious way.

Note that the identity endomorphism $\mathrm{id} \in \mathfrak{gl}(\mathbf{T}(M)^*)$ gives rise to the vector fields

$$d = l(\mathrm{id}) \in \mathfrak{v}(M, \Omega)_1 \text{ and } \varepsilon = i(\mathrm{id}) \in \mathfrak{v}(M, \Omega)_0,$$

the first one being the exterior derivative and the second one the grading derivation.

Let $\psi \in \Gamma(M, \Omega^p)$ be a holomorphic p-form on M. Using formula (2.5) we can define a vector-valued form $j(\psi) \in \Gamma(M, \Omega^{p+1} \otimes \Theta)$.

3.2 Proposition (What is $l \circ j$). *We have*

$$l(j(\psi)) = \psi d + (-1)^{p+1}(d\psi)\varepsilon \quad \textit{for any} \quad \psi \in \Gamma(M, \Omega^p).$$

Proof. Follows immediately from (1.6) and (2.11). □

We are now going to discuss the problem of calculating the cohomology algebra $H^{\bullet}(M, \mathcal{T})$ endowed with the bracket induced by the Lie bracket in $\mathcal{T}$ (for details, see [40]). First, it follows from the decomposition (3.26) that

$$H^{\bullet}(M, \mathcal{T}) = i^*(H^{\bullet}(M, \Omega \otimes \Theta)) \oplus l^*(H^{\bullet}(M, \Omega \otimes \Theta)).$$

To calculate $H^{\bullet}(M, \Omega \otimes \Theta)$, one can use the standard Dolbeault–Serre resolution consisting of smooth vector-valued forms on M (actually, it was first considered in [13]). Denote $\Phi := \bigoplus_{p,q \geq 0} \Phi^{p,q}$, where $\Phi^{p,q}$ is the sheaf of complex-valued smooth (p,q)-forms on M. Then, for any $p \geq 0$, the differential graded sheaf $(\Phi^{p,*} \otimes \Theta, \bar{\partial})$ is a fine resolution of $\Omega^p \otimes \Theta$, whence

$$H^q(M, \Omega^p \otimes \Theta) \simeq H^q(\Gamma(M, \Phi^{p,*} \otimes \Theta), \bar{\partial}). \tag{3.32}$$

The algebraic bracket and the FN-bracket in $\Omega \otimes \Theta$ induce certain brackets in the graded vector space $H^{\bullet}(M, \Omega \otimes \Theta)$. By the isomorphism (3.32), they correspond, respectively, to the algebraic bracket and the FN-bracket in $\Phi \otimes \Theta$ defined in [12]. As to the cohomology of $\mathcal{T}$, we obtain the following result.

3.3 Proposition (Decomposing $H^{\bullet}(M, \mathcal{T})$ using i and l). *We have*

$$\begin{aligned} H^{\bullet}(M, \mathcal{T}) &= i^*(H^{\bullet}(M, \Omega \otimes \Theta)) \oplus l^*(H^{\bullet}(M, \Omega \otimes \Theta)) \\ &\simeq H(\Gamma(M, \Phi \otimes \Theta), \bar{\partial}) \oplus H(\Gamma(M, \Phi \otimes \Theta), \bar{\partial}). \end{aligned}$$

The bigrading in $H^{\bullet}(M, \mathcal{T})$ is given by the formula

$$H^q(M, \mathcal{T}_p) \simeq H^q(\Gamma(M, \Phi^{p+1,\bullet} \otimes \Theta), \bar{\partial}) \oplus H^q(\Gamma(M, \Phi^{p,\bullet} \otimes \Theta), \bar{\partial}), \quad p \geq -1, \ q \geq 0,$$

and the bracket $[\alpha, \beta]$, where $\alpha, \beta \in H^{\bullet}(M, \mathcal{T})$, is determined by the algebraic bracket of smooth vector-valued forms in the left summand, by the FN-bracket in the right one and by the formula (3.28) in the case where α, β belong to different summands.

3.7 Actions of Lie superalgebras on supermanifolds. Transitive and $\bar{0}$-transitive supermanifolds Let $(M, \mathcal{O})$ be a supermanifold and $\mathfrak{g}$ a Lie superalgebra. An *action* of $\mathfrak{g}$ on $(M, \mathcal{O})$ is an arbitrary Lie superalgebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{v}(M, \mathcal{O})$. Actions of Lie superalgebras usually appear as the differentials of actions of Lie supergroups, but we will avoid to consider the general (rather technical) Lie theory for supermanifolds, referring to [27]. Actually, we will deal only with the standard actions of classical Lie supergroups on super-Grassmannians (see Section 6).

If an action $\varphi : \mathfrak{g} \longrightarrow \mathfrak{v}(M, \mathcal{O})$ is given, then with any $x \in M$ the linear mapping

$$\varphi^x = \operatorname{ev}_x \varphi : \mathfrak{g} \longrightarrow T_x(M, \mathcal{O})$$

is associated. The set $\mathfrak{g}_x = \operatorname{Ker} \varphi^x$ is a subalgebra of $\mathfrak{g}$, called the *stabilizer* of x. The action φ is called *transitive* if φ^x is surjective for any $x \in M$. In this case one also says that $(M, \mathcal{O})$ is a *homogeneous space* of the Lie superalgebra $\mathfrak{g}$.

Restricting an action $\varphi : \mathfrak{g} \longrightarrow \mathfrak{v}(M, \mathcal{O})$ to the even component, we get a homomorphism $\varphi_0 : \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$. If M is compact, then, as in the classical Lie theory, it is possible to integrate φ_0, getting a homomorphism $\Phi : G \to \operatorname{Aut}(M, \mathcal{O})$, where G is the simply connected complex Lie group with tangent algebra $\mathfrak{g}_{\bar{0}}$. This homomorphism induces an action $\Phi_0 : G \to \operatorname{Bih} M$ of G on M. We say that the action φ is $\bar{0}$-*transitive* if Φ_0 is a transitive action in the usual sense. Clearly, this is equivalent to the following condition: $\varphi^x : \mathfrak{g}_{\bar{0}} \longrightarrow T_x(M)$ is surjective for any $x \in M$. Any transitive action is $\bar{0}$-transitive.

Let again φ be an action of $\mathfrak{g}$ on $(M, \mathcal{O})$, where M is compact. Then, G acts on the sheaf $\mathcal{T}$ by the automorphisms

$$g_* : v \mapsto (\Phi(g)^{-1})^* v \Phi(g)^* \text{ for any } g \in G.$$

One immediately verifies that

$$\operatorname{ev}_{gx} g_* = d_x \Phi_0(g) \operatorname{ev}_x, \quad g \in G,\ x \in M.$$

It follows that

$$\varphi^{gx} \operatorname{Ad}_g = d_x \Phi_0(g) \varphi^x, \quad g \in G,\ x \in M.$$

As a corollary, we get the following proposition.

3.4 Proposition (Transitive and $\bar{0}$-transitive actions). *Let φ be a $\bar{0}$-transitive action of $\mathfrak{g}$ on $(M, \mathcal{O})$, where M is compact. Then,*

(1) *The stabilizers $\mathfrak{g}_x$, where $x \in M$, of φ are conjugate by inner automorphisms of $\mathfrak{g}$* (i.e., by the automorphisms of the form Ad_g for $g \in G$).

(2) *The action φ is transitive if and only if the mapping $\varphi_0^x : \mathfrak{g}_{\bar{1}} \longrightarrow T_{x_0}(M, \mathcal{O})_{\bar{1}}$ is surjective for a certain $x_0 \in M$.*

3.9 Homogeneous and $\bar{0}$-homogeneous supermanifolds Let now $(M, \mathcal{O})$ be a supermanifold, where M is compact. Then, there is a natural action $\varphi = \operatorname{id}$ of the finite-dimensional Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ on $(M, \mathcal{O})$. The supermanifold $(M, \mathcal{O})$ is called *homogeneous* (respectively, $\bar{0}$-*homogeneous*) if this action is transitive (respectively, $\bar{0}$-transitive). This means that the mapping $\operatorname{ev}_x : \mathfrak{v}(M, \mathcal{O}) \longrightarrow T_x(M, \mathcal{O})$ (respectively, the even component of this mapping) is surjective for any $x \in M$. Proposition 3.4 implies that a $\bar{0}$-homogeneous supermanifold is homogeneous if and only if the odd component of the mapping $\operatorname{ev}_{x_0} : \mathfrak{v}(M, \mathcal{O}) \longrightarrow T_{x_0}(M, \mathcal{O})$ is surjective for a certain point $x_0 \in M$.

Suppose that an action φ of a Lie superalgebra $\mathfrak{g}$ on a supermanifold $(M,\mathcal{O})$ is given. We are going to define an action on the split supermanifold $(M,\operatorname{gr}\mathcal{O})$. To do this, we note that the filtration (3.16) gives rise to the filtration

$$\mathfrak{g}=\mathfrak{g}_{(-1)}\supset\mathfrak{g}_{(0)}\supset\ldots\supset\mathfrak{g}_{(m)}\supset\mathfrak{g}_{(m+1)}=0,$$

defined by the formula

$$\mathfrak{g}_{(p)}=\mathfrak{g}\cap\varphi^{-1}(\Gamma(M,\mathcal{T}_{(p)}))=\{u\in\mathfrak{g}\mid\varphi(u)(\mathcal{O})\subset\mathcal{J}^p,\ \varphi(u)(\mathcal{J})\subset\mathcal{J}^{p+1}\}.$$

Clearly, $\mathfrak{g}$ is a filtered Lie superalgebra, and φ determines a homomorphism $\widetilde{\varphi}$ of the correspondent graded Lie superalgebra $\widetilde{\mathfrak{g}}$ into the graded Lie superalgebra $\mathfrak{v}(M,\operatorname{gr}\mathcal{O})$, i.e., an action of $\widetilde{\mathfrak{g}}$ on $(M,\operatorname{gr}\mathcal{O})$.

In particular, consider the natural action $\varphi=\mathrm{id}$ of $\mathfrak{g}=\mathfrak{v}(M,\mathcal{O})$ on a compact supermanifold $(M,\mathcal{O})$. We see that $\mathfrak{g}_{(p)}=\Gamma(M,\mathcal{T}_{(p)})$, and $\widetilde{\varphi}$ is an injective homomorphism $\widetilde{\mathfrak{g}}\longrightarrow\mathfrak{v}(M,\operatorname{gr}\mathcal{O})$ induced by σ_p (see Proposition 3.1).

3.5 Example (Super-Grassmannians). Consider the super-Grassmannian $\mathrm{Gr}_{k|l}^{n|m}$ defined in Example 2.7. The general linear Lie supergroup $\mathrm{GL}_{n|m}(\mathbb{C})$ acts on $\mathrm{Gr}_{k|l}^{n|m}$ by multiplying the coordinate matrix Z (see formula (2.14)) on the left by a matrix of $\mathrm{GL}_{n|m}(\mathbb{C})$. The differential of this action is an action of the Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C})$ on this supermanifold. One easily checks that it is transitive.

The above action induces transitive actions of the subsuperalgebras $\mathfrak{osp}_{n|m}(\mathbb{C})$, $\mathfrak{pe}_{n|n}(\mathbb{C})$, and $\mathfrak{q}_n(\mathbb{C})$ of $\mathfrak{gl}_{n|m}(\mathbb{C})$ on the subsupermanifolds $\mathrm{I}\,\mathrm{Gr}_{k|l}^{n|m}$, $\mathrm{I}_{\mathrm{odd}}\,\mathrm{Gr}_{k|l}^{n|n}$, and $\Pi\,\mathrm{Gr}_{s|s}^{n|n}$ of the general super-Grassmannian, considered in Examples 2.10, 2.11, and 2.12, respectively (for $\Pi\,\mathrm{Gr}_{s|s}^{n|n}$, the transitivity will be proved in Proposition 6.6).

4 Classification of non-split supermanifolds

4.1 Sheaves of automorphisms and the classification theorem Let $(M,\mathcal{O})$ be a supermanifold. Consider the sheaf $\mathcal{A}ut\ \mathcal{O}$ of automorphisms of the structure sheaf $\mathcal{O}$ of $(M,\mathcal{O})$ (as usual, any automorphism is even and maps each stalk $\mathcal{O}_x$, where $x\in M$, onto itself). This is a sheaf of groups. For any $F=(f,\varphi)\in\mathrm{Aut}(M,\mathcal{O})$, the mapping $\mathrm{Int}\,F:a\mapsto\varphi a\varphi^{-1}$ is an automorphism of $\mathcal{A}ut\,\mathcal{O}$ which gives an action Int of the group $\mathrm{Aut}(M,\mathcal{O})$ on $\mathcal{A}ut\ \mathcal{O}$ by automorphisms of this sheaf.

Clearly, any $a\in\mathcal{A}ut\ \mathcal{O}$ maps $\mathcal{J}$ onto itself, and hence preserves the filtration (2.12) and induces a germ of an automorphism of $\operatorname{gr}\mathcal{O}$. By definition, a induces the identity mapping on $\mathcal{F}=\mathcal{O}/\mathcal{J}$. Consider the filtration

$$\mathcal{A}ut\ \mathcal{O}=\mathcal{A}ut_{(0)}\mathcal{O}\supset\mathcal{A}ut_{(2)}\mathcal{O}\supset\ldots,\tag{4.33}$$

where

$$\mathcal{A}ut_{(2p)}\mathcal{O}=\{a\in\mathcal{A}ut\ \mathcal{O}\mid a(u)-u\in\mathcal{J}^{2p}\text{ for all }\ u\in\mathcal{O}\}.$$

One easily sees that the $\mathcal{A}ut_{(2p)}\mathcal{O}$ are subsheaves of normal subgroups of $\mathcal{A}ut\ \mathcal{O}$. They also are invariant under the action Int of $\mathrm{Aut}(M,\mathcal{O})$ defined above.

Following [17] and [50*], [47], we will use the sheaves of automorphisms in order to describe the family of all supermanifolds, having as their retract a given split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$. The 1-cohomology sets $H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}_{\mathrm{gr}})$ for $p \geq 1$ play the main role in this description. We recall (see [19]) that for any sheaf of (not necessarily abelian) groups $\mathcal{G}$ on M the 1-cohomology set $H^1(M, \mathcal{G})$ is defined. It has no natural group structure, but has a distinguished element e, also called the *unit element*. We will express the cohomology class by its Čech cocycle in a sufficiently fine open cover of M. The unit element is determined by the unit Čech cocycle.

Let $\mathbf{E}$ be a holomorphic vector bundle over a complex manifold M and $\mathcal{E}$ be the sheaf of holomorphic sections of $\mathbf{E}$. Then, we can consider the split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$, where $\mathcal{O}_{\mathrm{gr}} = \bigwedge \mathcal{E}$. Let $\operatorname{Aut}\mathbf{E}$ be the group of all automorphisms of the vector bundle $\mathbf{E}$. Clearly, any automorphism of $\mathbf{E}$ gives rise to an automorphism of $(M, \mathcal{O}_{\mathrm{gr}})$, and thus we get a natural inclusion $\operatorname{Aut}\mathbf{E} \subset \operatorname{Aut}(M, \mathcal{O}_{\mathrm{gr}})$. It follows that $\operatorname{Aut}\mathbf{E}$ acts on the sheaves $\mathcal{A}ut_{(2p)}\mathcal{O}_{\mathrm{gr}}$ by the action Int. Hence, this group acts on each 1-cohomology set $H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}_{\mathrm{gr}})$ for $p \geq 0$, leaving the unit element e fixed.

4.1 Theorem (The role of $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$). *To any supermanifold $(M, \mathcal{O})$ that has $(M, \mathcal{O}_{\mathrm{gr}})$ as its retract there corresponds an element of the set $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. This correspondence gives rise to a bijection between the isomorphism classes of supermanifolds, satisfying the above condition, and the orbits of the group* $\operatorname{Aut}\mathbf{E}$ *on $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ under the action* Int *of the group* $\operatorname{Aut}\mathbf{E}$. *The given split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ corresponds to the unit element $e \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$.*

4.2 Corollary (On splitness). *The following conditions are equivalent:*

(1) *The supermanifold is split, i.e., isomorphic to its retract.*

(2) $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}) = \{e\}$.

Let us describe the correspondence mentioned in Theorem 4.1. Let $(M, \mathcal{O})$ be a supermanifold such that $\operatorname{gr}\mathcal{O} = \mathcal{O}_{\mathrm{gr}}$. We can choose an open cover $\mathfrak{U} = (U_i)_{i \in I}$ of M such that the exact sequences (2.13) split over each U_i. Then, we get isomorphisms $\sigma_i : \mathcal{O}|_{U_i} \longrightarrow \mathcal{O}_{\mathrm{gr}}|_{U_i}$, where $i \in I$, inducing the identity isomorphisms of the $\mathbb{Z}$-graded sheaves. Setting $g_{ij} := \sigma_i \sigma_j^{-1}$, we obtain a 1-cocycle $g = (g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. Its cohomology class $\gamma \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ does not depend of the choice of σ_i; this is the class desired.

The above cover $\mathfrak{U}$ can be chosen in such a way that $\mathbf{E}$ is trivial over U_i for any $i \in I$. Then, for any $i \in I$, we have an isomorphism

$$\rho_i : \mathcal{O}_{\mathrm{gr}}|_{U_i} \longrightarrow \bigwedge_{\mathcal{F}_n(x^{(i)})} (\xi_1^{(i)}, \ldots, \xi_m^{(i)})|_{U_i},$$

providing U_i with the local coordinate system $x_1^{(i)}, \dots, x_n^{(i)}, \xi_1^{(i)}, \dots, \xi_m^{(i)}$. For any pair $i,\ j$ such that $U_i \cap U_j \neq \emptyset$ we get the isomorphism

$$\varphi_{ij} = \rho_i \rho_j^{-1} : \bigwedge_{\mathcal{F}_n(x^{(j)})} (\xi_1^{(j)}, \dots, \xi_m^{(j)})|_{U_j} \to \bigwedge_{\mathcal{F}_n(x^{(i)})} (\xi_1^{(i)}, \dots, \xi_m^{(i)})|_{U_i}$$

which is expressed by the transition functions of $(M, \mathcal{O}_{\mathrm{gr}})$. One can ask: "*how to write the transition functions of* $(M, \mathcal{O})$ *in terms of the transition functions of the retract and the cocycle* g?"

To answer this question, we have to consider the isomorphisms

$$\psi_{ij} = \rho_i \sigma_i \sigma_j^{-1} \rho_j^{-1} = \rho_i g_{ij} \rho_j^{-1}.$$

Clearly, $\psi_{ij} = (\rho_i g_{ij} \rho_i^{-1}) \varphi_{ij}$. This means that the transition functions of $(M, \mathcal{O})$ can be obtained from the transition functions of $(M, \mathcal{O}_{\mathrm{gr}})$ by applying the automorphism g_{ij} expressed in terms of the coordinates $x^{(i)},\ \xi^{(i)}$.

4.4 The exponential mapping and its applications In general, to explicitly describe the set $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ is a difficult problem. But we will see below that, under certain strong conditions, this set coincides with the 1-cohomology of a locally free analytic sheaf on M. This simple case is sufficient for further applications.

We will use the linearization method proposed in [47]. Let $(M, \mathcal{O})$ be an arbitrary supermanifold of dimension $n|m$. As in the classical Lie theory, there exists a natural relationship between automorphisms and derivations of the sheaf $\mathcal{O}$. From formula (3.16) we get the filtration

$$\mathcal{T}_{(2)\bar{0}} \supset \mathcal{T}_{(4)\bar{0}} \supset \dots, \tag{4.34}$$

where

$$\mathcal{T}_{(2p)\bar{0}} = \mathcal{T}_{(2p)} \cap \mathcal{T}_{\bar{0}} = \mathcal{T}_{(2p-1)} \cap \mathcal{T}_{\bar{0}} = \{\delta \in \mathcal{T}_{\bar{0}} \mid \delta(\mathcal{O}) \subset \mathcal{J}^{2p}\}.$$

Then, we have the exponential mapping

$$\exp : \mathcal{T}_{(2)\bar{0}} \longrightarrow \mathcal{A}ut_{(2)}\mathcal{O}.$$

It is expressed by the usual exponential series which is actually a polynomial, since $v^k = 0$ for any $v \in \mathcal{T}_{(2)\bar{0}}$ and any $k > \left[\frac{m}{2}\right]$. One proves that exp is bijective [7] and maps $\mathcal{T}_{(2p)\bar{0}}$ onto $\mathcal{A}ut_{(2p)}\mathcal{O}$ for any $p = 1, 2. \dots$. Thus, it is an isomorphism of sheaves of sets (but in general not of groups). We denote $\log := \exp^{-1}$.

4.3 Proposition (Necessary conditions of splitness). *For any* $p \geq 1$*, there is the following exact sequence of the sheaves of groups:*

$$0 \longrightarrow \mathcal{A}ut_{(2p+2)}\mathcal{O} \longrightarrow \mathcal{A}ut_{(2p)}\mathcal{O} \overset{\lambda_{2p}}{\to} (\mathcal{T}_{\mathrm{gr}})_{2p} \longrightarrow 0, \tag{4.35}$$

where $\mathcal{T}_{\mathrm{gr}} = \mathcal{D}er\, \mathcal{O}_{\mathrm{gr}}$ *is the* $\mathbb{Z}$*-graded tangent sheaf of* $(M, \mathcal{O}_{\mathrm{gr}})$ *and* λ_p *is the composition of the following mappings:*

$$\lambda_{2p} : \mathcal{A}ut_{(2p)}\mathcal{O} \overset{\log}{\longrightarrow} \mathcal{T}_{(2p)\bar{0}} \overset{\pi_p}{\longrightarrow} \mathcal{T}_{(2p)\bar{0}}/\mathcal{T}_{(2p+2)\bar{0}} \overset{h_p}{\to} (\mathcal{T}_{\mathrm{gr}})_{2p},$$

π_p being the canonical projection and h_p the natural isomorphism implied by Proposition 3.1. If $(M, \mathcal{O}) = (M, \mathcal{O}_{\mathrm{gr}})$ is split, then λ_{2p} maps any germ $\exp u \in \mathcal{A}ut_{(2p)}\mathcal{O}$ onto the $(2p)$-component of $u \in \mathcal{T}_{(2p)}$.

Proof. Consider the mapping $\widetilde{\lambda}_{2p} = \pi_p \log$. Using the Campbell–Hausdorff formula, we get

$$\widetilde{\lambda}_{2p}((\exp u)(\exp v)) = \widetilde{\lambda}_{2p}(\exp(u + v + \frac{1}{2}[u, v] + \dots)) = \pi_p(u) + \pi_p(v) = \\ = \widetilde{\lambda}_{2p}(\exp u) + \widetilde{\lambda}_{2p}(\exp v).$$

Hence, $\widetilde{\lambda}_{2p}$ is a homomorphism of sheaves of groups. Clearly, $\operatorname{Ker} \widetilde{\lambda}_{2p} = \mathcal{A}ut_{(2p+2)}\mathcal{O}$, and we get the exact sequence of sheaves of groups

$$0 \longrightarrow \mathcal{A}ut_{(2p+2)}\mathcal{O} \longrightarrow \mathcal{A}ut_{(2p)}\mathcal{O} \xrightarrow{\widetilde{\lambda}_{2p}} \mathcal{T}_{(2p)\bar{0}}/\mathcal{T}_{(2p+2)\bar{0}} \longrightarrow 0. \tag{4.36}$$

Clearly, $\mathcal{T}_{(2p+2)\bar{0}} = \mathcal{T}_{(2p+1)\bar{0}}$. Using Proposition 3.1, we get

$$\mathcal{T}_{(2p)\bar{0}}/\mathcal{T}_{(2p+2)\bar{0}} \simeq \mathcal{T}_{(2p)\bar{0}}/\mathcal{T}_{(2p+1)\bar{0}} \simeq (\mathcal{T}_{\mathrm{gr}})_{2p}.$$

Now the sequence (4.35) follows from the sequence (4.36). □

4.4. Lemma. *For any $p \geq 2$, if $H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p}) = 0$, then $H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) = \{e\}$.*

Proof. We will use the induction on p. Clearly, the claim is true for all p sufficiently big. We have to prove that if it is true for a certain $p \geq 3$, then it is true for $p - 1$ as well. The exact sequence (4.35) gives the cohomology exact sequence (see [19])

$$H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) \longrightarrow H^1(M, \mathcal{A}ut_{(2p-2)}\mathcal{O}) \xrightarrow{\lambda^*_{2p-2}} H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p-2}).$$

Clearly, $H^1(M, \mathcal{A}ut_{(2p-2)}\mathcal{O}) = \{e\}$ follows from

$$H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p-2}) = 0, \quad H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) = \{e\}.$$

□

4.5. Proposition. *Suppose that $H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p}) = 0$ for any $p \geq 2$. Then, the mapping*

$$\lambda^*_2 : H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}) \to H^1(M, (\mathcal{T}_{\mathrm{gr}})_2)$$

is injective.

Proof. The sequence (4.35) for the sheaf $\mathcal{O} = \mathcal{O}_{\mathrm{gr}}$ gives the cohomology exact sequence

$$H^1(\mathcal{A}ut_{(4)}\mathcal{O}_{\mathrm{gr}}) \longrightarrow H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}) \xrightarrow{\lambda_2^*} H^1(M, (\mathcal{T}_{\mathrm{gr}})_2) \tag{4.37}$$

Suppose that $\gamma, \eta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ and that $\lambda_2^*(\gamma) = \lambda_2^*(\eta)$. Let γ be determined by the cocycle (g_{ij}) and η by the cocycle (h_{ij}) in a cover $\mathfrak{U} = (U_i)_{i\in I}$ of M. Then, our assumption implies that

$$\lambda_2(g_{ij}) = \lambda_2(h_{ij}) + c_j - c_i, \text{ where } c_i \in ((\mathcal{T}_{\mathrm{gr}})_2)_{U_i}.$$

We can assume that $c_i = \lambda_2(g_i)$, where $g_i \in (\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})_{U_i}$. Then, $\lambda_2(g_i g_{ij} g_j^{-1}) = \lambda_2(h_{ij})$. Thus, we can suppose from the beginning that $\lambda_2(g_{ij}) = \lambda_2(h_{ij})$.

Consider the cochain $f \in C^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$, given by the formula $f_{ij} = h_{ij} g_{ij}^{-1}$. Then, $\lambda_2(f) = 0$.

Let $(M, \mathcal{O})$ be the supermanifold corresponding to the cohomology class of γ due to Theorem 4.1. Then, $g_{ij} = h_i h_j^{-1}$, where $h_i : \mathcal{O}|_{U_i} \longrightarrow \mathcal{O}_{\mathrm{gr}}|_{U_i}$ for $i \in I$, are certain isomorphisms of sheaves of superalgebras inducing the identity mappings on $\mathcal{O}_{\mathrm{gr}}|_{U_i}$. The equalities $h_{ij} = f_{ij} g_{ij} = f_{ij} h_i h_j^{-1}$ and $h_{ij} h_{jk} = h_{ik}$ imply that

$$f_{ij} h_i h_j^{-1} f_{jk} h_j h_k^{-1} = f_{ik} h_i h_k^{-1}$$

or

$$(h_i^{-1} f_{ij} h_i)(h_j^{-1} f_{jk} h_j) = h_i^{-1} f_{ik} h_i.$$

Clearly, $\lambda_2(h_i^{-1} f_{ij} h_i) = 0$, whence $(h_i^{-1} f_{ij} h_i) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(4)}\mathcal{O})$. Therefore,

$$(h_i^{-1} f_{ij} h_i) \in Z^1(M, \mathcal{A}ut_{(4)}\mathcal{O}).$$

By Lemma 4.4, this latter cocycle is cohomologous to e, i.e.,

$$h_i^{-1} f_{ij} h_i = u_i u_j^{-1}, \text{ where } u_i \in (\mathcal{A}ut_{(4)}\mathcal{O})_{U_i}.$$

Thus, $f_{ij} = h_i u_i u_j^{-1} h_i^{-1}$. It follows that

$$\begin{aligned} h_{ij} = f_{ij} g_{ij} = h_i u_i u_j^{-1} h_i^{-1} g_{ij} = h_i u_i u_j^{-1} h_j^{-1} = (h_i u_i h_i^{-1})(h_i h_j^{-1})(h_j u_j^{-1} h_j^{-1}) = \\ = v_i g_{ij} v_j^{-1}, \end{aligned}$$

where $v_i = h_i u_i h_i^{-1} \in (\mathcal{A}ut_{(4)}\mathcal{O}_{\mathrm{gr}})_{U_i}$. This implies that $\gamma = \eta$. □

4.6. Theorem. *Suppose that* $H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p}) = H^2(M, (\mathcal{T}_{\mathrm{gr}})_{2p}) = 0$ *for any* $p \geq 2$. *Then, the mapping* $\lambda_2^* : H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}) \longrightarrow H^1(M, (\mathcal{T}_{\mathrm{gr}})_2)$ *is bijective.*

Proof. The injectivity follows from Proposition 4.5, while the surjectivity is implied by Theorem 3 of [47]. □

To calculate the quotient of $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ by $\operatorname{Aut}\mathbf{E}$, the following assertion is useful. For any $c \in \mathbb{C}^\times$, denote by A_c the automorphism of $\mathbf{E}$ given by the multiplication by the scalar c.

4.7 Lemma (Technical). *We have*

$$\lambda_2^* \circ \operatorname{Int} A_c = c^2 \lambda_2^*, \ \textit{where } c \in \mathbb{C}^\times.$$

Proof. Consider the grading vector field $\varepsilon \in \Gamma(M, (\mathcal{T}_{\mathrm{gr}})_0)$ on $(M, \mathcal{O})$. Clearly, A_c gives rise to the automorphism $\alpha_c = \exp(b\varepsilon)$, where $c = \exp b$, of the structure sheaf $\mathcal{O}_{\mathrm{gr}}$. Let $g = (g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ be a cocycle of a cover $\mathfrak{U} = (U_i)_{i\in I}$ of M and $w_{ij} = \log g_{ij}$. Then,

$$\begin{aligned}\alpha_c g_{ij} \alpha_c^{-1} &= (\exp(b\varepsilon))(\exp w_{ij})(\exp(b\varepsilon))^{-1} = \exp((\operatorname{Ad}\exp(b\varepsilon))(w_{ij})) = \\ &= \exp((\exp(b \operatorname{ad}\varepsilon))(w_{ij})) = \exp(w_{ij} + b[\varepsilon, w_{ij}] + \frac{1}{2!}b^2[\varepsilon,[\varepsilon,w_{ij}]] + \dots).\end{aligned}$$

Applying λ_2 and denoting the 2-component of the vector field w_{ij} by $w_{ij}^{(2)}$, we get

$$\lambda_2(\alpha_c g_{ij}\alpha_c^{-1}) = w_{ij}^{(2)} + 2bw_{ij}^{(2)} + \frac{4b^2}{2!}w_{ij}^{(2)} + \dots = (\exp 2b)w_{ij}^{(2)} = c^2\lambda_2(g_{ij}).$$

Thus, $\lambda_2^*((\operatorname{Int}\alpha_c)\gamma) = c^2\lambda_2^*(\gamma)$, where γ is the cohomology class of g. □

This yields, in particular, the following simple fact.

4.8 Proposition (Uniqueness of non-split supermanifold with given retract). *Suppose that*

$$\begin{gathered} H^1(M, (\mathcal{T}_{\mathrm{gr}})_2) \simeq \mathbb{C}, \\ H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p}) = 0 \textit{ for any } p \geq 2, \\ H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}) \neq \{e\}. \end{gathered}$$

Then, λ_2^* is bijective and $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})/\operatorname{Aut}\mathbf{E}$ consists of two elements. Thus, there exists precisely one non-split supermanifold having $(M, \mathcal{O}_{\mathrm{gr}})$ as its retract.

Proof. By Proposition 4.5, $\lambda_2^* : H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}) \longrightarrow \mathbb{C}$ is injective. Therefore, $\operatorname{Im}\lambda_2^*$ contains a non-zero element, and Lemma 4.7 implies that λ_2^* is surjective. By the same lemma, the group $\operatorname{Aut}\mathbf{E}$ has precisely two orbits on $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. □

4.9. Remark. The conditions of Theorem 4.6 are fulfilled, in particular, if $m = 2$ or 3. The corresponding special cases were proved in [30, Ch. 4], and [7], respectively. In the general case, the class $\lambda_2^*(\gamma)$ is closely related to the first obstruction to splitting the sequences (2.13) considered in [30, Ch. 4]. If $\lambda_2^*(\gamma) = 0$, then $\gamma \in \operatorname{Im} H^1(M, \mathcal{A}ut_{(4)})$, and we can apply λ_4^*, and so on. The resulting obstruction theory is discussed in [2], [10], [30], [45]. On the other hand, any non-split supermanifold can be regarded as a result of deformation of its retract, and λ_2^* can be interpreted as the corresponding Kodaira–Spencer mapping (for details, see [10], [11]).

4.11 A family of non-split supermanifolds with retract (M, Ω) Here we consider the case where $\mathcal{O}_{\mathrm{gr}}$ is the sheaf of holomorphic forms Ω on M. Using closed (1,1)-forms on M, we will construct an abelian subsheaf of the sheaf of groups $\mathcal{A}ut_{(2)}\Omega$. The 1-cohomology of this subsheaf determines a family of supermanifolds with retract (M, Ω). This family is non-trivial whenever M is a compact Kähler manifold of $\dim M > 1$ and $H^{1,1}(M, \mathbb{C}) \neq 0$.

Let $\mathcal{Z}\Omega^1$ denote the subsheaf of Ω^1 consisting of closed forms. Consider the following sequence of sheaves and their homomorphisms:

$$\mathcal{Z}\Omega^1 \xrightarrow{\beta} \Omega^1 \xrightarrow{\nu} \mathcal{T}_2 \overset{\exp}{\to} \mathcal{A}ut_{(2)}\Omega,$$

where β is the identical inclusion and ν is given by the formula

$$\nu(\psi) = \psi d. \tag{4.38}$$

We claim that the composition mapping $\mu : \mathcal{Z}\Omega^1 \longrightarrow \mathcal{A}ut_{(2)}\Omega$ is a homomorphism of sheaves of groups. By formula (4.38), we see that $\mu(\psi) = \exp(\psi d)$. Clearly, for any $\psi_1, \psi_2 \in \mathcal{Z}\Omega^1$, we have $(\psi_1 d)(\psi_2 d) = 0$. Therefore,

$$\mu(\psi) = \mathrm{id} + \psi d,$$

and

$$\mu(\psi_1 + \psi_2) = \mu(\psi_1)\mu(\psi_2).$$

It follows that we have the cohomology mapping taking 0 to the unit element

$$\mu^* : H^1(M, \mathcal{Z}\Omega^1) \longrightarrow H^1(M, \mathcal{A}ut_{(2)}\Omega).$$

Consider the homomorphism of sheaves of groups $\lambda_2 : \mathcal{A}ut_{(2)}\Omega \longrightarrow \mathcal{T}_2$ defined in Proposition 4.3.

4.10 Proposition (Technical). *The relation $\lambda_2\mu = \nu\beta$ holds. Suppose that $\dim M > 1$. If $\mu^*(\zeta) = \mu^*(\zeta')$ for some $\zeta, \zeta' \in H^1(M, \mathcal{Z}\Omega^1)$, then $\beta^*(\zeta) = \beta^*(\zeta')$.*

Proof. Since $\lambda_2 \exp = \mathrm{id}$ on $\mathcal{T}_2$, we see that $\lambda_2\mu = \nu\beta$. Now, it follows from Proposition 3.2 that $\nu\beta = lj\beta$, where $j : \Omega^1 \to \Omega^2 \otimes \Theta = \mathcal{H}om(\Omega^1, \Omega^2)$ is given by the formula

$$j(\psi)(\alpha) = \psi\alpha. \tag{4.39}$$

Thus, $\lambda_2\mu = lj\beta$, whence $\lambda_2^*\mu^* = l^*j^*\beta^*$.

Suppose that $\dim M > 1$. We claim that l^* and j^* are injective. Indeed, both l and j are injections onto a direct summand. In the first case this follows from formula (3.26). Further, if $\dim M > 1$, then $j(\Omega^1)$ admits a direct complement in $\Omega^2 \otimes \Theta$, namely, the kernel of the contraction mapping $c : \Omega^2 \otimes \Theta \longrightarrow \Omega^1$ (see formula (2.9)). Thus, l^*j^* is injective, which implies our second assertion. □

Let $\mathfrak{U} = (U_i)$ be an open cover of M and let $\psi = (\psi_{ij}) \in Z^1(\mathfrak{U}, \mathcal{Z}\Omega^1)$. Then, the above construction assigns to ψ the supermanifold given by the cocycle

$$g = (g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\Omega), \quad \text{where} \quad g_{ij} = \mathrm{id} + \psi_{ij} d. \tag{4.40}$$

Suppose that $\dim M > 1$. Due to Theorem 4.1, we see from Proposition 4.10 that this supermanifold is non-split whenever the cohomology class of ψ in $H^1(M, \Omega^1)$ is non-zero.

Now we pass to an important case, where a "closed cocycle" ψ appears. Let ω be a (1,1)-form on M satisfying $d\omega = 0$. Then, clearly, $\bar{\partial}\omega = 0$, and by the Dolbeault theorem ω determines a cohomology class in $H^1(M, \Omega^1)$. It turns out that it can be given by a closed Čech cocycle.

4.11 Lemma (Technical). *We have the exact sequence of sheaves:*

$$0 \longrightarrow \mathcal{Z}\Omega^1 \longrightarrow \Phi^{1,0}_{\partial} \xrightarrow{\bar{\partial}} \mathcal{Z}\Phi^{1,1} \longrightarrow 0, \tag{4.41}$$

where $\Phi^{1,0}_{\partial} \subset \Phi^{1,0}$ *is the subsheaf of* $\bar{\partial}$*-closed* $(1,0)$*-forms and* $\mathcal{Z}\Phi^{1,1} \subset \Phi^{1,1}$ *the subsheaf of* d*-closed* $(1,1)$*-forms.*

Proof. Clearly, we have the following exact sequence:

$$0 \longrightarrow \Omega^1 \longrightarrow \Phi^{1,0} \xrightarrow{\bar{\partial}} \Phi^{1,1}_{\bar{\partial}} \longrightarrow 0,$$

where $\Phi^{1,1}_{\bar{\partial}} \subset \Phi^{1,1}$ is the subsheaf of $\bar{\partial}$-closed $(1,1)$-forms. By definition, $\mathcal{Z}\Omega^1 = \Omega^1 \cap \Phi^{1,0}_{\partial}$. Therefore, we only have to prove that $\bar{\partial}(\Phi^{1,0}_{\partial}) = \mathcal{Z}\Phi^{1,1}$.

If $\varphi \in \Phi^{1,0}$ and $\partial\varphi = 0$, then $d\bar{\partial}\varphi = \frac{\partial}{\partial\bar{\partial}}\varphi = -\bar{\partial}\partial\varphi = 0$. Conversely, suppose that the form $\bar{\partial}\varphi$, where $\varphi \in \Phi^{1,0}$, satisfies $d\bar{\partial}\varphi = 0$. Then, $\bar{\partial}\partial\varphi = 0$. Since $\partial\varphi \in \Phi^{2,0}$, this form is holomorphic and closed. Therefore, $\partial\varphi = \partial\varphi_1$, where $\varphi_1 \in \Omega^1$. Hence, $\varphi - \varphi_1 \in \Phi^{1,0}_{\partial}$, and $\bar{\partial}\varphi = \bar{\partial}(\varphi - \varphi_1)$. □

4.14 An isomorphism *D* Now, consider the cohomology exact sequence, corresponding to (4.41):

$$\Gamma(M, \Phi^{1,0}_{\partial}) \xrightarrow{\bar{\partial}} \Gamma(M, \mathcal{Z}\Phi^{1,1}) \xrightarrow{\delta^*} H^1(M, \mathcal{Z}\Omega^1).$$

Using δ^*, we get the mapping

$$\mu^*\delta^* : \Gamma(M, \mathcal{Z}\Phi^{1,1}) \longrightarrow H^1(M, \mathcal{A}ut_{(2)}\Omega).$$

Thus, any $(1,1)$-form ω on M such that $d\omega = 0$ determines a supermanifold with retract (M, Ω). To obtain an expression of the corresponding cocycle g, we have to find a cocycle ψ determining $\delta^*(\omega)$. Consider an open cover $\mathfrak{U} = (U_i)$ of M such that $\omega = \bar{\partial}\psi_i$ in any U_i, where $\psi_i \in \Phi^{1,0}_{\partial}(U_i)$. By definition of the connecting homomorphism δ^*, the desired cocycle is $\psi = (\psi_{ij}) \in Z^1(\mathfrak{U}, \mathcal{Z}\Omega^1)$, where $\psi_{ij} = \psi_j - \psi_i$ in $U_i \cap U_j \neq \emptyset$. Finally, the cocycle g is given by the formula (4.40).

Note that any $\omega \in \Gamma(M, \mathcal{Z}\Phi^{1,1})$ satisfies the condition $\bar{\partial}\omega = 0$, and hence determines an element $[\omega]$ of the Dolbeault cohomology group

$$H^{1,1}(M, \mathbb{C}) = \Gamma(M, \mathcal{Z}\Phi^{1,1})/\bar{\partial}\Gamma(M, \Phi^{1,0}).$$

Further, by the Dolbeault theorem, we have an isomorphism

$$D : H^{1,1}(M, \mathbb{C}) \to H^1(M, \Omega^1).$$

4.12 Proposition (An isomorphism D). *We have*

$$D([\omega]) = \beta^*\delta^*\omega, \quad \textit{for any} \;\; \omega \in \Gamma(M, \mathcal{Z}\Phi^{1,1}).$$

If $\dim M > 1$*, then, for any two* $\omega, \omega' \in \Gamma(M, \mathcal{Z}\Phi^{1,1})$*, the equation* $\mu^*\delta^*\omega = \mu^*\delta^*\omega'$ *implies* $[\omega] = [\omega']$*. In particular, the supermanifold corresponding to* $[\omega]$ *is non-split whenever* $[\omega] \neq 0$*.*

Any Kähler form ω *on a compact manifold* M *of dimension* > 1 *determines a non-split supermanifold with retract* (M, Ω)*.*

Proof. The usual proof of the Dolbeault theorem (see, e.g., [18]) shows that $D([\omega])$ is the cohomology class of the cocycle (ψ) described above. Thus, $D([\omega]) = \beta^*\delta^*\omega$. If $\mu^*\delta^*\omega = \mu^*\delta^*\omega'$, then $\beta^*\delta^*\omega = \beta^*\delta^*\omega'$ by Proposition 4.10, and hence $D([\omega]) = D([\omega'])$, and $[\omega] = [\omega']$.

If M is compact and ω is a Kähler form, then the de Rham class of ω is non-zero. Since M is Kähler, this implies that $[\omega] \neq 0$. □

The situation is much more simple in the case where M is a compact Kähler manifold.

4.13 Theorem (M is a compact Kähler manifold). *If* M *is a compact Kähler manifold, then we have a linear mapping*

$$\hat{\delta} : H^{1,1}(M, \mathbb{C}) \longrightarrow H^1(M, \mathcal{Z}\Omega^1)$$

such that $\beta^*\hat{\delta} = D$*. If* $\dim M > 1$*, then the mapping*

$$\mu^*\hat{\delta} : H^{1,1}(M, \mathbb{C}) \to H^1(M, \mathcal{A}ut_{(2)}\Omega)$$

is injective.

Proof. Since M is compact Kähler, any cohomology class in $H^{1,1}(M, \mathbb{C})$ contains a closed $(1,1)$-form ω, e.g., a harmonic one. We set

$$\hat{\delta}([\omega]) := \delta^*(\omega).$$

To check the correctness of this definition, consider a closed form

$$\omega' = \omega + \bar{\partial}\alpha, \text{ where } \alpha \in \Gamma(M, \Phi^{0,1}).$$

Then, by $\partial\bar{\partial}$-Lemma (see [18]), $\omega' - \omega = \bar{\partial}\partial\varphi$, where φ is a C^∞ function on M. If $\omega = \bar{\partial}\psi_i$ in U_i, then $\omega' = \bar{\partial}(\psi_i + \partial\varphi)$. Now it is clear that $\delta^*(\omega) = \delta^*(\omega')$. By Proposition 4.12, $\beta^*\hat{\delta} = D$. The injectivity of $\mu^*\hat{\delta}$ follows from the same proposition. □

We return to the sheaf homomorphism $j : \Omega^1 \longrightarrow \Omega^2 \otimes \Theta$ defined by the formula (4.39). Due to the formula (2.8), j can be also expressed by

$$j(\psi)(u_1, u_2) = \psi(u_1)u_2 - \psi(u_2)u_1 \quad \text{for any} \quad u_1, u_2 \in \Theta. \tag{4.42}$$

We want to express the corresponding homomorphism $j^* : H^1(M, \Omega^1) \to H^1(M, \Omega^2 \otimes \Theta)$ in terms of differential forms. Together with the Dolbeault resolution of Ω^1, we use the Dolbeault–Serre resolution $(\Phi^{2,*} \otimes \Theta, \bar{\partial})$ of $\Omega^2 \otimes \Theta$ formed by smooth vector-valued forms. Clearly, j extends to the homomorphism of resolutions

$$\widetilde{\jmath} = \mathrm{id} \otimes j : \Phi^{0,*} \otimes \Omega^1 \longrightarrow \Phi^{0,*} \otimes \Omega^2 \otimes \Theta.$$

Identifying $\Phi^{0,1} \otimes \Omega^1$ with $\Phi^{1,1}$ and $\Phi^{0,1} \otimes \Omega^2 \otimes \Theta$ with $\Phi^{2,1} \otimes \Theta$, respectively, we see from formula (4.40) that $\widetilde{\jmath} : \Phi^{1,1} \longrightarrow \Phi^{2,1} \otimes \Theta$ is expressed by

$$\widetilde{\jmath}(\omega)(u_1, u_2, v) = \omega(u_1, v)u_2 - \omega(u_2, v)u_1, \quad \text{where} \quad u_1, u_2 \in \Theta,\ v \in \bar{\Theta}. \tag{4.43}$$

This implies the following Proposition.

4.14 Proposition (A useful formula). *For any $\bar{\partial}$-closed form $\omega \in \Gamma(M, \Phi^{1,1})$, the class $j^*(D[\omega]) \in H^1(M, \Omega^2 \otimes \Theta)$ is determined by the $\bar{\partial}$-closed form $\widetilde{\jmath}(\omega)$ given by the formula (4.43).*

Now, we apply our construction to the *canonical form* ω defined by Koszul, see [28]. Let V be a volume form on a complex manifold M. Then, one associates with V a closed (1,1)-form ω in the following way. Let $\mathfrak{U} = (U_i)$ be a coordinate cover of M and $x_1^{(i)}, \dots, x_n^{(i)}$ holomorphic coordinates in U_i. Then, in any U_i, we have

$$V = V_i dx_1^{(i)} \dots dx_n^{(i)} d\overline{x}_1^{(i)} \dots d\overline{x}_n^{(i)},$$

where V_i is a positive C^∞ function in U_i, unique up to a constant factor and independent of i. Denote

$$J_{ij} := \frac{D(x_1^{(i)}, \dots, x_n^{(i)})}{D(x_1^{(j)}, \dots, x_n^{(j)})},$$

then

$$V_j = |J_{ij}|^2 V_i \quad \text{in } U_i \cap U_j.$$

The canonical form ω is defined by the formula

$$\omega = \bar{\partial}\partial \log V_i \quad \text{in} \quad U_i$$

(this definition differs by a sign from that due to Koszul). Clearly, $d\omega = 0$.

4.15 Theorem (The canonical supermanifold). *The supermanifold with retract (M, Ω), corresponding to the canonical form ω, does not depend of the choice of V. It is determined by the following cocycle $g \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\Omega)$:*

$$g_{ij} = \mathrm{id} + \frac{1}{J_{ij}}(dJ_{ij})d. \tag{4.44}$$

Proof. The cocycle ψ corresponding to $[\omega]$ has the form

$$\psi_{ij} = \partial \log V_j - \partial \log V_i = \partial \log \frac{V_j}{V_i} = \partial \log |J_{ij}|^2 = d \log J_{ij} = \frac{1}{J_{ij}}(dJ_{ij}).$$

This implies our assertion. □

The supermanifold, described in Theorem 4.15, will be called the *canonical supermanifold*, corresponding to M. It is not necessarily non-split.

4.19 Lifting of vector fields Let $(M, \mathcal{O})$ be a supermanifold having $(M, \mathcal{O}_{\mathrm{gr}})$ as its retract. The filtration (3.16) of $\mathcal{T} = \mathcal{D}er\, \mathcal{O}$ gives rise to the filtration

$$\mathfrak{v}(M, \mathcal{O}) = \mathfrak{v}(M, \mathcal{O})_{(-1)} \supset \mathfrak{v}(M, \mathcal{O})_{(0)} \supset \mathfrak{v}(M, \mathcal{O})_{(1)} \supset \dots,$$

where $\mathfrak{v}(M, \mathcal{O})_{(p)} = \Gamma(M, \mathcal{T}_{(p)})$. By Proposition 3.1, we get the exact sequences

$$0 \longrightarrow \mathfrak{v}(M, \mathcal{O})_{(p+1)} \longrightarrow \mathfrak{v}(M, \mathcal{O})_{(p)} \xrightarrow{\sigma_p} \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_p \text{ for } p \geq -1. \tag{4.45}$$

We say that a vector field $u \in \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_p$ *lifts to* $(M, \mathcal{O})$, if u belongs to $\mathrm{Im}\, \sigma_p$. In this case, one can suppose that $u = \sigma_p(\hat{u})$, where $\hat{u}$ has the same parity as p. We are going to express this property in cohomological terms. Let $(M, \mathcal{O})$ be determined by a class $\gamma \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$.

Suppose that we have an open cover $\mathfrak{U} = (U_i)_{i \in I}$ of M and a system of isomorphisms of the sheaves of superalgebras $f_i : \mathcal{O}|_{U_i} \to \mathcal{O}_{\mathrm{gr}}|_{U_i}$ such that $f_i(\varphi) = \varphi + \mathcal{J}^{q+1} \in (\mathcal{O}_{\mathrm{gr}})_q$ for $\varphi \in \mathcal{J}^q$. Then, $g = (g_{ij})$, where $g_{ij} = f_i f_j^{-1}$, is a 1-cocycle defining γ.

4.16 Proposition (Conditions on lifting). *A vector field $v \in \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_p$ lifts to $(M, \mathcal{O})$ if and only if there exists a 0-cochain $(v_i) \in C^0(M, (\mathcal{T}_{\mathrm{gr}})_{(p)})$ such that*

$$v_i \equiv v \bmod (\mathcal{T}_{\mathrm{gr}})_{(p+1)}(U_i), \tag{4.46}$$

$$g_{ij} v_j = v_i g_{ij} \text{ in } U_i \cap U_j \neq \emptyset. \tag{4.47}$$

In this case, we have

$$[\lambda_2^*(\gamma), v] = 0. \tag{4.48}$$

Proof. Suppose that v lifts to $(M, \mathcal{O})$ and $\hat{v} \in \mathfrak{v}(M, \mathcal{O})_{(p)}$ satisfies $\sigma_p(\hat{v}) = v$.

Define $v_i \in (\mathcal{T}_{\mathrm{gr}})_{(p)}(U_i)$ by the formula $v_i = f_i \hat{v} f_i^{-1}$. Clearly, v_i satisfies condition (4.47). Now, for any $\varphi \in \mathcal{J}^q$, denote $\varphi_i := f_i^{-1}(\varphi + \mathcal{J}^{q+1}) \in \mathcal{J}^q$. Then, $\hat{v}(\varphi_i) = g_i + h_i$, where $g_i \in f_i^{-1}(\mathrm{Gr}^{p+q}(\mathcal{O}))$ and $h_i \in f_i^{-1}(\mathrm{gr}\, \mathcal{O})_{(p+q+1)}) = \mathcal{J}^{(p+q+1)}$. Hence,

$$v_i(\varphi + \mathcal{J}^{q+1}) = (f_i \hat{v} f_i^{-1})(\varphi + \mathcal{J}^{q+1}) = f_i \hat{v}(\varphi_i) \equiv f_i(g_i) \bmod (\mathrm{gr}\, \mathcal{O})_{(p+q+1)}.$$

On the other hand, we have by definition

$$v(\varphi + \mathcal{J}^{q+1}) = \hat{v}(\varphi_i) + \mathcal{J}^{p+q+1} = g_i + \mathcal{J}^{p+q+1} = f_i(g_i).$$

Thus, condition (4.46) is proved.

Conversely, suppose a cochain $(v_i) \in C^0(M, (\mathcal{T}_{\mathrm{gr}})_{(p)})$ satisfying conditions (4.46) and (4.47) be given. By condition (4.47), we have

$$f_j^{-1} v_j f_j = f_i^{-1} g_{ij} v_j f_j = f_i^{-1} v_i g_{ij} f_j = f_i^{-1} v_i f_i.$$

Then, $\hat{v} = f_i^{-1} v_i f_i$ is a global section of $\mathcal{T}_{(p)}$. For any $\varphi \in \mathcal{J}^q$, we have

$$\hat{v}(\varphi) = (f_i^{-1} v_i f_i)(\varphi) = f_i^{-1} v_i(\varphi + \mathcal{J}^{q+1}) = f_i^{-1}(v(\varphi + \mathcal{J}^{q+1}) + \psi),$$

where $\psi \in (\mathrm{gr}\,\mathcal{O})_{(p+q+1)}$. It follows that $\hat{v}(\varphi)$ lies in $v(\varphi + \mathcal{J}^{q+1}) \in \mathcal{J}^{p+q}/\mathcal{J}^{p+q+1}$.

Thus, $\hat{v}(\varphi) + \mathcal{J}^{p+q+1} = v(\varphi + \mathcal{J}^{q+1})$, and hence $\sigma_p(\hat{v}) = v$.

To prove formula (4.48), we denote $w_{ij} = \log g_{ij}$ and deduce from condition (4.47) that

$$\begin{aligned} v_i &= g_{ij} v_j g_{ij}^{-1} = (\exp w_{ij}) v_j (\exp w_{ij})^{-1} = \mathrm{Ad}_{\exp w_{ij}} v_j \\ &= \exp(\mathrm{ad}_{w_{ij}})(v_j) = v_j + [w_{ij}, v_j] + \tfrac{1}{2!}[w_{ij}, [w_{ij}, v_j]] + \dots . \end{aligned} \tag{4.49}$$

Write $v_i = v_i^{(p)} + v_i^{(p+2)} + \dots$, where $v_i^{(k)} \in (\mathcal{T}_{\mathrm{gr}})_k(U_i)$. By condition (4.46), $v_i^{(p)} = v$. Then, formula (4.49) implies that $v_i^{(p+2)} = v_j^{(p+2)} + [\lambda_2(g)_{ij}, v]$. Thus, formula (4.48) is proved. □

Now, we return to the case where $\mathcal{O}_{\mathrm{gr}} = \Omega$ and $(M, \mathcal{O})$ is determined by a class $\zeta \in H^1(M, \mathcal{Z}\Omega^1)$ as in Theorem 4.13. Let a vector field $u \in \mathfrak{v}(M, \Omega)_p$ be given. We would like to know, whether u lifts to a vector field on the supermanifold $(M, \mathcal{O})$. This problem will be studied in the following three cases: $u = d$, and $u = l(v)$ as well as $u = i(v)$, where $v \in \mathfrak{v}(M)$.

Denote a cocycle representing the class ζ by $\psi = (\psi_{ij}) \in Z^1(\mathfrak{U}, \mathcal{Z}\Omega^1)$. Then, $(M, \mathcal{O})$ corresponds to the cohomology class $\gamma = \mu^*(\zeta)$ of the cocycle $g = (g_{ij})$ given by the formula (4.46). We can suppose that there exist isomorphisms $f_i : \mathcal{O}|_{U_i} \longrightarrow \Omega|_{U_i}$ for $i \in I$, inducing the identity isomorphisms of the $\mathbb{Z}$-graded sheaves and such that $g_{ij} = f_i f_j^{-1}$ over $U_i \cap U_j \neq \emptyset$.

4.17 Proposition (The lift of d). *The derivation $d \in \mathfrak{v}(M, \Omega)_1$ always lifts to $(M, \mathcal{O})$.*

Proof. We have

$$g_{ij} d g_{ij}^{-1} = (\mathrm{id} + \psi_{ij} d) d (\mathrm{id} - \psi_{ij} d) = d.$$

Applying Proposition 4.16 (with $v = v_i = d$), we get our assertion. □

4.18 Proposition (Technical). *If $u \in \mathfrak{v}(M)$ and $l(u)^*(\zeta) = 0$, then $l(u)$ lifts to $(M, \mathcal{O})$.*

Proof. We can assume that $l(u)(\psi_{ij}) = \alpha_j - \alpha_i$ in $U_i \cap U_j$, where $\alpha_i \in \mathcal{Z}\Omega^1(U_i)$. Set

$$v_i = l(u) + \alpha_i d.$$

Then,

$$\begin{aligned} g_{ij}v_jg_{ij}^{-1} &= (\mathrm{id} + \psi_{ij}d)v_j(\mathrm{id} - \psi_{ij}d) = v_j + [\psi_{ij}d, v_j] - \psi_{ij}dv_j(\psi_{ij}d) \\ &= v_j + [\psi_{ij}d, v_j] = l(u) + \alpha_j d - l(u)(\psi_{ij})d = l(u) + \alpha_i d = v_i. \end{aligned}$$

Thus, Proposition 4.16 can be applied. □

4.19. Corollary. *[The lift of $u \in \mathfrak{v}(M)$ on the canonical M] If $(M, \mathcal{O})$ is the canonical supermanifold, then $l(u)$ lifts to $(M, \mathcal{O})$ for any $u \in \mathfrak{v}(M)$.*

Proof. We want to prove that, if $\psi_{ij} = d \log J_{ij}$, then $l(u)$ satisfies the condition of Proposition 4.18 (see (4.44)). Denoting $w_i := dx_1^{(i)} \dots dx_n^{(i)}$, we have

$$w_i = J_{ij}w_j \text{ in } U_i \cap U_j.$$

Applying $l(u)$, we get

$$l(u)(w_i) = l(u)(J_{ij})w_j + J_{ij}l(u)(w_j).$$

Clearly, $l(u)(w_i) = \varphi_i w_i$, where $\varphi_i \in \mathcal{F}(U_i)$. It follows that

$$\varphi_i = \frac{1}{J_{ij}} l(u)(J_{ij}) + \varphi_j,$$

whence

$$l(u)(\psi_{ij}) = dl(u)(\log J_{ij}) = d\left(\frac{1}{J_{ij}} l(u)(J_{ij})\right) = d\varphi_i - d\varphi_j.$$

This yields our assertion. □

4.24 A spectral sequence In this subsection, we consider the following problem, more general than the one studied in Subsection 4.1. Let $(M, \mathcal{O})$ be a supermanifold with retract $(M, \mathcal{O}_{\mathrm{gr}})$. Suppose that $(M, \mathcal{O})$ is determined by a class $\gamma \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. Let us denote $\mathcal{T} := \mathcal{D}er\,\mathcal{O}$, $\mathcal{T}_{\mathrm{gr}} := \mathcal{D}er\,\mathcal{O}_{\mathrm{gr}}$. We want to describe $H^\bullet(M, \mathcal{T})$ under the assumption that the bigraded algebra $H^\bullet(M, \mathcal{T}_{\mathrm{gr}})$ is known.

We fix an open Stein cover $\mathfrak{U} = (U_i)_{i \in I}$ of M and consider the corresponding Čech cochain complex $C^*(\mathfrak{U}, \mathcal{T}) = \bigoplus_{p \geq 0} C^p(\mathfrak{U}, \mathcal{T})$. The filtration (3.16) gives rise to the filtration

$$C^*(\mathfrak{U}, \mathcal{T}) = C_{(-1)} \supset C_{(0)} \supset \dots \supset C_{(p)} \supset \dots \supset C_{(m+1)} = 0 \tag{4.50}$$

of this complex by the subcomplexes

$$C_{(p)} := C^*(\mathfrak{U}, \mathcal{T}_{(p)}).$$

Denoting the image of the natural mapping $H^\bullet(M, \mathcal{T}_{(p)}) \longrightarrow H^\bullet(M, \mathcal{T})$ by $H(M, \mathcal{T})_{(p)}$, we get the filtration

$$H^\bullet(M, \mathcal{T}) = H(M, \mathcal{T})_{(-1)} \supset \dots \supset H(M, \mathcal{T})_{(p)} \supset \dots \supset H(M, \mathcal{T})_{(m+1)} = 0. \tag{4.51}$$

Note that the filtration (4.50) is a filtration of the graded differential algebra $C^*(\mathfrak{U}, \mathcal{T})$ (under a bracket determined by the Lie bracket in $\mathcal{T}$) by graded differential subalgebras, and hence the filtration (4.51) is a filtration of the graded algebra $H^\bullet(M, \mathcal{T})$ by graded subalgebras. Denote by $\operatorname{gr} H^\bullet(M, \mathcal{T})$ the bigraded algebra associated with the filtration (4.51); its bigrading is given by the formula

$$\operatorname{gr} H^\bullet(M, \mathcal{T}) = \bigoplus_{\substack{p \geq -1 \\ q \geq 0}} \operatorname{Gr}^p H^q(M, \mathcal{T}).$$

By a general procedure invented by J. Leray (see [14*, Ch. III.7]), the filtration (4.50) gives rise to a spectral sequence of bigraded algebras (E_r, d_r) converging to $E_\infty \simeq \operatorname{gr} H^\bullet(M, \mathcal{T})$. We have

$$d_r(E_r^{p,q}) \subset E_r^{p+r, q-r+1} \text{ for any } r,\ p,\ q. \tag{4.52}$$

The algebra E_{r+1} is identified with the homology algebra $H(E_r, d_r)$. If we denote Z_r by $\operatorname{Ker} d_r$, then we have the natural homomorphism $\varkappa_{r+1}^r : Z_r \to Z_{r+1}$. For any $s > r$, denote $\varkappa_s^r := \varkappa_s^{s-1} \dots \varkappa_{r+1}^r$ (this composition is not defined on the entire Z_r).

The following theorem is proved in [39].

4.20 Theorem (The first three terms of the spectral sequence).

(1) *The first three terms of the spectral sequence* (E_r) *can be identified with the following bigraded algebras:*

$$E_0 = C^*(\mathfrak{U}, \mathcal{T}_{\mathrm{gr}}),$$
$$E_1 = E_2 = H^\bullet(M, \mathcal{T}_{\mathrm{gr}}).$$

Here

$$E_0^{p,q} = C^{p+q}(\mathfrak{U}, (\mathcal{T}_{\mathrm{gr}})_p),$$
$$E_1^{p,q} = E_2^{p,q} = H^{p+q}(M, (\mathcal{T}_{\mathrm{gr}})_p).$$

(2) $d_{2k+1} = 0$, *and hence* $E_{2k+1} = E_{2k+2}$ *for all* $k \geq 0$.

(3) $d_2 = \operatorname{ad}_{\lambda_2^*(\gamma)}$.

Proposition 3.1 implies the cohomology exact sequence

$$H^{p+q}(M, \mathcal{T}_{(p+1)}) \longrightarrow H^{p+q}(M, \mathcal{T}_{(p)}) \overset{\sigma_p^*}{\to} H^{p+q}(M, (\mathcal{T}_{\mathrm{gr}})_p) = E_2^{p,q}.$$

We would like to describe the subspace $\operatorname{Im} \sigma_p^* \subset H^{p+q}(M, (\mathcal{T}_{\mathrm{gr}})_p)$ by means of our spectral sequence. An element $a \in E_2^{p,q}$ will be called a *permanent cocycle* if

$$d_2 a = 0,\ d_4(\varkappa_4^2 a) = 0,\ \ d_6(\varkappa_6^2 a) = 0, \text{ etc.}$$

Let us denote the subspace of permanent cocycles by $Z_\infty^{p,q}$.

4.21 Proposition (Technical, [39]). *We have*

$$\sigma_p^*(H^{p+q}(M, \mathcal{T}_{(p)})) \subset Z_\infty^{p,q},$$
$$\sigma_p(H^0(M, \mathcal{T}_{(p)})) = Z_\infty^{p,-p}.$$

Thus, a vector field $v \in \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_p$ lifts to $(M, \mathcal{O})$ if and only if v is a permanent cocycle of our spectral sequence (and, in particular, satisfies the condition $d_2 v = [\lambda_2^*(\gamma), v] = 0$, cf. Proposition 4.16).

5 Applications to flag manifolds

5.1 Flag manifolds and homogeneous vector bundles A flag manifold of a connected semisimple complex Lie group G is, by definition, a complex homogeneous space $M = G/P$, where P is a parabolic subgroup of G. In this subsection, we fix the notation and summarize some facts about flag manifolds. For proofs, see [1], [4], [38].

Let P be a parabolic subgroup of G, i.e., a subgroup containing a Borel subgroup of G. We choose a maximal algebraic torus T of G lying in P and a pair of mutually opposite Borel subgroups $B,\ B_- \supset T$ such that $B_- \subset P$. Let Δ denote the root system of G with respect to T, let Δ_+ be the system of positive roots corresponding to B, and $\Pi \subset \Delta_+$ the subsystem of simple roots. Denote

$$\gamma := \tfrac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

If G is simple, then we denote by δ the highest (or maximal) root, i.e., the highest weight of the adjoint representation Ad of G. This root is the only maximal element of Δ relative the following partial order in the vector space $\mathfrak{t}(\mathbb{R})^*$:

$$\lambda \succeq \mu \text{ if and only if } \lambda - \mu = \sum_{\alpha \in \Pi} k_\alpha \alpha, \quad \text{where all } k_\alpha \text{ are non-negative integers.}$$

In particular, we have the decomposition

$$\delta = \sum_{\alpha \in \Pi} n_\alpha \alpha, \tag{5.53}$$

where all n_α are positive integers.

Denote by $\Delta(Q)$ the root system of any Lie subgroup Q of G normalized by T; this is a subsystem of $\Delta = \Delta(G)$. In particular, we have

$$\Delta(B_\pm) = \Delta_\pm$$

and

$$\Delta(P) = \Delta_- \cup [S],$$

where $[S]$ is the set of all roots that can be expressed as linear combination of elements in the subset $S \subset \Pi$. Here, $S \neq \Pi$ if $\dim M > 0$.

We have the semidirect decomposition

$$P = R \rtimes N_-,$$

where R is the maximal reductive subgroup and N_- is the nilradical of P. Here,

$$\begin{gathered}\Delta(R) = [S],\\ \Delta(N_-) = \Delta_- \setminus [S],\end{gathered}$$

and S is the system of simple roots of R, corresponding to $B \cap R$. Denote by N_+ the unipotent subgroup generated by the root vectors, the roots of which belong to the set $\Delta(N_+) = -\Delta(N_-)$. Then, for the corresponding Lie algebras, we have the following decompositions:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{n}_- \oplus \mathfrak{r} \oplus \mathfrak{n}_+ = \mathfrak{p} \oplus \mathfrak{n}_+,\\ \mathfrak{p} &= \mathfrak{n}_- \oplus \mathfrak{r}.\end{aligned} \tag{5.54}$$

Denote by o the point $P \in M = G/P$. Due to formulas (5.54), the holomorphic tangent space $T_o(M) = T_o^{1,0}(M) = \mathfrak{g}/\mathfrak{p}$ can be identified with $\mathfrak{n}_+$. The isotropy representation τ of P in $T_o(M)$ is induced by the adjoint representation Ad_P of P in $\mathfrak{g}$. Since $\mathfrak{n}_+$ is invariant under Ad_R, then $\tau|_R$ is identified with the representation Ad_R in $\mathfrak{n}_+$. It follows that $\Delta(N_+)$ is the system of weights of τ relative to $\mathfrak{t}$.

On the other hand, it is usual to identify $T_o^{0,1}(M)$ with $\mathfrak{n}_-$ (see [4]).

Denote by $(-,-)$ the Killing form on $\mathfrak{g}$. We suppose that in any root subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}$, a basis vector e_α is chosen so that $(e_\alpha, e_{-\alpha}) = 1$ for $\alpha \in \Delta$. Then, the $h_\alpha = [e_\alpha, e_{-\alpha}]$ for $\alpha \in \Pi$, form a basis of $\mathfrak{t}$. We also will use the notation

$$\langle \alpha, \beta \rangle = \tfrac{2(\alpha,\beta)}{(\beta,\beta)} \quad \text{for any} \quad \alpha, \beta \in \Delta.$$

The Killing form determines an R-invariant duality between $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Identifying $\mathfrak{n}_-$ with $T_o^{0,1}(M)$, we see that the isotropy representation of R in $T_o^{0,1}(M)$ is induced by Ad_R and coincides with $\tau^*|_R$. Its system of weights is $\Delta(N_-)$.

5.2 Montgomery's theorem Since M is compact and simply connected, any maximal compact subgroup K of G acts on M transitively, due to Montgomery's theorem, see [33*]: "*If G is a connected Lie group which acts transitively on a compact manifold M, and if the stabilizer G_x of the point $x \in M$ is connected, then G contains a compact subgroup which acts transitively on M.*"

Montgomery's theorem implies the following

Corollary. *If G is a connected Lie group which acts transitively on a compact simply connected manifold M, then G contains a compact subgroup which also acts transitively on M.* See also `https://en.wikipedia.org/wiki/Maximal_compact_subgroup#Existence_and_uniqueness`.

The Cartan-Iwasawa-Malcev theorem asserts that every connected Lie group (and indeed every connected locally compact group) admits maximal compact subgroups and that they are all conjugate to one another. For any semisimple Lie group, uniqueness is a consequence of the Cartan fixed point theorem, which asserts that if a compact group acts by isometries on a complete simply connected negatively curved Riemannian manifold, then it has a fixed point.

Maximal compact subgroups of connected Lie groups are usually not unique, but they are unique up to conjugation.

Therefore, if G contains a compact subgroup that acts transitively, it also contains a maximal (under inclusion) compact subgroup which acts transitively. Now, we have one maximal compact subgroup K which acts transitively. Any other maximal compact subgroup has the form $K' = gKg^{-1}$, where $g \in G$. The groups K' also acts transitively. Indeed, for any $x \in M$ we have $Kx = M$. Therefore,

$$K'(x) = gK(g^{-1}x) = gK(y) = M.$$

Then, $M = G/P = K/L$, where $L = P \cap K$. We can suppose that the corresponding real Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is spanned by ih_α for $\alpha \in \Pi$, and $e_\alpha - e_{-\alpha}$, $i(e_\alpha + e_{-\alpha})$ for $\alpha \in \Delta_+$. Then, we have

$$\mathfrak{g} = \mathfrak{k}(\mathbb{C}), \quad \mathfrak{r} = \mathfrak{l}(\mathbb{C}).$$

The subgroup L is the centralizer of its center in K, and hence is a subgroup of maximal rank. Hence, the Poincaré polynomial of M is expressed by the *Hirsch formula*, see [32*]. On the other hand, the Dolbeault cohomology groups of M satisfy

$$H^{p,q}(M, \mathbb{C}) = 0 \quad \text{for } p \neq q \tag{5.55}$$

(see, e.g., [4]). Since M is a Kähler manifold, the Hodge decomposition yields

$$H^s(M, \mathbb{C}) \simeq \begin{cases} H^{p,p}(M, \mathbb{C}) & \text{for } s = 2p \\ 0 & \text{for } s = 2p + 1. \end{cases}$$

It follows, in particular, that

$$H^{1,1}(M, \mathbb{C}) \simeq H^2(M, \mathbb{C}) \simeq \mathfrak{z}(\mathfrak{r}) \simeq \mathbb{C}^r, \quad \text{where } r = |\Pi \setminus S|. \tag{5.56}$$

We can suppose that $G = (\operatorname{Bih} M)^\circ$. Then, (see [38])

$$\operatorname{Bih} M = G \rtimes \Sigma, \tag{5.57}$$

where Σ is a finite group, naturally isomorphic to the subgroup of $\operatorname{Aut} \Pi$ leaving S invariant.

It is well known that with any holomorphic linear representation $\varphi : P \to \mathrm{GL}(E)$ one can associate a holomorphic vector bundle $\mathbf{E}_\varphi$ over M. The total space of this bundle is the quotient

$$G \times_\varphi E := (G \times E)/P$$

of $G \times E$ by the diagonal action of P. The group G acts on $\mathbf{E}_\varphi$ by automorphisms, covering the given action on M. This bundle is called the *homogeneous vector bundle* determined by φ. For example, take the bundle $\mathbf{E}_\tau$ isomorphic to $\mathbf{T}(M)$.

The cohomology $H^\bullet(M, \mathcal{E}_\varphi) = \bigoplus_{p\geq 0} H^\bullet(M, \mathcal{E}_\varphi)$ admits a natural G-module structure. The corresponding representation Φ of G is called *induced*. If φ is irreducible (or completely reducible), then the induced representation can be calculated with the help of an algorithm found by Bott.

Denote by W the Weyl group of G. This group is generated by reflections σ_α, corresponding to the roots $\alpha \in \Delta$, but as a system of generators one can choose $\{\sigma_\alpha \mid \alpha \in \Pi\}$. As usual, we call a weight λ of G *dominant* (resp. *strictly dominant*) if $(\lambda, \alpha) \geq 0$ (resp. > 0) for all $\alpha \in \Pi$. The Bott algorithm is the following operation $\xi \mapsto \xi^*$:

$$\xi^* = \sigma(\xi + \gamma) - \gamma, \tag{5.58}$$

where $\xi + \gamma$ is regular and $\sigma \in W$ is chosen in such a way that ξ^* is strictly dominant (or $\sigma(\xi + \gamma)$ is dominant, which is the same). The *index* of $\xi + \gamma$ is the number of roots in $\Phi_\sigma = \sigma\Delta_- \cap \Delta_+$ or, which is the same, the minimal number of factors in a decomposition of σ into the product of σ_α for $\alpha \in \Pi$. The index is also equal to the number of positive roots α such that $(\xi + \gamma, \alpha) < 0$.

The result of Bott is as follows (see [1], [4], [26]):

5.1 Theorem (Bott's theorem). *Let $\varphi : P \longrightarrow \mathrm{GL}(E)$ be an irreducible holomorphic representation with highest weight Λ. Then, the induced representation can be determined as follows:*

(1) *If $\Lambda + \gamma$ is singular, then $H^\bullet(M, \mathcal{E}_\varphi) = 0$.*

(2) *If $\Lambda + \gamma$ is regular of index p, then $H^q(M, \mathcal{E}_\varphi) = 0$ for $q \neq p$ and $H^p(M, \mathcal{E}_\varphi)$ is an irreducible G-module with highest weight Λ^*.*

This theorem gives, in particular, a description of the vector space

$$\Gamma(\mathbf{E}_\varphi) = \Gamma(M, \mathcal{E}_\varphi) = H^0(M, \mathcal{E}_\varphi)$$

of holomorphic sections of $\mathbf{E}_\varphi$. Note that the induced representation $\Phi : G \longrightarrow \mathrm{GL}(\Gamma(\mathbf{E}_\varphi))$ acts as follows:

$$(\Phi(g)s)(x) = gs(g^{-1}x) \text{ for any } g \in G, \quad s \in \Gamma(\mathbf{E}_\varphi) \text{ and } x \in M. \tag{5.59}$$

5.2. Corollary. *Under the assumptions of Theorem 5.1, $\Gamma(\mathbf{E}_\varphi) \neq 0$ if and only if Λ is dominant, and in this case $\Gamma(\mathbf{E}_\varphi)$ is an irreducible G-module with highest weight Λ.*

If φ is completely reducible, then the induced representation can be calculated as well, by decomposing φ into irreducible components and applying Theorem 5.1 to the corresponding homogeneous vector bundles. As to the general case, we only make the following useful remark.

5.3. Corollary. *Let $\varphi : P \longrightarrow \mathrm{GL}(E)$ be an arbitrary holomorphic representation and let Λ be a highest weight of the induced representation Φ of G in $\Gamma(\mathbf{E}_\varphi)$. Then, Λ is a highest weight of φ.*

Proof. Note that a highest weight of φ is the same as a highest weight of the completely reducible representation $\varphi|_R$. By Corollary 5.2, our assertion is true whenever φ is irreducible. Suppose that it is true for $\dim E < m$ and let us prove it for $\dim E = m$. Let E' be a maximal P-submodule of E and denote $E'' := E/E'$. Then, we have the exact sequence of G-sheaves

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

and the corresponding exact sequence of cohomology with G-equivariant mappings

$$0 \longrightarrow \Gamma(\mathcal{E}') \longrightarrow \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E}'').$$

Let Λ be a highest weight of the G-module $\Gamma(\mathcal{E})$. Since Φ is completely reducible, Λ is a highest weight of $\Gamma(\mathcal{E}')$ or $\Gamma(\mathcal{E}'')$. Using the inductive hypothesis and the complete reducibility of $\varphi|_R$, we see that Λ is a highest weight of φ. $\square$

5.6 Vector fields on (M, Ω) Here we will study the split supermanifold (M, Ω), assuming that G is simple. Our goal is to calculate the graded Lie superalgebra of vector fields $\mathfrak{v}(M, \Omega)$ (see [35*]).

It is known (see [1], [38]) that the Lie group $(\mathrm{Bih}\ M)^\circ$ is simple and its isotropy subgroup is parabolic. Thus, we can assume that $G = (\mathrm{Bih}\ M)^\circ$ and $\mathfrak{g} = \mathfrak{v}(M)$. Thanks to (3.29) and (3.30), we have

$$\begin{aligned} \mathfrak{v}(M, \Omega)_{-1} &= i(\mathfrak{g}), \\ \mathfrak{v}(M, \Omega)_0 &= i(\mathfrak{gl}(\mathbf{T}(M)^*)) \ni l(\mathfrak{g}). \end{aligned} \tag{5.60}$$

As in Subsection 2.1, denote by ad_p the adjoint representation of $\mathfrak{v}(M, \Omega)_0$ in $\mathfrak{v}(M, \Omega)_p$. The following lemma was first proved in [23].

5.4. Lemma. *If $\mathfrak{g}$ is simple, then $\mathfrak{gl}(\mathbf{T}(M)^*) = \mathfrak{gl}(\mathbf{T}(M)) = \langle \mathrm{id} \rangle$. The representation ad_{-1} is irreducible and faithful.*

Proof. From the classical relation (see also (3.28))

$$[l(u), i(v)] = i([u, v]), \quad u, v \in \mathfrak{g},$$

we see that ad_{-1} is irreducible and faithful on $l(\mathfrak{g})$.

Further, let us regard $\mathfrak{gl}(\mathbf{T}(M)^*)$ as $H^0(M, \Omega^1 \otimes \Theta) = \mathfrak{gl}(\mathbf{T}(M))$. Then, (2.4) implies that

$$[i(\eta), i(v)] = i(\{\eta, v\}) = -\eta \,\bar{\wedge}\, v, \qquad \text{for any } \eta \in \mathfrak{gl}(\mathbf{T}(M)^*) \text{ and } v \in \mathfrak{g}.$$

If $\mathrm{ad}_{-1}\, i(\eta) = 0$, then $\eta(v) = 0$ for any $v \in \mathfrak{g}$. Since G acts on M transitively, we have

$$\mathrm{ev}_x(\mathfrak{g}) = T_x(M) \text{ for all } x \in M,$$

and hence $\eta = 0$. Thus, ad_{-1} is faithful on $i(\mathfrak{gl}(\mathbf{T}(M)^*))$.

Let $\mathfrak{a}$ denote the radical of $i(\mathfrak{gl}(\mathbf{T}(M)^*))$. It is non-trivial: it contains $\langle\varepsilon\rangle = \langle i(\mathrm{id})\rangle$. Since ad_{-1} is irreducible, its image is a reductive Lie algebra with radical $\langle\mathrm{id}\rangle = \langle\mathrm{ad}_{-1}\varepsilon\rangle$. By (5.60), $\mathfrak{a}$ coincides with the radical of $\mathfrak{v}(M,\Omega)_0$, and hence $\mathfrak{a} = \langle\varepsilon\rangle$. It follows that $i(\mathfrak{gl}(\mathbf{T}(M)^*))$ is reductive, and

$$i(\mathfrak{gl}(\mathbf{T}(M)^*)) = \langle\varepsilon\rangle \oplus \mathfrak{s},$$

where $\mathfrak{s}$ is a semisimple Lie algebra. We have to prove that $\mathfrak{s} = 0$.

Clearly, $\mathfrak{s}$ is invariant under $\mathrm{ad}_0(l(\mathfrak{g}))$, and hence we get the homomorphism

$$\mathrm{ad}_0(l) : \mathfrak{g} \longrightarrow \mathfrak{Der}\,\mathfrak{s} = \mathrm{ad}\,\mathfrak{s}$$

which is injective if $[l(\mathfrak{g}),\mathfrak{s}] \neq 0$. In this latter case, we obtain therefore an injective homomorphism $h : \mathfrak{g} \longrightarrow \mathfrak{s}$ satisfying

$$[h(u), z] = [l(u), z] \ \text{ for any } u \in \mathfrak{g},\ z \in \mathfrak{s}.$$

In particular,

$$h([u,v]) = [h(u), h(v)] = [l(u), h(v)] \ \text{ for any } u, v \in \mathfrak{g}. \tag{5.61}$$

Now, we note that $\mathfrak{gl}(\mathbf{T}(M)^*) = \Gamma(\mathbf{T}(M)^* \otimes \mathbf{T}(M))$ is the vector space of holomorphic sections of the homogeneous vector bundle $\mathbf{T}(M)^* \otimes \mathbf{T}(M) = \mathbf{E}_\varphi$, where $\varphi = \tau^*\tau$. From (5.59) we deduce that the induced representation Φ of G in $\Gamma(\mathbf{T}(M)^* \otimes \mathbf{T}(M))$ satisfies the following condition

$$i(d\Phi(u)\eta) = [l(u), i(\eta)] \quad \text{for any } u \in \mathfrak{g},\ \eta \in \mathfrak{gl}(\mathbf{T}(M)^*). \tag{5.62}$$

Suppose that $[l(\mathfrak{g}),\mathfrak{s}] \neq 0$. Then, eqs. (5.61) and (5.62) imply that $\mathrm{Im}\,h$ determines a G-submodule of $\mathfrak{gl}(\mathbf{T}(M)^*)$, where the adjoint representation of G is realized. Thus, the highest root δ is a highest weight of Φ. By Corollary 5.3, δ is a highest weight of φ. But the weights of φ have the form $\alpha - \beta$, where $\alpha, \beta \in \Delta_+$. This yields a contradiction.

Thus, we have proved that $[l(\mathfrak{g}),\mathfrak{s}] = 0$. It follows that $\mathrm{ad}_{-1}(\mathfrak{s})$ commutes with the irreducible linear Lie algebra $\mathrm{ad}_{-1} l(\mathfrak{g})$. By the Schur lemma, $\mathrm{ad}_{-1}(\mathfrak{s}) = 0$, and hence $\mathfrak{s} = 0$. □

A graded Lie superalgebra of the form

$$\mathfrak{v} = \bigoplus_{p \geq 1} \mathfrak{v}_p \tag{5.63}$$

is called *transitive* if for any $p \geq 1$ it satisfies

$$\{x \in \mathfrak{v}_p \mid [x, \mathfrak{v}_{-1}] = 0\} = 0. \tag{5.64}$$

A graded Lie superalgebra of the form (5.63) is called *irreducible* if the representation ad_{-1} of $\mathfrak{v}_0$ is irreducible. All irreducible transitive complex graded Lie superalgebras of finite dimension were classified in [24] (see also [48]).

5.5 Theorem ($\mathfrak{v}(G/P, \Omega)$ for G simple).

1) *For any flag manifold $M = G/P$ of a simple complex Lie group G, the graded Lie superalgebra $\mathfrak{v}(M, \Omega)$ is transitive and irreducible.*

2) *Under the above assumptions, suppose that $\mathfrak{v}(M, \mathcal{O}) = \mathfrak{g}$. Then,*

$$\begin{aligned}\mathfrak{v}(M,\Omega)_{-1} &= i(\mathfrak{g}),\\ \mathfrak{v}(M,\Omega)_0 &= \langle\varepsilon\rangle \oplus l(\mathfrak{g}),\\ \mathfrak{v}(M,\Omega)_1 &= \langle d\rangle,\\ \mathfrak{v}(M,\Omega)_p &= 0 \textit{ for any } p \geq 2.\end{aligned}$$

Proof. By Lemma 5.4, $\mathfrak{v}(M, \Omega)$ is irreducible and satisfies condition (5.64) for $p = 0$. Thus, we need to prove (5.64) for any $p > 0$.

We will use the following fact: if $\varphi \in \Omega^p$, where $p > 0$, and if $i(v)\varphi = 0$ for all $v \in \mathfrak{g}$, then $\varphi = 0$. To prove this, we note that

$$(i(v)\varphi)_x(v_1, \ldots, v_p) = \varphi_x(\mathrm{ev}_x(v), v_1, \ldots, v_p), \quad v_i \in T_x(M) \text{ for } x \in M.$$

Since $\mathfrak{g}$ acts transitively, the condition $i(v)\varphi = 0$ yields $\varphi_x = 0$ for any $x \in M$.

Now, suppose that a vector field $u \in \mathfrak{v}(M, \Omega)_p$ for $p > 0$ satisfies $[u, i(v)] = 0$ for all $v \in \mathfrak{g}$. Then, for any $f \in \mathcal{F}$, we have

$$[i(v), u](f) = i(v)u(f) = 0.$$

By the above, we have $u(f) = 0$. Therefore, for any $\varphi \in \Omega^1$, we get

$$[i(v), u](\varphi) = i(v)u(\varphi) + (-1)^{p+1}u(i(v)(\varphi)) = i(v)u(\varphi) = 0.$$

Since $u(\varphi) \in \Omega^{p+1}$, this implies $u(\varphi) = 0$. Thus, $u = 0$, and item (1) is proved.

The item (2) for $p = -1, 0$ follows from (5.60) and Lemma 5.4. It is also clear that $\langle d\rangle \subset \mathfrak{v}(M, \Omega)_1$. In particular, we see that the representation ad_{-1} of $\mathfrak{v}(M, \Omega)_0 \simeq \mathfrak{g}$ is the adjoint one, while the representation ad_1 of this Lie algebra contains a trivial component of dimension 1. The classification of transitive irreducible graded Lie superalgebras $\mathfrak{v}$ given in [24, Theorem 4], shows that if $\mathfrak{v}$ satisfies the above conditions, then $\dim \mathfrak{v}_1 = 1$ and $\mathfrak{v}_p = 0$ for $p \geq 2$. Thus, item (2) follows from item (1). □

5.8 A family of non-split supermanifolds Here, we apply the construction of Subsection 5.2 to the case where $M = G/P$ is a flag manifold of a simple complex Lie group G. We show that in this situation one always obtains a non-empty family of non-split supermanifolds having (M, Ω) as their retract. We also study holomorphic vector fields on these supermanifolds.

5.6. Theorem. *Let $M = G/P$ be a flag manifold, where G is simple and $\dim M > 1$, and denote $r := |\Pi \setminus S|$. Then, there exists a family of distinct non-split supermanifolds that have (M, Ω) as their retract, parametrized by $\mathbb{CP}^{r-1}/\Sigma$. Here Σ is the finite group from (5.57).*

If P is maximal, then this family consists of a unique supermanifold, which is isomorphic to the canonical one.

Proof. The group $\operatorname{Aut}\mathbf{T}(M)^*$ naturally acts on $H^{1,1}(M,\mathbb{C})$, and the mapping

$$\mu^*\hat{\delta} : H^{1,1}(M,\mathbb{C}) \longrightarrow H^1(M, \mathcal{A}ut_{(2)}\Omega)$$

is equivariant. By Theorems 4.1 and 4.13, this mapping determines a family of distinct non-split supermanifolds having (M, Ω) as their retract which is parametrized by the set $(H^{1,1}(M,\mathbb{C}) \setminus \{0\}) / \operatorname{Aut}\mathbf{T}(M)^*$. On the other hand, $\mathrm{GL}(\mathbf{T}(M)^*) = \mathbb{C}^\times$ (see Lemma 5.4), and (3.31) yields

$$\operatorname{Aut}\mathbf{T}(M)^* = \mathbb{C}^\times \times \operatorname{Bih} M.$$

Thanks to (5.57), we see that

$$\operatorname{Aut}\mathbf{T}(M)^* = \mathbb{C}^\times \times (G \rtimes \Sigma).$$

Clearly, the action of G on $H^{1,1}(M,\mathbb{C})$ is trivial. Using Lemma 4.11, we deduce that

$$(H^{1,1}(M,\mathbb{C}) \setminus \{0\}) / \operatorname{Aut}\mathbf{T}(M)^* = (H^{1,1}(M,\mathbb{C}) \setminus \{0\})/(\mathbb{C}^\times \times \Sigma) = \mathbb{P}(H^{1,1}(M,\mathbb{C}))/\Sigma.$$

Due to (5.56), this implies our first assertion.

To prove the second claim, we note that the canonical supermanifold corresponding to M is non-split, since in our case the canonical form ω is positive-definite [28] and hence $[\omega] \neq 0$. Thus, it enters the family just constructed. But if P is maximal, then $r = 1$, and $\mathbb{CP}^{r-1}$ contains only one point. □

Now we will study holomorphic vector fields on supermanifolds $(M, \mathcal{O})$ of the family constructed above, applying Proposition 3.1 to $\mathcal{O}_{\mathrm{gr}} = \Omega$. We have to settle what derivations of Ω described in Theorem 5.5 can be lifted to $(M, \mathcal{O})$. We need the following lemma.

5.7. Lemma. *Let $M = G/P$ be a flag manifold. Then, both homomorphisms of the sequence*

$$H^{1,1}(M,\mathbb{C}) \xrightarrow{\hat{\delta}} H^1(M, \mathcal{Z}\Omega^1) \xrightarrow{\beta^*} H^1(M, \Omega^1)$$

(see Theorem 4.13) *are isomorphisms, and $H^1(M, \mathcal{Z}\Omega^1)$ is a trivial G-module.*

Proof. Clearly, we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{Z}\Omega^1 \xrightarrow{\beta} \Omega^1 \xrightarrow{d} \mathcal{Z}\Omega^2 \longrightarrow 0,$$

where $\mathcal{Z}\Omega^2$ is the sheaf of closed forms from Ω^2. Consider the corresponding cohomology exact sequence:

$$H^0(M, \mathcal{Z}\Omega^2) \longrightarrow H^1(M, \mathcal{Z}\Omega^1) \xrightarrow{\beta^*} H^1(M, \Omega^1).$$

Since M is Kähler, all holomorphic forms on it are closed, and hence

$$H^0(M, \mathcal{Z}\Omega^2) = H^0(M, \Omega^2) \simeq H^{2,0}(M, \mathbb{C}).$$

By (5.55), this group is trivial, and therefore β^* is injective. It is also surjective, since $\beta^*\hat{\delta}$ is the Dolbeault isomorphism. It follows that β^* and $\hat{\delta}$ are isomorphisms. The natural G-action on $H^1(M, \mathcal{Z}\Omega^1)$ is trivial, since this is true for $H^{1,1}(M, \mathbb{C})$. □

5.8 Proposition (Technical). *Let $M = G/P$ be as in Theorem 5.6, $\mathfrak{g} = \mathfrak{v}(M)$, and let $(M, \mathcal{O})$ be any non-split supermanifold of the family described in Theorem 5.6. Then, $l(v)$ for $v \in \mathfrak{g}$, and d can be lifted to $(M, \mathcal{O})$, and we have*

$$\begin{aligned}
\mathfrak{v}(M, \mathcal{O})_{(0)} &= \mathfrak{v}(M, \mathcal{O})_{\bar{0}} \oplus \langle \hat{d} \rangle, \\
\mathfrak{v}(M, \mathcal{O})_{(1)} &= \langle \hat{d} \rangle, \\
\mathfrak{v}(M, \mathcal{O})_{(p)} &= 0 \text{ for } p \geq 2.
\end{aligned}$$

Here $\sigma_0 : \mathfrak{v}(M, \mathcal{O})_{\bar{0}} \longrightarrow \mathfrak{g}$ is an isomorphism, $\hat{d} \neq 0$, $\sigma_1(\hat{d}) = d$ and $[\hat{d}, \hat{d}] = [\hat{d}, v] = 0$ for all $v \in \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$.

Proof. Consider the exact sequence (4.45) for $\mathcal{O}_{\mathrm{gr}} = \Omega$. From Theorem 5.5 we deduce that $\mathfrak{v}(M, \mathcal{O})_{(p)} = 0$ for $p \geq 2$ and that $\sigma_1 : \mathfrak{v}(M, \mathcal{O})_{(1)} \longrightarrow \mathfrak{v}(M, \Omega)_1 = \langle d \rangle$ is injective. By Proposition 4.17, we see that $\mathfrak{v}(M, \mathcal{O})_{(1)} = \langle \hat{d} \rangle$, where $\hat{d}$ is odd and $\sigma_1(\hat{d}) = d$. For $p = 0$, the exact sequence has the form

$$0 \longrightarrow \mathfrak{v}(M, \mathcal{O})_{(1)} \longrightarrow \mathfrak{v}(M, \mathcal{O})_{(0)} \xrightarrow{\sigma_0} \mathfrak{v}(M, \Omega)_0 = \langle \varepsilon \rangle \oplus l(\mathfrak{g}).$$

By Lemma 5.7 and Corollary 4.19, any $v \in l(\mathfrak{g})$ lifts to $(M, \mathcal{O})$. On the other hand, ε does not lift by Proposition 4.16, since $[\varepsilon, \lambda_2(\gamma)] = 2\lambda_2(\gamma) \neq 0$ by eq. (2.2). Hence, $\operatorname{Im} \sigma_0 = l(\mathfrak{g})$. This implies our assertion concerning $\mathfrak{v}(M, \mathcal{O})_{(0)}$.

Since $\mathfrak{v}(M, \mathcal{O})_{(0)}$ is a subalgebra of $\mathfrak{v}(M, \mathcal{O})$, it follows that $\hat{d}$ is a weight vector of the representation $\mathrm{ad}_{\bar{1}}$ of $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}$, but the corresponding weight is 0, since $\mathfrak{g}$ is simple. Thus, $[\hat{d}, v] = 0$ for all $v \in \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$. It follows that $[\hat{d}, \hat{d}]$ lies in the center of $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}$, whence $[\hat{d}, \hat{d}] = 0$. □

One can ask whether $\mathfrak{v}(M, \mathcal{O})$ coincides with it subalgebra $\mathfrak{v}(M, \mathcal{O})_{(0)}$ calculated in Proposition 5.8. This is not true in general, and in Theorem 5.33 I give the complete answer for the case where M is an irreducible Hermitian symmetric space.

5.10 Irreducible Hermitian symmetric spaces A *Hermitian symmetric space* is, by definition, a connected complex manifold M, endowed with a Hermitian structure and satisfying the following condition: for any $x \in M$, there exists a holomorphic isometry s_x of M such that $d_x s_x = -\operatorname{id}$.

Let M be a compact Hermitian symmetric space. Let K be the identity component of the group of all holomorphic isometries of M; this is a compact Lie group. It is known (see [21]) that M is a homogeneous space of K, and hence can be regarded as the coset space K/L, where L is the stabilizer K_o of a point $o \in M$.

In what follows, we suppose that M is simply connected and irreducible (as a Hermitian space). It is known (see [21]) that if M is simply connected, then L is the centralizer of a torus in K, containing the symmetry s_o. Now, $G = (\operatorname{Bih} M)^\circ$ is the complexification $G = K(\mathbb{C})$, and $M = G/P$, where $P = G_o$ is a parabolic subgroup of G. Thus, M is a flag manifold of a special type. Now, a simply connected compact Hermitian symmetric space M is irreducible if and only if K and G are simple. In this case, P is maximal.

Let G be a connected simple complex Lie group. We retain the notation of Subsection 5.1 and suppose that a maximal torus T and a Borel subgroup $B \supset T$ of G are chosen. Consider the decomposition (5.53) of the highest root δ. A simple root $\alpha \in \Pi$ will be called *special* if $n_\alpha = 1$.

Let P be a parabolic subgroup of G containing the Borel subgroup B_-. In the above notation, the flag manifold $M = G/P$ is Hermitian symmetric if and only if the subset $S \subset \Pi$ defining P has the form $S = \Pi \setminus \{\alpha_0\}$, where α_0 is a special simple root. Thus, in this case,

$$\Delta(P) = \Delta_- \cup [\Pi \setminus \{\alpha_0\}].$$

It follows that

$$\Delta(N_-) = \Delta_- \setminus [\Pi \setminus \{\alpha_0\}],$$

i.e., this is the set of those negative roots $-\beta$ of G, whose expression through simple roots contains α_0 (necessarily with coefficient 1). The subgroups N_+ and N_- are commutative. The isotropy representation $\tau : P \longrightarrow \mathrm{GL}(\mathfrak{n}_+)$ is irreducible; in particular, $\tau|N_-$ is trivial, and τ is completely determined by its restriction onto R.

For all simple Lie groups G of rank > 1 that have special simple roots, [36, Table 6] shows the Dynkin diagrams of extended systems of simple roots $\widetilde{\Pi} = \Pi \cup \{-\delta\}$, where for any $\alpha \in \Pi$ the coefficient n_α is indicated.

Note that a simple root is special if and only if it lies in the same orbit as $-\delta$ under the symmetry group of $\widetilde{\Pi}$.

On the other hand, the nontrivial symmetries of Π, existing for the types A_l, D_l, E_6, transform special roots into special roots, and we have to consider special roots up to these symmetries. In all cases, we have $(\alpha_0, \alpha_0) = 2$ for a special root α_0.

5.11 The three cases We also see that any irreducible symmetric Hermitian space of dimension ≥ 2 satisfies to one of the following three conditions, depending on the choice of G and of a special simple root α_0:

I. $(\delta, \alpha_0) = 0$, and there exists a unique $\alpha_1 \in \Pi$ such that $(\alpha_0, \alpha_1) \neq 0$.

This case occurs for the groups G of types $B_l,\ C_l,\ D_l,\ E_6,\ E_7$ and for any special rootα_0; we have $n_{\alpha_1}=2$.

II. $(\delta,\alpha_0)=0$, and there exist two different simple roots $\alpha_1,\ \alpha_2$, such that $(\alpha_0,\alpha_i)\neq 0$ for $i=1,2$.

This case occurs for the groups G of type $A_l,\ l\geq 3$, for any α_0 corresponding to the interior vertices of the (non-extended) Dynkin diagram. Here, we have $n_{\alpha_1}=n_{\alpha_2}=1$. The manifolds M are the Grassmannians Gr_s^{l+1} for $1<s<l$.

III. $(\delta,\alpha_0)\neq 0$.

This case occurs for the groups G of type A_l for $l\geq 2$, for any of the two roots α_0 corresponding to the end vertices of the (non-extended) Dynkin diagram. There exists a unique $\alpha_1\in\Pi$ such that $(\alpha_0,\alpha_1)\neq 0$, and we have $n_{\alpha_1}=1$. The manifolds M are the projective spaces $\mathbb{CP}^l$, where $l\geq 2$.

The roots $\alpha_1,\ \alpha_2\in\Pi$, described above, will be called the *neighbors* of α_0. We admit a numbering of simple roots $\Pi=\{\alpha_0,\alpha_1,\dots,\alpha_{l-1}\}$ using this notation. For a weight λ of G, we will denote by $m(\lambda)$ the coefficient at α_1 (or the sum of the coefficients of $\alpha_1,\ \alpha_2$) in the expression of λ in terms of Π. In particular, we see that

$$m(\delta)=\begin{cases}2 \text{ in the cases I, II}\\ 1 \text{ in the case III.}\end{cases}$$

Clearly, the weight system of the irreducible representation τ coincides with $\Delta(N_+)$, the highest weight being δ and the lowest one α_0. Similarly, the weight system of τ^* is $\Delta(N_-)$, the highest weight being $-\alpha_0$ and the lowest one $-\delta$.

5.12 Invariant vector-valued forms In this Subsection, we discuss invariant vector-valued forms on flag manifolds and the invariant cohomology $H^{\bullet}(M,\Omega\otimes\Theta)^G$ of irreducible Hermitian symmetric spaces.

Retaining the notation of Subsection 5.1, consider a flag manifold $M=G/P=K/L$. Clearly, K naturally acts on the vector space $\Gamma(M,\Phi\otimes\Theta)$ of all smooth vector-valued forms on M. The well-known É. Cartan principle of reducing invariants of a transitive action to invariants of the isotropy group (see, e.g., [38, Theorem 4.2]) gives

5.9. Proposition. *The evaluation mapping of* $\Gamma(M,\Phi\otimes\Theta)$ *onto*

$$\bigwedge(T_o^{1,0}(M)\oplus T_o^{0,1}(M))^*\otimes T_o^{1,0}(M)$$

given by $\varphi\mapsto\varphi_o$ *determines an isomorphism of the bigraded vector spaces*

$$\begin{aligned}\Gamma(M,\Phi\otimes\Theta)^K &\to (\bigwedge(T_o^{1,0}(M)\oplus T_o^{0,1}(M))^*\otimes T_o^{1,0}(M))^L\\ &=(\bigwedge(\mathfrak{n}_+\oplus\mathfrak{n}_-))^*\otimes\mathfrak{n}_+)^L\\ &=(\bigwedge(\mathfrak{n}_+\oplus\mathfrak{n}_-))^*\otimes\mathfrak{n}_+)^R\end{aligned}$$

preserving the operations $\overline{\wedge}$ *and* $\{-,-\}$. *This isomorphism maps* $\Gamma(M,\Phi^{p,q}\otimes\Theta)^K$ *onto*

$$((\bigwedge^p\mathfrak{n}_+\otimes\bigwedge^q\mathfrak{n}_-)^*\otimes\mathfrak{n}_+)^R=(\bigwedge^p\mathfrak{n}_+^*\otimes\bigwedge^q\mathfrak{n}_-^*\otimes\mathfrak{n}_+)^R=(\bigwedge^p\mathfrak{n}_-\otimes\bigwedge^q\mathfrak{n}_+\otimes\mathfrak{n}_+)^R.$$

Now we give examples of invariant vector-valued forms.

5.10 Example ($\omega \in \Gamma(M, \Phi^{1,1})$ on a complex manifold M). Let M be a complex manifold and $\omega \in \Gamma(M, \Phi^{1,1})$. Consider the form

$$\theta_2 = \widetilde{\jmath}(\omega) \in \Gamma(M, \Phi^{2,1} \otimes \Theta)$$

given by the formula (4.43). Thus,

$$\theta_2(u_1, u_2, v) = \omega(u_1, v)u_2 - \omega(u_2, v)u_1, \quad u_1, u_2 \in \Theta,\ v \in \bar{\Theta}. \tag{5.65}$$

More generally, we can construct the following vector-valued $(p, p-1)$-form θ_p for $p \geq 1$:

$$\theta_p(u_1, \dots, u_p, v_1, \dots, v_{p-1}) = (p-1)! \begin{vmatrix} \omega(u_1, v_1) & \dots & \omega(u_1, v_{p-1}) & u_1 \\ \omega(u_2, v_1) & \dots & \omega(u_2, v_{p-1}) & u_2 \\ \vdots & & \vdots & \vdots \\ \omega(u_p, v_1) & \dots & \omega(u_p, v_{p-1}) & u_p \end{vmatrix}, \tag{5.66}$$

where $u_i \in \Theta$ and $v_j \in \bar{\Theta}$. In particular, $\theta_1 = \mathrm{id}$ and θ_2 is as in (5.65). Clearly, $\theta_p \neq 0$ for $p \leq n = \dim M$. We note that (see [40])

$$\theta_p \mathbin{\bar{\wedge}} \theta_q = p\theta_{p+q-1}. \tag{5.67}$$

By Proposition 5.9, the form θ_p is completely determined by its value at $o \in M$ which is expressed by the same formula (5.66) through the value ω_o at o. As ω_o, we can choose any L-invariant $(1,1)$-form at o. For example, the Killing form on $\mathfrak{g}$ determines an invariant form ω satisfying

$$\omega_o(u, v) = (u, v), \quad u \in \mathfrak{n}_+,\ v \in \mathfrak{n}_-. \tag{5.68}$$

In what follows, we consider the case where M is an irreducible compact Hermitian symmetric space. Then, the isotropy representation is irreducible, and hence the form (5.68) is the only (up to a constant factor) L-invariant $(1,1)$-form on $T_o(M)$.

5.11 Example ($M = \mathrm{Gr}_s^n$). Consider the complex Grassmannian $M = \mathrm{Gr}_s^n$, where s is an integer in $\{1, \dots, n-1\}$. This is an irreducible compact Hermitian symmetric space with $G = \mathrm{SL}_n(\mathbb{C})$ and $K = \mathrm{SU}_n$. It is convenient to regard M as a homogeneous space of the group $G_0 = \mathrm{GL}_n(\mathbb{C})$ with a natural action on M (this action is not effective). As usual, we choose in G_0 the maximal torus T consisting of all diagonal matrices and the Borel subgroup B consisting of all upper triangular matrices. Then, B_- is the subgroup of all lower triangular matrices. Denote $r := n - s$. For o we take the point $\langle e_{r+1}, \dots, e_n \rangle \in \mathrm{Gr}_s^n$. Then, the isotropy subgroup P of G at o is parabolic and contains B_-; it consists of all matrices of the form

$$\begin{pmatrix} A_1 & 0 \\ V & A_2 \end{pmatrix}, \tag{5.69}$$

where $A_1 \in \mathrm{GL}_r(\mathbb{C})$ and $A_2 \in \mathrm{GL}_s(\mathbb{C})$. Its maximal reductive subgroup R consists of matrices of the form (5.69) with $V = 0$ and can be identified with $\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_s(\mathbb{C})$, while the unipotent radical N_- is abelian and consists of matrices of the form (5.69) with $A_1 = I_r$ and $A_2 = I_s$. The subalgebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$ consist of matrices of the form

$$\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix},$$

respectively, v being an $(s \times r)$-matrix and u an $(r \times s)$-matrix. We will identify $\mathfrak{n}_-$ and $\mathfrak{n}_+$ with the vector spaces of matrices $\mathrm{M}_{s,r}(\mathbb{C})$ and $\mathrm{M}_{r,s}(\mathbb{C})$, respectively. The isotropy representation τ of P on $\mathfrak{n}_+ = T_o(M)$ is as follows:

$$\tau\left(\begin{pmatrix} A_1 & 0 \\ V & A_2 \end{pmatrix}\right)(u) = A_1 u A_2^{-1}. \tag{5.70}$$

Let us replace the Killing form by the following invariant bilinear form on $\mathfrak{gl}_n(\mathbb{C})$:

$$(X, Y) = \mathrm{tr}\ XY.$$

Then, using (5.68), we can define the K-invariant vector-valued forms θ_p on M by (5.66). Now, we construct new examples of K-invariant vector-valued (2,1)- and (3,2)-forms. Note that the same method permits to construct certain invariant vector-valued $(p, p-1)$-forms for any $p \geq 1$.

Define the K-invariant vector-valued (2,1)-form η by its L-invariant value

$$\eta_o(u_1, u_2, v) = u_1 v u_2 - u_2 v u_1, \quad u_1, u_2 \in \mathfrak{n}_+, v \in \mathfrak{n}_-. \tag{5.71}$$

The forms θ_2 and η are linearly independent whenever $1 < s < n-1$, and they coincide for $s = 1$ or $s = n-1$.

Similarly, we define the K-invariant vector-valued (3,2)-forms η_1, η_2, η_3, whose L-invariant values at o are as follows:

$$\begin{aligned}(\eta_1)_o &= \mathrm{Alt}\ \mathrm{tr}(u_1 v_1 u_2 v_2) u_3 = \mathrm{Alt}\,(u_1 v_1, u_2 v_2) u_3 \\ &= 2((u_1 v_1, u_2 v_2) u_3 + (u_2 v_1, u_3 v_2) u_1 + (u_3 v_1, u_1 v_2) u_2 \\ &- (u_2 v_1, u_1 v_2) u_3 - (u_3 v_1, u_2 v_2) u_1 - (u_1 v_1, u_3 v_2) u_2),\end{aligned} \tag{5.72}$$

$$\begin{aligned}(\eta_2)_o &= \mathrm{Alt}\ \mathrm{tr}(u_1 v_1) u_2 v_2 u_3 = \mathrm{Alt}\,(u_1, v_1) u_2 v_2 u_3 \\ &= (u_1, v_1) u_2 v_2 u_3 + (u_2, v_1) u_3 v_2 u_1 + (u_3, v_1) u_1 v_2 u_2 \\ &- (u_2, v_1) u_1 v_2 u_3 - (u_3, v_1) u_2 v_2 u_1 - (u_1, v_1) u_3 v_2 u_2 \\ &- (u_1, v_2) u_2 v_1 u_3 - (u_2, v_2) u_3 v_1 u_1 - (u_3, v_2) u_1 v_1 u_2 \\ &+ (u_2, v_2) u_1 v_1 u_3 + (u_3, v_2) u_2 v_1 u_1 + (u_1, v_2) u_3 v_1 u_2,\end{aligned} \tag{5.73}$$

$$\begin{aligned}(\eta_3)_o = \mathrm{Alt}\ u_1 v_1 u_2 v_2 u_3 &= u_1 v_1 u_2 v_2 u_3 + u_2 v_1 u_3 v_2 u_1 + u_3 v_1 u_1 v_2 u_2 \\ &- u_2 v_1 u_1 v_2 u_3 - u_3 v_1 u_2 v_2 u_1 - u_1 v_1 u_3 v_2 u_2 \\ &- u_1 v_2 u_2 v_1 u_3 - u_2 v_2 u_3 v_1 u_1 - u_3 v_2 u_1 v_1 u_2 \\ &+ u_2 v_2 u_1 v_1 u_3 + u_3 v_2 u_2 v_1 u_1 + u_1 v_2 u_3 v_1 u_2.\end{aligned} \tag{5.74}$$

Here, $u_1, u_2, u_3 \in \mathfrak{n}_+$ for $v_1, v_2 \in \mathfrak{n}_-$.

We will need the following properties of the forms introduced in Examples 5.10 and 5.11.

5.12. Lemma. *Suppose that* $M = \mathrm{Gr}_s^n$ *and (recall,* $r := n - s$*)*

(1) *The forms* θ_3, η_1, η_2, η_3 *are linearly independent whenever* $s, r \geq 3$.

(2) *If* $r = 2$ *for* $s \geq 3$, *then* θ_3, η_1, η_2 *are linearly independent, while*

$$\eta_3 = \eta_2 + \frac{1}{2}\eta_1 - \frac{1}{2}\theta_3. \tag{5.75}$$

(3) *If* $s = 2$ *for* $r \geq 3$, *then* θ_3, η_1, η_2 *are linearly independent, while*

$$\eta_3 = -\eta_2 - \frac{1}{2}\eta_1 - \frac{1}{2}\theta_3. \tag{5.76}$$

(4) *If* $s = r = 2$, *then* θ_3, η_1 *are linearly independent, while*

$$\eta_2 = -\frac{1}{2}\eta_1, \quad \eta_3 = -\frac{1}{2}\theta_3. \tag{5.77}$$

Proof. To check the relations (5.75), (5.76), (5.77), we use the following simple fact: for any two 2×2-matrices A, B we have

$$AB + BA = (\operatorname{tr} A)B + (\operatorname{tr} B)A + (\operatorname{tr} AB - (\operatorname{tr} A)(\operatorname{tr} B))I. \tag{5.78}$$

In the case (2), we obtain (5.75) by applying (5.78) to $A = u_1v_1$ and $B = u_2v_2$, and alternating the resulting expression of

$$u_1v_1u_2v_2u_3 + u_2v_2u_1v_1u_3 = (u_1v_1u_2v_2 + u_2v_2u_1v_1)u_3.$$

Similarly, in the case (3) we apply (5.78) to $A = v_1u_2$ and $B = v_2u_3$ and alternate the resulting expression of

$$u_1v_1u_2v_2u_3 + u_1v_2u_3v_1u_2 = u_1(v_1u_2v_2u_3 + v_2u_3v_1u_2).$$

Now, (5.75) and (5.76) imply (5.77) in the case (4).

I skip the proof of linear independence. □

It is well known (thanks to É. Cartan) that the real cohomology algebra of a compact Riemannian symmetric space $M = K/L$ is naturally isomorphic to the algebra of K-invariant differential forms on M (see, e.g., [38, Corollary of Theorem 9.7]). We want to prove a similar assertion concerning the invariant cohomology $H^{\bullet}(M, \Omega \otimes \Theta)^K$ of a simply connected compact Hermitian symmetric space M.

We use the fine resolution $(\Phi\otimes\Theta, \bar{\partial})$ of the sheaf $\Omega\otimes\Theta$. By the Dolbeault-Serre theorem, the sheaf cohomology $H^{\bullet}(M, \Omega \otimes \Theta)$ and the cohomology of the complex $(\Gamma(M, \Phi \otimes \Theta), \bar{\partial})$ are isomorphic. Actually, we have

$$H^q(M, \Omega^p \otimes \Theta) \simeq H^{p,q}(\Gamma(M, \Phi \otimes \Theta), \bar{\partial}).$$

Under this isomorphism, the algebraic and the FN-bracket in $H^{\bullet}(M, \Omega\otimes\Theta)$ are induced by the same operations in $\Gamma(M, \Phi\otimes\Theta)$. Denote the operator in $\Gamma(M, \Phi\otimes\Theta)$ conjugate to $\bar\partial$ with respect to the given K-invariant Hermitian metric on M by $\bar\partial^*$ and the Beltrami–Laplace operator by $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. As usual, a form $\varphi \in \Gamma(M, \Phi\otimes\Theta)$ is called *harmonic* if $\Box\varphi = 0$. For a harmonic φ, we have $\bar\partial\varphi = 0$, and any cohomology class contains precisely one harmonic form.

5.13. Proposition. *Let M be a simply connected compact Hermitian symmetric space, K the identity component of the group of all holomorphic isometries of M. Then,*

$$\Gamma(M, \Phi^r\otimes\Theta)^K = 0 \quad \text{whenever } r \text{ is even.}$$

Moreover, any $\varphi \in \Gamma(M, \Phi\otimes\Theta)^K$ is harmonic, and hence $\bar\partial$-closed. Assigning to a form $\varphi \in \Gamma(M, \Phi\otimes\Theta)^K$ its cohomology class, we get an isomorphism of bigraded algebras

$$\lambda : \Gamma(M, \Phi\otimes\Theta)^K \longrightarrow H^{\bullet}(M, \Omega\otimes\Theta)^G$$

both for the algebraic and the FN-brackets.

The FN-bracket in $H^{\bullet}(M, \Omega\otimes\Theta)^G$ is identically 0.

Proof. For any form $\varphi \in \Gamma(M, \Phi^r\otimes\Theta)^K$ we have $s^*\varphi = \varphi$. Since $ds_o = -\mathrm{id}$, we see that $(s^*\varphi)_o = (-1)^{r+1}\varphi_o$. If r is even, then $\varphi_o = 0$, and hence $\varphi = 0$. This proves the first assertion.

Moreover, in the same situation we have $\bar\partial\varphi \in \Gamma(M, \Phi^{r+1}\otimes\Theta)^K$. If r is odd, then $\bar\partial\varphi = 0$. Similarly, $\bar\partial^*\varphi = 0$, and hence φ is harmonic. It follows that

$$\lambda : \Gamma(M, \Phi\otimes\Theta)^K \longrightarrow H^{\bullet}(M, \Omega\otimes\Theta)^K = H^{\bullet}(M, \Omega\otimes\Theta)^G$$

is defined and injective. To prove that λ is surjective, suppose that $\varphi \in \Gamma(M, \Phi\otimes\Theta)$ is a harmonic form representing a G-invariant cohomology class. Then, for any $k \in K$, the form $k^*\varphi$ is harmonic and lies in the same cohomology class as φ. Therefore, $k^*\varphi = \varphi$ for $k \in K$, so $\varphi \in \Gamma(M, \Phi\otimes\Theta)^K$.

Clearly, $\Gamma(M, \Phi\otimes\Theta)^K$ is a subalgebra under both brackets and λ is an isomorphism of algebras. The FN-bracket is 0, since $H^q(M, \Omega^p\otimes\Theta)^G = 0$ whenever $p+q$ is even. $\Box$

5.14. Corollary. *Under assumptions of Proposition 5.13, we have* (recall definition of R under (5.69))

$$H^q(M, \Omega^p\otimes\Theta)^G \simeq (\bigwedge^p \mathfrak{n}_- \otimes \bigwedge^q \mathfrak{n}_+ \otimes \mathfrak{n}_+)^R.$$

Now we are going to calculate certain invariant cohomology groups assuming that M is irreducible. First of all, we will find the degrees for which they are non-zero.

5.15. Proposition. *For any simply connected irreducible compact Hermitian symmetric space M of dimension $n \geq 2$ we have $H^q(M, \Omega^p\otimes\Theta)^G \neq 0$ if and only if $q = p-1$ for any $p = 1, \dots, n$.*

Proof. Let ω be the Kähler form on M corresponding to the given Kähler metric. Consider the invariant forms θ_p for $p = 1, \dots, n$, given by the formula (4.14). By Proposition 5.13, they determine non-zero cohomology classes in $H^{p-1}(M, \Omega^p \otimes \Theta)^G$.

By Corollary 5.14, it is sufficient to show that the representation of R induced in $\bigwedge^p \mathfrak{n}_- \otimes \bigwedge^q \mathfrak{n}_+ \otimes \mathfrak{n}_+$ has no zero weights whenever $q \neq p-1$. But each weight of this representation has the form

$$\lambda = (-p+q+1)\alpha_0 + \sum_{j=1}^{l-1} k_j \alpha_j.$$

If $\lambda = 0$, then $q = p-1$. □

It was proved in [40] that

$$H^{p-1}(M, \Omega^p \otimes \Theta)^G \simeq \mathbb{C}, \text{ where } p = 1, \dots, n,$$

for $M = \mathbb{CP}^n$ (case III). We investigate now the degrees $p = 1, 2, 3$ in the general case.

By Lemma 5.4, we have

$$H^0(M, \Omega^1 \otimes \Theta)^G \simeq \mathbb{C}.$$

For the case $p = 2$ we need the following fact, implied by a result of Kostant (see [26])]. We will denote by σ_j the reflection $\sigma_{\alpha_j} \in W$ corresponding to the simple root α_j.

5.16. Lemma. *The irreducible components of the R-module $\bigwedge^2 \mathfrak{n}_+$ correspond one-to-one to those simple roots α_k of G that are neighbors of α_0. The component that corresponds to α_k has the lowest weight $2\alpha_0 + \alpha_k$ and the lowest weight vector $e_{\alpha_0} \wedge e_{\alpha_0+\alpha_k}$.*

Thus, $\bigwedge^2 \mathfrak{n}_+$ is irreducible in the cases I, III and has two irreducible components in the case II.

Proof. By [26, Section 8], the irreducible components of $\bigwedge^2 \mathfrak{n}_+$ correspond to those elements $\sigma \in W$ satisfying

1. $\sigma = \sigma_j \sigma_k$ for $j \neq k$,
2. $\Phi_\sigma = \sigma\Delta_- \cap \Delta_+ \subset \Delta(N_+)$.

The set $\Phi_\sigma = \{\alpha, \beta\}$ can be determined from the relation

$$\sigma\gamma = \gamma - \alpha - \beta.$$

The lowest weight of the component corresponding to σ is $\alpha + \beta$, and the lowest weight vector is $e_\alpha \wedge e_\beta$.

Clearly,

$$\sigma\gamma = \sigma_j \sigma_k \gamma = \sigma_j(\gamma - \alpha_k) = \gamma - \alpha_j - \sigma_j \alpha_k.$$

Hence, $\alpha+\beta=\alpha_j+\sigma_j\alpha_k$, where $\sigma_j\alpha_k=\alpha_k-\langle\alpha_k,\alpha_j\rangle\alpha_j$. Since α and β contain α_0 with coefficient 1, the same property must have the roots α_j and $\sigma_j\alpha_k$. It follows that $j=0$ and $\langle\alpha_k,\alpha_0\rangle\neq 0$, i.e., α_k is a neighbor of α_0. Since $\langle\alpha_k,\alpha_0\rangle=-1$, we have

$$\alpha+\beta=\alpha_0+(\alpha_0+\alpha_k).$$

This easily implies that $\alpha=\alpha_0$ and $\beta=\alpha_0+\alpha_k$ (or vice versa). □

5.17 Proposition ($H^1(M,\Omega^2\otimes\Theta)^G$). *We have*

$$H^1(M,\Omega^2\otimes\Theta)^G\simeq\begin{cases}\mathbb{C} & \text{in the cases I, III}\\ \mathbb{C}^2 & \text{in the case II.}\end{cases}$$

Proof. By Corollary 5.2,

$$H^1(M,\Omega^2\otimes\Theta)^G\simeq(\bigwedge^2\mathfrak{n}_-\otimes\mathfrak{n}_+\otimes\mathfrak{n}_+)^R.$$

Now,

$$\mathfrak{n}_+\otimes\mathfrak{n}_+=\bigwedge^2\mathfrak{n}_+\oplus \mathrm{S}^2\,\mathfrak{n}_+.$$

By Lemma 5.16, $\bigwedge^2\mathfrak{n}_+$ is the irreducible R-module with lowest weight $2\alpha_0+\alpha_1$ in the cases I, III. It is easy to prove that it is not isomorphic to any submodule of $\mathrm{S}^2\,\mathfrak{n}_+$. Indeed, the lowest weight vector of such a submodule must be $e_{\alpha_0}e_{\alpha_0+\alpha_1}$, which is impossible. Thus, $\mathfrak{n}_+\otimes\mathfrak{n}_+$ contains precisely one component dual to $\bigwedge^2\mathfrak{n}_-$, implying the result. The case II is considered similarly. □

5.18. Remark. Clearly, in the cases I, III, a basic element of $H^1(M,\Omega^2\otimes\Theta)^G$ is determined by the invariant form θ_2 given by the formula (5.65), ω being determined by the formula (5.68). In the case II, a basis of $H^1(M,\Omega^2\otimes\Theta)^G$ is formed by the cohomology classes of θ_2 and η, where η is given by the formula (5.71), see Example 5.11.

The following proposition can be proved by case-by-case verification using the decompositions into irreducible components. We omit the proof, since we will not use the result.

5.19 Proposition ($\dim H^2(M,\Omega^3\otimes\Theta)^G$). *The dimension* $k=\dim H^2(M,\Omega^3\otimes\Theta)^G$ *is as follows:*

(1) $k=0$ *in the case III for* $l=2$ ($M=\mathbb{CP}^2$)*;*

(2) $k=1$ *in the case I for types* B_l,E_l *and* D_l *for* $l>4$*, if* M *is a quadric, and in the case III for* $l>2$*;*

(3) *$k = 2$ in the case I for types C_l and D_l, if M is the isotropic Grassmannian of maximal type, and in the case II, if $l = s + 1 = 3$ ($M = \mathrm{Gr}_2^4$);*

(4) *$k = 3$ in the case II whenever $2 < s = l - 1$ or $2 = s < l - 1$;*

(5) *$k = 4$ in the case II whenever $2 < s < l - 1$.*

5.20. Remark. In the case II, a basis of $H^2(M, \Omega^3 \otimes \Theta)^G$ is given by the cohomology classes of the following forms:

$$\begin{aligned}
&\theta_2, \eta_1 \text{ for } s = t = 2,\\
&\theta_2, \eta_1, \eta_2 \text{ for } s = 2 \text{ for } t \geq 3 \text{ or } s \geq 3,\ t = 2,\\
&\theta_2, \eta_1, \eta_2, \eta_3 \text{ for } s, t \geq 3
\end{aligned}$$

(see Example 5.11 and Lemma 5.12).

5.22 An application of a theorem of Bott Let again M be an irreducible simply connected compact Hermitian symmetric space. In this subsection, we apply Theorem 5.1 to calculation of the cohomology $H^q(M, \Omega^p \otimes \Theta)$ for $q = 1, 2$. We regard $\Omega^p \otimes \Theta$ as the sheaf of holomorphic sections of the homogeneous vector bundle $\bigwedge^p \mathbf{T}(M)^* \otimes \mathbf{T}(M)$ corresponding to the completely reducible representation $\tau \bigwedge^p \tau^*$ of P.

The following well-known property of dominant weights will be used (see [22, S 13, Exercise 8]).

5.21. Lemma. *If λ is a non-zero dominant weight of a simple group G, then in the expression*

$$\lambda = \sum_{i=0}^{l-1} k_i \alpha_i$$

we have $k_i > 0$ for all $i = 0, \dots, l - 1$.

A weight λ of G will be called *R-dominant* if $(\lambda, \alpha_i) \geq 0$ for all $i = 1, \dots, l - 1$. Any highest weight of a representation of R is, evidently, R-dominant.

Recall that in the theory of Bott the operation $\xi \mapsto \xi^*$ given by the formula (5.58) is essential. Note that if $\sigma = \sigma_i$ is the reflection corresponding to the simple root α_i, then

$$\xi^* = \sigma_i \xi - \alpha_i = \xi - (1 + \langle \xi, \alpha_i \rangle)\alpha_i. \tag{5.79}$$

We also need the following lemmas.

5.22. Lemma. *Let λ be an R-dominant weight of G. The weight $\lambda + \gamma$ has index 1 if and only if $\lambda^* = \sigma_0(\lambda + \gamma) - \gamma$ is dominant.*

Proof. Clearly, the condition is sufficient. Now suppose that $\lambda+\gamma$ has index 1. Then, α_0 is the only positive root of G such that $(\lambda+\gamma,\alpha_0)<0$. For any $i>0$, we have

$$(\lambda^*,\alpha_i)=(\sigma_0(\lambda+\gamma),\alpha_i)-1=(\lambda+\gamma,\sigma_0\alpha_i)-1.$$

Since $\sigma_0\alpha_i=\alpha_i-\langle\alpha_i,\alpha_0\rangle\alpha_0$ is a positive root not equal to α_0, this number is non-negative. Also

$$(\lambda^*,\alpha_0)=(\sigma_0(\lambda+\gamma),\alpha_0)-1=-(\lambda+\gamma,\alpha_0)-1\geq 0.$$

Thus, λ^* is dominant. □

5.23. Lemma. (1) *A root $\alpha\in\Delta(N_+)$ satisfies $m(\alpha)=0$ if and only if $\alpha=\alpha_0$.*

(2) *Let λ be a weight of the representation $\tau\bigwedge^p\tau^*$ for $p\geq 1$, i.e.,*

$$\lambda=\alpha-\beta_1-\ldots-\beta_p, \tag{5.80}$$

where $\alpha,\beta_i\in\Delta(N_+)$, β_i are all distinct. Then,

$$m(\lambda)\leq\begin{cases}3-p \text{ in the cases } I, II\\ 2-p \text{ in the case } III.\end{cases}$$

If the equality takes place here, then $m(\alpha)=m(\delta)$, one of β_i coincides with α_0, and we have $m(\beta_i)=1$ for all $\beta_i\neq\alpha_0$.

Proof. (1) If α does not coincide with the lowest root α_0 of τ, then there exists a sequence of simple roots $\alpha_{j_1},\ldots,\alpha_{j_k}$ such that

$$\alpha=(\ldots((\alpha_0+\alpha_{j_1})+\alpha_{j_2})+\ldots)+\alpha_{j_k},$$

where any sum in parentheses is a root. In particular, we have that $\alpha_0+\alpha_{j_1}\in\Delta(N_+)$; whence $(\alpha_0,\alpha_{j_1})<0$, and α_{j_1} is a neighbor of α_0.

(2) The number $m(\lambda)$ attains its maximum whenever $m(\alpha)$ is maximal (that is, whenever $m(\alpha)=m(\delta)$) and $m(\beta_i)$ are minimal (that is, $=0,1$). Due to item (1), $m(\beta_i)=0$ for only one root $\beta_i=\alpha_0$. Therefore,

$$m(\lambda)\leq\begin{cases}2-(p-1)=3-p \text{ in the cases I, II}\\ 1-(p-1)=2-p \text{ in the case III,}\end{cases}$$

and the equality takes place in the situation described above. □

5.24 Proposition ($H^p(M,\Omega^1\otimes\Theta)$). *We have*

$$H^1(M,\Omega^1\otimes\Theta)\simeq\begin{cases}\mathfrak{g} \text{ in the cases I, II,}\\ 0 \text{ in the case III,}\end{cases}$$

$$H^p(M,\Omega^1\otimes\Theta)=0 \text{ for } p\geq 2.$$

Proof. The representation $\tau^*\tau$ contains a unique irreducible component with highest weight $\lambda_0 = \delta - \alpha_0$. By (5.79),

$$\lambda_0^* = \sigma_0(\lambda_0 + \gamma) - \gamma = \sigma_0\delta = \delta - \langle \delta, \alpha_0 \rangle \alpha_0.$$

In the cases I and II, $\lambda_0^* = \delta$ is dominant. By Lemma 5.22, $\lambda_0 + \gamma$ has index 1, and, by Bott's theorem, we get a unique G-submodule of $H^1(M, \Omega^1 \otimes \Theta)$ isomorphic to $\mathfrak{g}$. In the case III, we have $\langle \lambda_0, \alpha_0 \rangle = -1$. Therefore, $\lambda_0 + \gamma$ is singular, and, by Bott's theorem, our component gives nothing to the cohomology.

Now, it suffices to prove that any non-dominant highest weight λ of $\tau^*\tau$, such that $\lambda + \gamma$ is regular, coincides with λ_0. Clearly, $\lambda = \alpha - \beta$, where $\alpha, \beta \in \Delta(N_+)$ for $\alpha \neq \beta$. Since λ does not contain α_0, we have

$$\lambda = \sum_{j=1}^{l-1} k_j \alpha_j,$$

where $k_j \in \mathbb{Z}$. Since $\lambda \prec \alpha \preceq \delta$, we have $k_j \leq n_{\alpha_j}$ for $j = 1, \dots, l-1$. In particular, $m(\lambda) \leq m(\delta)$. Since λ is R-dominant, but not dominant, and $\lambda + \gamma$ is regular, it follows that $\langle \lambda, \alpha_0 \rangle \leq -2$. On the other hand, $\langle \lambda, \alpha_0 \rangle = -m(\lambda)$, whence $m(\lambda) \geq 2$. We see that the case III is impossible and that in the cases I and II we have $m(\lambda) = m(\delta) = 2$. Then, Lemma 5.23(2) implies that $\beta = \alpha_0$.

Thus, $\lambda = \alpha - \alpha_0$ is the only expression of the weight λ as a difference of two roots from $\Delta(N_+)$. It follows that the corresponding highest vector $v \in \mathfrak{n}_+ \otimes \mathfrak{n}_-$ has the form

$$v = e_\alpha \otimes e_{-\alpha_0}.$$

But this vector cannot be a highest one if $\alpha \neq \delta$. Thus, $\lambda = \lambda_0$. □

The next proposition reduces calculation of $H^1(M, \Omega^p \otimes \Theta)$ for $p \geq 2$ to the results of Subsection 5.6, where its invariant part has been calculated.

5.25 Proposition ($H^1(M, \Omega^p \otimes \Theta) = H^1(M, \Omega^p \otimes \Theta)^G$). *For $p \geq 2$, we have*

$$H^1(M, \Omega^p \otimes \Theta) = H^1(M, \Omega^p \otimes \Theta)^G.$$

Proof. Let λ be a highest weight of $\tau \bigwedge^p \tau^*$ for $p \geq 2$. Then, λ has the form (5.80). Hence,

$$\lambda = (1-p)\alpha_0 + \mu, \tag{5.81}$$

where

$$\mu = \sum_{j=1}^{l-1} k_j \alpha_j \text{ for } k_j \in \mathbb{Z},\ k_j \leq n_{\alpha_j},\ j = 1, \dots, l-1. \tag{5.82}$$

Since $1 - p < 0$, it follows that λ is not dominant due to Lemma 5.22. Hence $(\lambda, \alpha_0) < 0$. But it is R-dominant, and hence $\lambda + \gamma$ has index 1 if and only if $\lambda^* = \sigma_0(\lambda + \gamma) - \gamma$ is dominant (see Lemma 5.21). Clearly,

$$\lambda^* = \sigma_0\lambda - \alpha_0 = (p-2)\alpha_0 + \sigma_0\mu = (p - 2 - \langle \mu, \alpha_0 \rangle)\alpha_0 + \mu = (p - 2 + m(\lambda))\alpha_0 + \mu.$$

By Bott's theorem we have to show that λ^* cannot be dominant and non-zero.

Suppose that the weight λ^* is dominant and non-zero. Then, by Lemma 5.21,

$$k_j > 0 \quad \text{for } j = 1, \ldots, l-1; \quad m(\lambda) > 2 - p.$$

Applying Lemma 5.23(2), we see that the case III is impossible and that in the cases I and II we have $m(\lambda) = 3 - p$. Since $m(\lambda) > 0$, it follows that $p = 2$ and $m(\lambda) = 1$. Thus,

$$\lambda^* = \alpha_0 + \mu, \quad \lambda = -\alpha_0 + \mu.$$

In the case II, we have $m(\lambda) = k_1 + k_2 = 2$ which gives a contradiction. Now me must consider the case I only.

Clearly, $\alpha_0 + \alpha_1 = \sigma_0\alpha_1$ is a positive root of G. Since $\lambda + \gamma$ has index 1 and $(\lambda, \alpha_0) < 0$, we get $(\lambda, \alpha_0 + \alpha_1) \geq 0$. On the other hand,

$$(\lambda, \alpha_0 + \alpha_1) = (-\alpha_0 + \alpha_1 + \sum_{j=2}^{l-1} k_j\alpha_j, \alpha_0 + \alpha_1) = -2 + (\alpha_1, \alpha_1) + \sum_{j=2}^{l-1} k_j(\alpha_j, \alpha_1).$$

If $l \geq 3$, we get $(\lambda, \alpha_0 + \alpha_1) < 0$ which is a contradiction. If $l = 2$, then G is of type B_2, and $(\lambda, \alpha_0 + \alpha_1) = -1 < 0$, too. □

5.26 Proposition ($H^2(M, \Omega^p \otimes \Theta)$). *For $p = 2$ or $p \geq 4$, we have*

$$H^2(M, \Omega^p \otimes \Theta) = 0.$$

Also,

$$H^2(M, \Omega^3 \otimes \Theta) = H^2(M, \Omega^3 \otimes \Theta)^G.$$

Proof. Let λ be a highest weight of $\tau \bigwedge^p \tau^*$ for $p \geq 2$. Then, as in Proposition 5.26, statements (5.80), (5.81) and (5.82) hold. Similarly, λ is R-dominant, but not dominant, and hence $(\lambda, \alpha_0) < 0$. Suppose that the index of $\lambda + \gamma$ is 2. As in the proof of Proposition 5.25,

$$\sigma_0(\lambda + \gamma) = (p - 2 + m(\lambda))\alpha_0 + \mu + \gamma.$$

We have

$$(\sigma_0(\lambda + \gamma), \alpha_0) = -(\lambda + \gamma, \alpha_0) > 0,$$
$$(\sigma_0(\lambda + \gamma), \alpha_j) = (\lambda + \gamma, \sigma_0\alpha_j) = (\lambda + \gamma, \alpha_j) > 0,$$

if α_j is not a neighbor of α_0. Since the index is equal to 2, $\sigma_0(\lambda + \gamma)$ is regular and non-dominant, and hence

$$(\sigma_0(\lambda + \gamma), \alpha_1) < 0$$

for a neighbor α_1 of α_0. Then, the weight

$$\lambda^* = \sigma_1\sigma_0(\lambda + \gamma) - \gamma$$

must be dominant. Using (5.79), we get

$$\lambda^* = (p-2+m(\lambda))\alpha_0 + ((-p+2-m(\lambda))\langle\alpha_0,\alpha_1\rangle - \langle\mu,\alpha_1\rangle + k_1 - 1)\alpha_1 + \mu', \quad (5.83)$$

where

$$\mu' = \sum_{j=2}^{l-1} k_j\alpha_j.$$

By Proposition 5.15, $\lambda^* \neq 0$, if $p \neq 3$. Suppose that $\lambda^* \neq 0$ for $p = 3$, too. Then, by Lemma 5.21, all the coefficients in (5.83) are positive. In particular,

$$k_j > 0 \text{ for } j = 2, \ldots, l-1 \text{ and } m(\lambda) > 2-p.$$

Applying Lemma 5.23(2), we see that the case III is impossible and that in the cases I and II we have $m(\lambda) = 3-p$.

Now consider the weight

$$\widetilde{\lambda} = \sigma_0(\lambda+\gamma) - \gamma.$$

Clearly, $\widetilde{\lambda}+\gamma$ is of index 1. As we saw, $(\widetilde{\lambda}+\gamma,\alpha_1) < 0$, and hence α_1 is the only positive root with this property. It follows from formula (5.79) that

$$\widetilde{\lambda} = \alpha_0 + \mu.$$

Therefore,

$$(\widetilde{\lambda},\alpha_0) = 2 - m(\lambda) = p-1.$$

To get a contradiction, we consider separately three cases.

1) Case II. We have, evidently, $k_2 = \ldots = k_{l-1} = 1$, and hence

$$\widetilde{\lambda} = \alpha_0 + (2-p)\alpha_1 + \alpha_2 + \ldots + \alpha_{l-1}.$$

Therefore,

$$(\widetilde{\lambda},\alpha_1) = \begin{cases} 3-2p \text{ if } \alpha_1 \text{ corresponds to an end vertex of the Dynkin diagram,} \\ 2-2p \text{ otherwise.} \end{cases}$$

Hence,

$$(\widetilde{\lambda},\alpha_1) < 0$$

for all $p \geq 2$. If $p = 2$, then the first case is impossible, because $\widetilde{\lambda}+\gamma$ is singular. Now,

$$(\widetilde{\lambda},\alpha_1+\alpha_0) = (p-1) + (2-2p) = 1-p < 0$$

for $p \geq 2$. This gives a contradiction.

2) Case I, the type of G is not C_l. We have

$$\widetilde{\lambda} = \alpha_0 + (3-p)\alpha_1 + \mu', \quad (5.84)$$

where $\mu' \neq 0$ (since $l \geq 3$) and α_1 is long. Hence,

$$(\widetilde{\lambda}, \alpha_1) = 5 - 2p + \sum_{j=2}^{l-1} k_j(\alpha_j, \alpha_1) < 5 - 2p,$$

and

$$(\widetilde{\lambda}, \alpha_0 + \alpha_1) < 4 - p.$$

This gives a contradiction whenever $p \geq 4$.

If $p = 2$, then

$$(\widetilde{\lambda}, \alpha_0 + \alpha_1) = 1 + (\widetilde{\lambda}, \alpha_1) \leq -1,$$

since $(\widetilde{\lambda}, \alpha_1) \leq -2$, and we get a contradiction as well.

For $p = 3$, the same argument shows that $(\widetilde{\lambda}, \alpha_1) = -2$. Then, we see from (5.84) that there exists precisely one root α_j (say, for $j = 2$) such that $(\alpha_j, \alpha_1) \neq 0$, and we have $k_2 = 1$ for $(\alpha_2, \alpha_1) = -1$. Then, $\alpha_1 + \alpha_2 \in \Delta_+$ and hence

$$0 \leq (\widetilde{\lambda}, \alpha_1 + \alpha_2) = -2 + (\widetilde{\lambda}, \alpha_2).$$

Thus, $(\widetilde{\lambda}, \alpha_2) \geq 2$. But

$$\widetilde{\lambda} = \alpha_0 + \alpha_1 + \alpha_2 + \sum_{j=3}^{l-1} k_j \alpha_j,$$

whence

$$(\widetilde{\lambda}, \alpha_2) = 1 + \sum_{j=3}^{l-1} k_j(\alpha_j, \alpha_2) \leq 1.$$

3) Case I, the type of G is C_l. Here, equality (5.84) holds as well, but α_1 is short. Hence,

$$(\widetilde{\lambda}, \alpha_1) = 2 - p + \sum_{j=2}^{l-1} k_j(\alpha_j, \alpha_1) \leq 2 - p.$$

But in this case, $\alpha_0 + 2\alpha_1 \in \Delta(N_+)$, and

$$(\widetilde{\lambda}, \alpha_0 + 2\alpha_1) \leq 3 - p.$$

This gives a contradiction whenever $p \geq 4$.

On the other hand, $\langle \widetilde{\lambda}, \alpha_1 \rangle \leq -2$, whence $(\widetilde{\lambda}, \alpha_1) \leq -1$, and $(\widetilde{\lambda}, \alpha_0 + 2\alpha_1) \leq p - 3$, which gives a contradiction for $p = 2$.

For $p = 3$, we see that the equality $(\widetilde{\lambda}, \alpha_1) \leq -1$ is compatible with (5.84) only for $\widetilde{\lambda} = \alpha_0$ for $l = 2$. But then $\lambda = -2\alpha_0$, and it is easy to see that in this case

$$-2\alpha_0 \neq \alpha - \alpha_0 - \beta_1 - \beta_2 \text{ for any } \alpha, \beta_1, \beta_2 \in \Delta(N_+).$$

□

5.26 Cohomology of $\mathcal{T}$ Summarizing the results of Subsections 5.6 and 5.32, we now describe the structure of the cohomology $H^q(M, \Omega^p \otimes \Theta)$ for $q = 0, 1, 2$ under our assumptions about M.

5.27. Proposition. *Suppose that M is a simply connected irreducible compact Hermitian symmetric space of dimension ≥ 2. The G-modules $H^q(M, \Omega^p \otimes \Theta)$ for $q = 0, 1, 2$, are listed in the following tables:*

Case I:

$q \backslash p$	0	1	2	3	4...
0	$\mathfrak{g}$	$\mathbb{C}$	0	0	0
1	0	$\mathfrak{g}$	$\mathbb{C}$	0	0
2	0	0	0	$\mathbb{C}^k$	0

Case II:

$q \backslash p$	0	1	2	3	4...
0	$\mathfrak{g}$	$\mathbb{C}$	0	0	0
1	0	$\mathfrak{g}$	$\mathbb{C}^2$	0	0
2	0	0	0	$\mathbb{C}^k$	0

Case III:

$q \backslash p$	0	1	2	3	4...
0	$\mathfrak{g}$	$\mathbb{C}$	0	0	0
1	0	0	$\mathbb{C}$	0	0
2	0	0	0	$\mathbb{C}^k$	0

where we denote by $\mathbb{C}$ the trivial G-module and by $\mathfrak{g}$ the adjoint one, and the number k is to be found in Proposition 5.19.

Due to Proposition 3.3, this result permits us to describe $H^q(M, \mathcal{T}_p)$ for $q = 0, 1, 2$.

5.28 Theorem (The G-modules $H^q(M, \mathcal{T}_p)$). *Suppose that M is a simply connected irreducible compact Hermitian symmetric space of dimension ≥ 2. The G-modules $H^q(M, \mathcal{T}_p)$ for $q = 0, 1, 2$, where $\mathcal{T} = \mathcal{D}er\, \Omega$, are listed in the following tables:*

Case I:

$q \backslash p$	-1	0	1	2	3	4...
0	$i^*(\mathfrak{g})$	$l^*(\mathfrak{g}) \oplus i^*(\mathbb{C})$	$l^*(\mathbb{C})$	0	0	0
1	0	$i^*(\mathfrak{g})$	$l^*(\mathfrak{g}) \oplus i^*(\mathbb{C})$	$l^*(\mathbb{C})$	0	0
2	0	0	0	$i^*(\mathbb{C}^k)$	$l^*(\mathbb{C}^k)$	0

Case II:

$q \backslash p$	-1	0	1	2	3	$4\dots$
0	$i^*(\mathfrak{g})$	$l^*(\mathfrak{g}) \oplus i^*(\mathbb{C})$	$l^*(\mathbb{C})$	0	0	0
1	0	$i^*(\mathfrak{g})$	$l^*(\mathfrak{g}) \oplus i^*(\mathbb{C}^2)$	$l^*(\mathbb{C}^2)$	0	0
2	0	0	0	$i^*(\mathbb{C}^k)$	$l^*(\mathbb{C}^k)$	0

Case III:

$q \backslash p$	-1	0	1	2	3	$4\dots$
0	$i^*(\mathfrak{g})$	$l^*(\mathfrak{g}) \oplus i^*(\mathbb{C})$	$l^*(\mathbb{C})$	0	0	0
1	0	0	$i^*(\mathbb{C})$	$l^*(\mathbb{C})$	0	0
2	0	0	0	$i^*(\mathbb{C})$	$l^*(\mathbb{C}^k)$	0

where we denote by $\mathbb{C}$ the trivial G-module and by $\mathfrak{g}$ the adjoint one, and the number k is to be found in Proposition 5.19.

Using Proposition 3.3, it is also possible to calculate the Lie bracket $[-,-]$ for the part of the algebra $H^{\bullet}(M,\mathcal{T})$ that is described in Theorem 5.28. Here we calculate only the adjoint operator $\operatorname{ad}\zeta$, where $\zeta \in H^1(M,\mathcal{T}_2)$.

Recall a result of Bott (see [4, Theorem I and Corollary 2 of Theorem W], and also [26]) that describes the cohomology of a flag manifold $M = G/P$ with values in the sheaf of holomorphic sections of a homogeneous vector bundle $\mathbf{E} \longrightarrow M$ in terms of the cohomology of the Lie algebra $\mathfrak{n}_-$. Suppose that $\mathbf{E} = \mathbf{E}_\varphi$, where φ is a holomorphic representation of P. In contrast to Theorem 5.1, this description is valid for arbitrary φ.

5.29. Theorem. *Let a holomorphic representation of G in a finite-dimensional vector space V be given. Then,*

$$\operatorname{Hom}_G(V, H^q(M,\mathcal{E})) \simeq H^q(\mathfrak{n}_-, \operatorname{Hom}(V, E_o))^R, \tag{5.85}$$

where the representation of $\mathfrak{n}_-$ in V is the restriction of the differential of the given representation of G, and that in E_o is the restriction of φ.

Proof. oof (a sketch of) By the Dolbeault–Serre theorem, $H^q(M,\mathcal{E})$ can be identified with the q-th cohomology of the complex $(\Gamma(M,\Phi^{0,*}\otimes\mathcal{E}),\bar\partial)$ of $\mathbf{E}$-valued forms of type $(0,*)$. The vector space $\Gamma(M,\Phi^{0,q}\otimes\mathcal{E})$ is the space of smooth sections of the homogeneous vector bundle $\bigwedge^q \mathbf{T}^{0,1}(M)^* \otimes \mathbf{E}$, whose fiber at o can be identified with $\bigwedge^q \mathfrak{n}_-^* \otimes E_o$. By the Frobenius reciprocity law,

$$\begin{aligned}\operatorname{Hom}_K(V, \Gamma(M,\Phi^{0,q}\otimes\mathcal{E}) &\simeq \operatorname{Hom}_L(V, \bigwedge^q \mathfrak{n}_-^* \otimes E_o)\\ &= \operatorname{Hom}_R(V, \bigwedge^q \mathfrak{n}_-^* \otimes E_o) = C^q(\mathfrak{n}_-, \operatorname{Hom}(V, E_o))^R.\end{aligned}$$

The isomorphism here is defined by the formula

$$h \mapsto \widetilde{h}, \text{ where } \widetilde{h}(v) = h(v)(o) \text{ for any } v \in V, \tag{5.86}$$

and we denote by $C^q(\mathfrak{n}_-, \mathrm{Hom}(V, E_o))$ the vector space of q-cochains of the Lie algebra $\mathfrak{n}_-$ with values in $\mathrm{Hom}(V, E_o)$. Passing to the cohomology, we get the isomorphism (5.85).

□

5.30. Proposition. *Let $\zeta = l^*([\theta]) \in H^1(M, \mathcal{T}_2)$, where $[\theta] \in H^1(M, \Omega^2 \otimes \Theta)$ is the cohomology class of the form $\theta \in \Gamma(M, \Phi^{2,1} \otimes \Theta)^K$. The map $\mathrm{ad}_\zeta : H^0(M, \mathcal{T}_{-1}) \to H^1(M, \mathcal{T}_1)$ is as follows:*

(1) *an isomorphism of the G-modules*

$$H^0(M, \mathcal{T}_{-1}) = i^*(H^0(M, \Theta)) \longrightarrow l^*(H^1(M, \Omega^1 \otimes \Theta))$$

for any $\theta \neq 0$ in the case I and for any $\theta = a\theta_2 + b\eta$, where $a \neq 0$, in the case II;

(2) *0 for $\theta = b\eta$ in the cases II and III.*

Proof. For any $w \in \mathfrak{g}$ we have, by (3.28),

$$[l(\theta), i(w)] = [i(w), l(\theta)] = l(\theta \,\overline{\wedge}\, w) - i([w, \theta]).$$

Since θ is K-invariant, we see that $[w, \theta] = 0$. By Proposition 3.3, $[l^*([\theta]), i^*(w)]$ is determined by the cocycle $l(\theta \,\overline{\wedge}\, w)$. Thus, our problem is reduced to the study of the mapping

$$H^0(M, \Theta) \to H^1(M, \Omega^1 \otimes \Theta)$$

defined on the cochain level by $w \mapsto \theta \,\overline{\wedge}\, w$. Recall that the form $\theta \,\overline{\wedge}\, w \in \Gamma(M, \Phi^{1,1} \otimes \Theta)$ is given by the formula

$$(\theta \,\overline{\wedge}\, w)(u, v) = \theta(w, u, v) \text{ for } u \in \Theta, v \in \bar{\Theta}.$$

We will use the isomorphism

$$\mathrm{Hom}_G(\mathfrak{g}, H^1(M, \Omega^1 \otimes \Theta)) \simeq H^1(\mathfrak{n}_-, \mathrm{Hom}(\mathfrak{g}, \mathfrak{n}_+^* \otimes \mathfrak{n}_+))^R \tag{5.87}$$

that follows from (5.85) if we identify the fiber E_o of the bundle $\mathbf{E} = \mathbf{T}(M)^* \otimes \mathbf{T}(M)$ with $\mathfrak{n}_+^* \otimes \mathfrak{n}_+$. As it was noticed above, this isomorphism on the cochain level is determined by (5.86). Let

$$h : \mathfrak{g} \longrightarrow \Gamma(M, \Phi^{1,1} \otimes \Theta) \; (= \Gamma(M, \Phi^{0,1} \otimes \Omega^1 \otimes \Theta))$$

be given by the formula $h(w) = \theta \,\overline{\wedge}\, w$. Then, h determines the mapping

$$\widetilde{h} : \mathfrak{g} \to (\mathfrak{n}_+^* \otimes \mathfrak{n}_-^*) \otimes \mathfrak{n}_+ = \mathrm{Hom}(\mathfrak{n}_+ \otimes \mathfrak{n}_-, \mathfrak{n}_+)$$

given by the formula

$$\widetilde{h}(w)(u,v) = \theta_o(\pi(w), u, v), \quad u \in \mathfrak{n}_+ \text{ for } v \in \mathfrak{n}_-,$$

where we identify the value $w(o)$ of the vector field w at o with $\pi(w)$, where $\pi : \mathfrak{g} \longrightarrow \mathfrak{n}_+$ is the projection along $\mathfrak{p}$ in the decomposition (5.54). In order to interprete $\widetilde{h}(w)$ as an element of

$$\mathfrak{n}_-^* \otimes (\mathfrak{n}_+^* \otimes \mathfrak{n}_+) = \mathrm{Hom}(\mathfrak{n}_-, \mathfrak{n}_+^* \otimes \mathfrak{n}_+),$$

we choose a basis $e_1, \dots, e_n$ of $\mathfrak{n}_+$ and denote by $e_1^*, \dots, e_n^*$ the dual basis of $\mathfrak{n}_+^*$. Then,

$$\widetilde{h}(w)(v) = \sum_{i=1}^n e_i^* \otimes \widetilde{h}(w)(u,v) = \sum_{i=1}^n e_i^* \otimes \theta_o(\pi(w), e_i, v), \quad v \in \mathfrak{n}_-.$$

Now, this form is viewed as the following cochain $c_\gamma \in C^1(\mathfrak{n}_+^*, \mathrm{Hom}(\mathfrak{g}, \mathfrak{n}_- \otimes \mathfrak{n}_+))$:

$$c_\theta(v)(w) = \sum_{i=1}^n e_i^* \otimes \theta_o(\pi(w), e_i, v), \quad v \in \mathfrak{n}_-, \ w \in \mathfrak{g}. \tag{5.88}$$

This cochain is an R-invariant cocycle of $\mathfrak{n}_-$, and we have to understand what is its cohomology class. By Proposition 5.27, we have

$$H^1(\mathfrak{n}_-, \mathrm{Hom}(\mathfrak{g}, \mathfrak{n}_+^* \otimes \mathfrak{n}_+))^R \simeq \begin{cases} \mathbb{C} & \text{in the cases I, II,} \\ 0 & \text{in the case III.} \end{cases}$$

It is convenient to identify $\mathfrak{n}_+^*$ with $\mathfrak{n}_-$ using the Killing form. Then, we have to consider the cochain complex $C^*(\mathfrak{n}_-, \mathrm{Hom}(\mathfrak{g}, \mathfrak{n}_- \otimes \mathfrak{n}_+))^R, \delta)$. Let us describe the space of 1-coboundaries $\delta C^0(\mathfrak{n}_-, \mathrm{Hom}(\mathfrak{g}, \mathfrak{n}_- \otimes \mathfrak{n}_+))^R$. Clearly,

$$C^0(\mathfrak{n}_-, \mathrm{Hom}(\mathfrak{g}, \mathfrak{n}_- \otimes \mathfrak{n}_+))^R = \mathrm{Hom}_R(\mathfrak{g}, \mathfrak{n}_- \otimes \mathfrak{n}_+).$$

For any $c \in \mathrm{Hom}_R(\mathfrak{g}, \mathfrak{n}_- \otimes \mathfrak{n}_+)$, we have $\delta c(y) = yc$ for any $y \in \mathfrak{n}_-$, i.e.,

$$\delta c(y)(z) = c([y,z]), \quad \text{for any } y \in \mathfrak{n}_-, \ z \in \mathfrak{g},$$

since $d\tau(\mathfrak{n}_-) = 0$. Clearly, $[\mathfrak{n}_-, \mathfrak{g}] = \mathfrak{n}_- \oplus \mathfrak{r}$. Since c is a homomorphism of R-modules, it follows that $(\delta c)(\mathfrak{n}_-)(\mathfrak{g})$ is contained in the vector subspace of $\mathfrak{n}_- \oplus \mathfrak{n}_+$ spanned by all $e_{-\alpha} \otimes e_\beta$, where $\alpha, \beta \in \Delta(N_+)$ and $\beta - \alpha \in \Delta(R)$ or $\alpha = \beta$.

In the cases I and II this subspace does not coincide with $\mathfrak{n}_- \oplus \mathfrak{n}_+$, i.e., there exist $\alpha, \beta \in \Delta(N_+)$ such that $\beta - \alpha \notin \Delta(R)$ and $\alpha \neq \beta$. Indeed, we can take $\beta = \delta$ and $\alpha = \alpha_0$.

Suppose that $\theta = \theta_2$. By (5.88),

$$\begin{aligned} c_{\theta_2}(v)(w) &= \sum_{i=1}^n e_i^* \otimes ((e_i, v)\pi(w) - (\pi(w), v)e_i) \\ &= \sum_{i=1}^n (e_i, v) e_i^* \otimes \pi(w) - (\pi(w), v) \sum_{i=1}^n e_i^* \otimes e_i \\ &= v \otimes \pi(w) - (\pi(w), v) \sum_{i=1}^n e_i^* \otimes e_i. \end{aligned}$$

In particular,

$$c_{\theta_2}(e_{-\alpha_0})(e_\delta) = e_{-\alpha_0} \otimes e_\delta.$$

It follows that $c_{\theta_2} \notin \delta C^0(\mathfrak{n}_-, \mathrm{Hom}(\mathfrak{g}, \mathfrak{n}_- \otimes \mathfrak{n}_+))^R$. Thus, $\theta = \theta_2$ defines a non-zero homomorphism in the cases I and II.

Now consider the case II, i.e., suppose that $M = \mathrm{Gr}_s^n(\mathbb{C})$, where $1 < s < n-1$. We will use the notation of Example 5.11. Then, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the following subspaces of $\mathfrak{gl}_n(\mathbb{C})$:

$$\begin{aligned} \mathfrak{n}_+ &= \langle E_{i\beta} \mid 1 \le i \le s,\ s+1 \le \beta \le m \rangle, \\ \mathfrak{n}_- &= \langle E_{\alpha j} \mid s+1 \le \alpha \le m,\ 1 \le j \le s \rangle. \end{aligned} \tag{5.89}$$

Here, $E_{\alpha j} = E^*_{j\alpha}$ form the basis dual to $E_{j\alpha}$. If $\theta = \eta$, then the cochain (5.88) has the form

$$c_\eta(v)(w) = \sum_{i,\alpha} E_{\alpha i} \otimes (E_{i\alpha} v \pi(w) - \pi(w) v E_{i\alpha}).$$

We write $v = \sum_{\beta j} v_{\beta j} E_{\beta j},\ \pi(w) = \sum_{j\beta} w_{j\beta} E_{j\beta}$. Then,

$$\begin{aligned} E_{i\alpha} v \pi(w) &= E_{i\alpha} (\sum_{\beta j \rho} v_{\beta j} w_{j\rho}) E_{\beta\rho} = \sum_{j\rho} v_{\alpha j} w_{j\rho} E_{i\rho}, \\ \pi(w) v E_{i\alpha}) &= (\sum_{jk\rho} w_{j\rho} v_{\rho k}) E_{jk} E_{i\alpha} = (\sum_{j\rho} w_{j\rho} v_{\rho i}) E_{j\alpha}. \end{aligned}$$

Hence,

$$c_\eta(v)(w) = \sum_{ij\alpha\rho} E_{\alpha i} \otimes \sum_{j\rho} (v_{\alpha j} w_{j\rho}) E_{i\rho} - \sum_{ij\alpha\rho} E_{\alpha i} \otimes (\sum_{j\rho} w_{j\rho} v_{\rho i}) E_{j\alpha}.$$

Consider the 0-cochain $c \in \mathrm{Hom}_R(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{n}_- \otimes \mathfrak{n}_+)$ given by the formula

$$\begin{aligned} c(\mathfrak{n}_+) &= c(\mathfrak{n}_-) = 0, \\ c(E_{ij}) &= \sum_\rho E_{\rho j} \otimes E_{i\rho}, \\ c(E_{\alpha\beta}) &= \sum_k E_{\alpha k} \otimes E_{k\rho} \end{aligned}$$

and restrict it to $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then, for any $v \in \mathfrak{n}_-$ and $w \in \mathfrak{g}$, we have

$$\begin{aligned} \delta c(v)(w) &= c([v, w]) = c([v, \pi(w)]) \\ &= c(\sum_{j\alpha\rho} v_{\alpha j} w_{j\rho} E_{\alpha\rho} - \sum_{ij\rho} v_{\rho i} w_{j\rho} E_{ji}) \\ &= \sum_{jk\alpha\rho} v_{\alpha j} w_{j\rho} E_{\alpha k} \otimes E_{k\rho} - \sum_{ij\alpha\rho} v_{\rho i} w_{j\rho} E_{\alpha i} \otimes E_{j\alpha} \\ &= c_\eta(v)(w). \end{aligned}$$

Thus, $c_\eta = \delta c$, and η defines the zero homomorphism. Evidently, this implies the statement. □

5.31. Proposition. *Let $M = \mathrm{Gr}^n_s$ for $2 \le s \le n-2$, and let $\theta, \varphi \in \Gamma(M, \Phi^{2,1} \otimes \Theta)^K$. If $n \ge 5$, then $\theta \barwedge \varphi = 0$ implies $\theta = 0$ or $\varphi = 0$. For $M = \mathrm{Gr}^4_2$, the only solutions of $\theta \barwedge \varphi = 0$, up to a constant factor, are $\theta = \sqrt{2}\theta_2 \pm \eta$, $\varphi = \theta_2 \pm \sqrt{2}\eta$.*

Proof. By (5.67),

$$\theta_2 \barwedge \theta_2 = 2\theta_3.$$

From (5.72), (5.73), and (5.74) we easily deduce the following relations:

$$\theta_2 \barwedge \eta = 2(\eta_1 + \eta_2),$$
$$\eta \barwedge \theta_2 = 4\eta_2,$$
$$\eta \barwedge \eta = 4\eta_3.$$

Write $\theta = a\theta_2 + b\eta$, $\varphi = c\theta_2 + d\eta$ with $a, b, c, d \in \mathbb{C}$. It follows that

$$\theta \barwedge \varphi = 2ac\theta_3 + 2ad\eta_1 + 2(ad + 2bc)\eta_2 + 4bd\eta_2.$$

Suppose that $3 \le s \le n-3$. By Lemma 5.12, $\theta \barwedge \varphi = 0$ yields

$$ac = ad = ad + bc = bd = 0.$$

Clearly, this implies $(a, b) = 0$ or $(c, d) = 0$.

If $n - s = 2$ and $s \ge 3$, then, by (5.75),

$$\theta \barwedge \varphi = 2(ac - bd)\theta_3 + 2(ad - bd)\eta_1 + 2(ad + 2bc + 2bd)\eta_2.$$

By Lemma 5.12, $\theta \barwedge \varphi = 0$ yields $ac - bd = ad + bd = ad + 2bc + 2bd = 0$. Clearly, this implies $(a, b) = 0$ or $(c, d) = 0$. The case $s = 2$, $n - s \ge 3$ is considered similarly.

Suppose now that $n = 4$ and $k = 2$. It follows from (5.77) that

$$\theta \barwedge \varphi = 2(ac - bd)\theta_3 + (ad - 2bc)\eta_1.$$

If $\theta \barwedge \varphi = 0$, then $ac - bd = ad - 2bc = 0$. If $(a, b) \ne 0$, then this implies

$$\begin{vmatrix} c & -d \\ d & -2c \end{vmatrix} = -2c^2 + d^2 = 0,$$

whence $d = \pm\sqrt{2}c$. If $(c, d) \ne 0$, then $a = \pm\sqrt{2}b$. □

5.31 Non-split supermanifolds In this subsection, we apply our results to the problem of classification of non-split supermanifolds. Theorem 5.28 implies that the split supermanifold (M, Ω) satisfies the conditions of Theorem 4.6. Thus, in this case the mapping

$$\lambda_2^* : H^1(M, \mathcal{A}ut_{(2)}\Omega) \longrightarrow H^1(M, \mathcal{T}_2)$$

is bijective. By Theorem 4.1, we can parametrize non-split supermanifolds with retract (M, Ω) (up to isomorphism) by orbits of the group $\mathrm{Aut}\,\mathbf{T}(M)^*$ in $H^1(M, \mathcal{T}_2) \setminus \{0\}$. By Propositions 5.25 and 5.9, one can identify $H^1(M, \mathcal{T}_2) = l^*(H^1(M, \Omega^2 \otimes \Theta))$ with the vector space of K-invariant vector-valued (2,1)-forms $\Gamma(M, \Phi^{2,1} \otimes \Theta)^K$ using the Dolbeault–Serre isomorphism. We use this parametrization in the statement of the following classification theorem.

5.32. Theorem. *Suppose that M is a simply connected irreducible compact Hermitian symmetric space of dimension ≥ 2.*

1) *If M is of type I or III, then there exists (up to an isomorphism) precisely one non-split supermanifold with retract (M,Ω), namely, the canonical one. The corresponding invariant vector-valued $(2,1)$-form is the form θ_2 given by the formula (5.65), ω being determined by (5.68).*

2) *If $M = \mathrm{Gr}_s^n$, $1 < s < n-1$ is of type II, then non-split supermanifolds with retract (M,Ω) are parametrized by $\mathbb{CP}^1/\Sigma$, where*

$$\Sigma = \begin{cases} \mathbb{Z}_2 & \text{if } n = 2s \\ \{e\} & \text{otherwise.} \end{cases}$$

The corresponding invariant vector-valued $(2,1)$-forms are $a\theta_2 + b\eta$, where η_o is given by the formula (5.71) and $a, b \in \mathbb{C}$ serve as homogeneous coordinates in $\mathbb{CP}^1$. For $n = 2s$, the action of the generator σ of Σ is expressed in these coordinates as follows: $\sigma(a : b) = (a : -b)$.

Proof. Similarly to the proof of Theorem 5.6, we have

$$(H^1(M,\mathcal{T}_2) \setminus \{0\})/\operatorname{Aut} \mathbf{T}(M)^* = \mathbb{P}(H^1(M,\mathcal{T}_2))/\Sigma = \mathbb{P}(\Gamma(M,\Phi^{2,1} \otimes \Theta)^K)/\Sigma.$$

Then, one applies Proposition 5.17.

Suppose that $M = \mathrm{Gr}_s^{2s}$ for $s \geq 2$. It is known (one deduces this from [38, S 15, Theorem 3]) that the generator σ of Σ, being regarded as a biholomorphic transformation of M, acts as follows:

$$\sigma(gP) = A(g)P, \quad g \in G = \mathrm{SL}_{2s}(\mathbb{C}),$$

where A is the automorphism of G given by the formula

$$A(g) = \begin{pmatrix} 0 & I_s \\ I_s & 0 \end{pmatrix} (g^\top)^{-1} \begin{pmatrix} 0 & I_s \\ I_s & 0 \end{pmatrix}.$$

We easily check that the automorphism d_eA acts on $\mathfrak{n}_\pm$ by

$$d_eA(u) = -u^\top, \quad u \in \mathfrak{n}_\pm.$$

By formula (5.71),

$$\eta_o(-u_1^\top, -u_2^\top, -v^\top) = \eta(u_1, u_2, v)^\top.$$

Therefore, $\sigma^*\eta = -\eta$. Clearly, $\sigma^*\theta_2 = \theta_2$. Thus, $\sigma^*(a\theta_2 + b\eta) = a\theta_2 + (-b)\eta$. □

Comparing Theorem 5.32 with Theorem 5.6, we see that the construction of Subsection 5.6 gives all non-split supermanifolds with retract Ω in the cases I and III, while this is not true in the case II (one uses Proposition 4.14, see Example 5.10).

Let us now fix a non-split supermanifold $(M,\mathcal{O})$ with retract (M,Ω), where M is a compact irreducible Hermitian symmetric space. Changing the notation, we will denote by $\mathcal{T}$ the tangent sheaf $\mathcal{D}er\,\mathcal{O}$ of $(M,\mathcal{O})$, setting $\mathcal{T}_{\rm gr}=\mathcal{D}er\,\Omega$. Our goal is to calculate the cohomology groups $H^q(M,\mathcal{T})$ for $q=0,1$. These groups depend on the non-zero form $\theta\in\Gamma(M,\Phi^{2,1}\otimes\Theta)^K$ which parametrizes the supermanifolds $(M,\mathcal{O})$, as it has been described above.

5.33. Theorem. *Let M be a simply connected irreducible compact Hermitian symmetric space of dimension ≥ 2. Let $(M,\mathcal{O})$ be a non-split supermanifold with retract (M,Ω), the tangent sheaf $\mathcal{T}$, and the corresponding vector-valued form $\theta\in\Gamma(M,\Phi^{2,1}\otimes\Theta)^K$.*

(1) *Let M be of type I and $\theta=\theta_2$, where ω is determined by* (5.68)*. Then,*

$$H^0(M,\mathcal{T}_{\bar 0})=\mathfrak{v}(M,\mathcal{O})_{\bar 0}\simeq\mathfrak{g}\ \ \text{(as Lie algebras)},$$

while (as $\mathfrak{g}$-modules)

$$H^0(M,\mathcal{T}_{\bar 1})=\mathfrak{v}(M,\mathcal{O})_{\bar 1}\simeq\mathbb{C}$$
$$H^1(M,\mathcal{T})=H^1(M,\mathcal{T}_{\bar 0})\simeq\mathfrak{g}.$$

The basic element $\hat d\in\mathfrak{v}(M,\mathcal{O})_{\bar 1}$ satisfies $[\hat d,\hat d]=0$.

(2) *Let M be of type II, i.e., $M=\mathrm{Gr}^n_s$ for $2\leq s\leq n-2$, and $\theta=a\theta_2+b\eta$, where $a\neq 0$. If $n\geq 5$, or $n=4$ and (a,b) is not proportional to $(\sqrt{2},\pm 1)$, then $H^0(M,\mathcal{T})$ is as in* (1)*, while*

$$H^1(M,\mathcal{T})=H^1(M,\mathcal{T}_{\bar 0})\simeq\mathfrak{g}\oplus\mathbb{C}\ \ \text{(as $\mathfrak{g}$-modules)}.$$

(3) *Let $M=\mathrm{Gr}^4_2$, $\theta=\sqrt{2}\theta_2+\eta$. Then, $H^0(M,\mathcal{T})$ is as in* (1)*, while* (as $\mathfrak{g}$-modules)

$$H^1(M,\mathcal{T}_{\bar 0})\simeq\mathfrak{g}\oplus\mathbb{C},$$
$$H^1(M,\mathcal{T}_{\bar 1})\simeq\mathbb{C}$$

(4) *Let M be of type II and $\theta=\eta$. Then,*

$$H^0(M,\mathcal{T}_{\bar 0})=\mathfrak{v}(M,\mathcal{O})_{\bar 0}\simeq\mathfrak{g}\ \ \text{(as Lie algebras)},$$

while (as $\mathfrak{g}$-modules)

$$H^0(M,\mathcal{T}_{\bar 1})=\mathfrak{v}(M,\mathcal{O})_{\bar 1}\simeq\mathfrak{g}\oplus\mathbb{C},$$
$$H^1(M,\mathcal{T}_{\bar 0})\simeq\mathfrak{g}\oplus\mathbb{C},$$
$$H^1(M,\mathcal{T}_{\bar 1})\simeq\mathfrak{g}.$$

(5) *Let M be of type III, i.e., $M=\mathbb{CP}^{n-1}$ for $n\geq 3$, and $\theta=\theta_2=\eta$. Then, $H^0(M,\mathcal{T})$ is as in* (4)*, while* (as $\mathfrak{g}$-modules)

$$H^1(M,\mathcal{T})=H^1(M,\mathcal{T}_{\bar 1})\simeq\begin{cases}0 & \text{for } n\geq 4,\\ \mathbb{C} & \text{for } n=3.\end{cases}$$

5.32.1 Comment and Open problem. Due to the isomorphism between $gr_p H^q(M,\mathcal{T})$ and $E_\infty^{p,q-p}$, in the proof we need $H^q(M,\mathcal{T})$ only for $q=0$ and 1, so for our purposes it is not necessary to compute $E^{p,q-p}$ for $q=2$ for any p. Therefore, some terms (denoted by "?" in the tables below) remain unknown and should be calculated for completeness.

Proof. Consider the spectral sequence (E_r) associated with $(M,\mathcal{O})$ due to Theorem 4.20. By this theorem, $E_2^{p,q-p}=H^q(M,(\mathcal{T}_{\rm gr})_p)$ and $d_2=\mathrm{ad}_l^*([\theta])$, where $[\theta]\in H^1(M,\Omega^2\otimes\Theta)^G$ is the cohomology class of θ. Clearly, d_2 is G-equivariant.

We are going to calculate d_2 on $E_2^{p,q-p}$ for $q=0,1$. The case $q=0,\ p=-1$ is settled by Proposition 5.30. In the case where $q=p=0$, we see that

$$
\begin{aligned}
[l^*([\theta]),l(v)]&=l^*([\theta,v])=l^*([[\theta,v]])=0,\\
[l^*([\theta]),\varepsilon]&=-2l^*([\theta]).
\end{aligned}
$$

Clearly, $d_2(E_2^{0,1})=d_2(E_2^{2,-1})=0$. The mapping $d_2:E_2^{0,1}\longrightarrow E_2^{2,0}$ is 0, too, since $E_2^{2,0}$ is a trivial G-module. Similarly, $d_2=0$ on $l^*(H^1(M,\Omega^1\otimes\Theta)\subset E_2^{1,0}$.

Now, for any $\varphi\in\Gamma(M,\Phi^{2,1}\otimes\Theta)^K$ we have

$$
[l(\theta),i(\varphi)]=[i(\varphi),l(\theta)]=l(\theta\,\overline{\wedge}\,\varphi),
$$

due to (3.28), since $[\varphi,\theta]=0$ by Proposition 5.9. By Theorem 4.13, $d_2i^*(\varphi)=l^*([\theta\,\overline{\wedge}\,\varphi])$. This class can be calculated with the help of Proposition 5.31 (note that, by Theorem 5.29, the forms $\sqrt{2}\theta_2\pm\eta$ determine isomorphic non-split supermanifolds). This settles the case $p=q=1$.

Summarizing, we see that the terms $E_3^{p,q-p}=E_4^{p,q-p}$ for $q=0,1,2$, are as follows (for the definition of s, see (5.89)):

Case I, $\theta=\theta_2$:

q \ p	-1	0	1	2	3	$4\ldots$
0	0	$\mathfrak{g}$	$\mathbb{C}$	0	0	0
1	0	$\mathfrak{g}$	0	0	0	0
2	0	0	0	?	$\mathbb{C}^{s-1}$	0

Case II, $\theta=a\theta_2+b\eta,\ a\neq 0:$

q \ p	-1	0	1	2	3	$4\ldots$
0	0	$\mathfrak{g}$	$\mathbb{C}$	0	0	0
1	0	$\mathfrak{g}$	0	$\mathbb{C}$	0	0
2	0	0	0	?	$\mathbb{C}^{s-2}$	0

Case II, $\theta = \eta$:

$q \backslash p$	-1	0	1	2	3	$4\dots$
0	$\mathfrak{g}$	$\mathfrak{g}$	$\mathbb{C}$	0	0	0
1	0	$\mathfrak{g}$	$\mathfrak{g}$	$\mathbb{C}$	0	0
2	0	0	0	?	$\mathbb{C}^{s-2}$	0

Case III, $n \geq 4$, $\theta = \theta_2 = \eta$:

$q \backslash p$	-1	0	1	2	3	$4\dots$
0	$\mathfrak{g}$	$\mathfrak{g}$	$\mathbb{C}$	0	0	0
1	0	0	0	0	0	0
2	0	0	0	?	0	0

Case III, $n = 3$, $\theta = \theta_2 = \eta$:

$q \backslash p$	-1	0	1	$2\dots$
0	$\mathfrak{g}$	$\mathfrak{g}$	$\mathbb{C}$	0
1	0	0	$\mathbb{C}$	0
2	0	0	0	0

Clearly, for $q = 0, 1$ we have $d_4 = d_6 = \ldots = 0$, and hence $E_3^{p,q-p} = E_\infty^{p,q-p}$ for all $p \geq 0$. This implies our theorem. □

5.34. Corollary. *Under assumptions of Theorem 5.33, we have*

$$\begin{aligned} \mathfrak{v}(M,\mathcal{O})_{\bar 0} &\simeq \mathfrak{g}, \\ \mathfrak{v}(M,\mathcal{O})_{(0)} &\simeq \mathfrak{g} \oplus \mathbb{C}, \\ \mathfrak{v}(M,\mathcal{O})_{(1)} &\simeq \mathbb{C}, \\ \mathfrak{v}(M,\mathcal{O})_{(p)} &= 0 \text{ for } p \geq 2. \end{aligned}$$

In the cases (1), (2), (3), $\mathfrak{v}(M,\mathcal{O}) = \mathfrak{v}(M,\mathcal{O})_{(0)}$, *and the supermanifold* $(M,\mathcal{O})$ *is not homogeneous. In the remaining cases,* $\mathfrak{v}(M,\mathcal{O}) \neq \mathfrak{v}(M,\mathcal{O})_{(0)}$.

Proof. The claims about $\mathfrak{v}(M,\mathcal{O})_{(p)}$ are implied by the calculation of the spectral sequence (E_r). It follows that $\mathfrak{v}(M,\mathcal{O})_{\bar 0} \simeq \mathfrak{g}$.

In the cases (1), (2), (3), we see that $\mathfrak{v}(M,\mathcal{O}) = \mathfrak{v}(M,\mathcal{O})_{(0)}$. Therefore, $\mathrm{ev}_x(v) = 0$ for all $v \in \mathfrak{v}(M,\mathcal{O})_{\bar 1}$, $x \in M$, and hence $(M,\mathcal{O})$ is not homogeneous. □

6 The Π-symmetric super-Grassmannian

Consider the supermanifold $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ defined in Example 2.9. Its reduction is the submanifold M of $\mathrm{Gr}^n_s \times \mathrm{Gr}^n_s$ consisting of the vector subsuperspaces $L \subset \mathbb{C}^{n|n}$ of dimension $s|s$ satisfying $L_{\bar 1} = \Pi(L_{\bar 0})$. Projecting M onto the first factor, we identify this manifold with Gr^n_s. Denoting $r = n - s$, we suppose that $r, s \geq 1$. Assume that Π is given in the standard basis by the matrix

$$\Pi = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

and define local coordinates in a neighborhood of the point $o = \langle e_{r+1}, \dots, e_n, f_1, \dots, f_s\rangle$ in M identified with $\langle e_{r+1}, \dots, e_n\rangle \in \mathrm{Gr}^n_s$. Clearly, the subsupermanifold $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ of $\mathrm{Gr}^{n|n}_{s|s}$ is defined in terms of the coordinate matrix (2.14) by the equations

$$Y = X, \quad \mathrm{H} = \Xi.$$

Thus, the coordinate matrix has the form

$$Z = \begin{pmatrix} X & \Xi \\ I_s & 0 \\ \Xi & X \\ 0 & I_s \end{pmatrix}, \tag{6.90}$$

where X and Ξ are $(r \times s)$-matrices. Denoting $X := (x_{i\alpha})$ and $\Xi := (\xi_{i\alpha})$, we get the even local coordinates $x_{i\alpha}$ and the odd ones $\xi_{i\alpha}$, $1 \leq i \leq r$, $1 \leq \alpha \leq s$, in a neighborhood of the point o.

Denote by $Q_n(\mathbb{C})$ the subsupergroup of $\mathrm{GL}_{n|n}(\mathbb{C})$ that preserves Π. Its coordinate matrix has the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{6.91}$$

where A and B are $(n \times n)$-matrices of even and odd coordinates, respectively, $\det A \neq 0$. The reduction G_0 of $Q_n(\mathbb{C})$ can be identified, in an obvious way, with $\mathrm{GL}_n(\mathbb{C})$. The Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$ of $Q_n(\mathbb{C})$ consists of all complex matrices of the form (6.91) with arbitrary $(n \times n)$-matrices A and B, and its even part $\mathfrak{g}_0$ can be identified with $\mathfrak{gl}_n(\mathbb{C})$.

The supermanifold $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ admits the standard action of $Q_n(\mathbb{C})$, which is expressed in coordinates as the multiplication of Z from the left by the coordinate matrix (6.91). This action induces, clearly, the standard transitive action of the Lie group $G_0 = \mathrm{GL}_n(\mathbb{C})$ on $M = \mathrm{Gr}^n_s$. Let P denote the isotropy subgroup of G_0 at the point $o \in M$; it consists of all matrices of the form (4.7). We will use the notation introduced in Example 5.11.

Let us denote by $a \mapsto a^*$ the differential of the standard action of $Q_n(\mathbb{C})$ on $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$. This is a homomorphism of the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$ into the Lie superalgebra $\mathfrak{v}(\Pi\,\mathrm{Gr}^{n|n}_{s|s})$ of holomorphic vector fields on $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$. In what follows, we need the expression of this homomorphism restricted to $\mathfrak{p}$. The holomorphic vector fields on $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ will be written in terms of the local coordinates in a neighborhood of o given by the matrix (6.90). Denote the elements of $\mathfrak{p}$ by

$$a_1 = \begin{pmatrix} (a_{ij}) & 0 \\ 0 & 0 \end{pmatrix},\ a_2 = \begin{pmatrix} 0 & 0 \\ 0 & (b_{\alpha\beta}) \end{pmatrix},\ v = \begin{pmatrix} 0 & 0 \\ (v_{\alpha j}) & 0 \end{pmatrix},$$

where $(a_{ij}) \in \mathfrak{gl}_r(\mathbb{C})$, $(b_{\alpha\beta}) \in \mathfrak{gl}_s(\mathbb{C})$, and $(v_{\alpha j})$ is an $(s \times r)$-matrix. We want to calculate the corresponding fundamental vector fields.

Clearly,

$$\begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} X & \Xi \\ I_s & 0 \\ \Xi & X \\ 0 & I_s \end{pmatrix} = \begin{pmatrix} A_1X & A_1\Xi \\ I_s & 0 \\ A_1\Xi & A_1X \\ 0 & I_s \end{pmatrix}.$$

By substituting $A_1 = \exp ta_1$ with $t \in \mathbb{C}$, by differentiating at $t = 0$ and changing the signs, we get

$$a_1^*(x_{i\alpha}) = -(a_1X)_{i\alpha},\quad a_1^*(\xi_{i\alpha}) = -(a_1\Xi)_{i\alpha}, \tag{6.92}$$

where we identify a_1 with (a_{ij}). Similarly, we find that

$$\begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix} \begin{pmatrix} X & \Xi \\ I_s & 0 \\ \Xi & X \\ 0 & I_s \end{pmatrix} \sim \begin{pmatrix} XA_2^{-1} & \Xi A_2^{-1} \\ I_s & 0 \\ \Xi A_2^{-1} & XA_2^{-1} \\ 0 & I_r \end{pmatrix},$$

whence

$$a_2^*(x_{i\alpha}) = (Xa_2)_{i\alpha},\quad a_2^*(\xi_{i\alpha}) = (\Xi a_2)_{i\alpha}, \tag{6.93}$$

where we identify a_2 with $(b_{\alpha\beta})$.

Further, for any $t \in \mathbb{C}$, we get

$$\begin{pmatrix} I_r & 0 & 0 & 0 \\ tv & I_s & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & tv & I_s \end{pmatrix} \begin{pmatrix} X & \Xi \\ I_s & 0 \\ \Xi & X \\ 0 & I_s \end{pmatrix} = \begin{pmatrix} X & \Xi \\ I_s + tvX & tv\Xi \\ \Xi & X \\ tv\Xi & I_r + tvX \end{pmatrix}.$$

Multiplying the result from the right by

$$\begin{pmatrix} I_s + tvX & tv\Xi \\ tv\Xi & I_r + tvX \end{pmatrix}^{-1} = \begin{pmatrix} I_s - tvX + \dots & -tv\Xi + \dots \\ -tv\Xi + \dots & I_r - tvX + \dots \end{pmatrix},$$

where the omitted terms are of order > 1 in t, we get the matrix

$$\begin{pmatrix} X - t(XvX + \Xi v\Xi) + \dots & \Xi - t(\Xi vX + Xv\Xi) + \dots \\ I_s & 0 \\ \Xi - t(\Xi vX + Xv\Xi) + \dots & X - t(XvX + \Xi v\Xi) + \dots \\ 0 & I_r \end{pmatrix}.$$

Therefore,

$$\begin{aligned} v^*(x_{i\alpha}) &= (XvX + \Xi v\Xi)_{i\alpha},\\ v^*(\xi_{i\alpha}) &= (\Xi vX + Xv\Xi)_{i\alpha}, \end{aligned} \tag{6.94}$$

where we identify v with $(v_{\alpha j})$.

From (6.92), (6.93), and(6.94) we get

6.1 Proposition (Explicit formulas of vector fields). *We have*

$$\begin{aligned} a_1^* &= -\sum_{i,k=1}^{r}\sum_{\alpha=1}^{s} a_{ik}x_{k\alpha}\frac{\partial}{\partial x_{i\alpha}} - \sum_{i,k=1}^{r}\sum_{\alpha=1}^{s} a_{ik}\xi_{k\alpha}\frac{\partial}{\partial \xi_{i\alpha}},\\ a_2^* &= \sum_{\alpha,\beta=1}^{s}\sum_{i=1}^{r} b_{\beta\alpha}x_{i\beta}\frac{\partial}{\partial x_{i\alpha}} + \sum_{\alpha\beta=1}^{s}\sum_{i=1}^{r} b_{\beta\alpha}\xi_{i\beta}\frac{\partial}{\partial \xi_{i\alpha}},\\ v^* &= \sum_{i,j=1}^{r}\sum_{\alpha,\beta=1}^{s} v_{\beta j}(x_{i\beta}x_{j\alpha} + \xi_{i\beta}\xi_{j\alpha})\frac{\partial}{\partial x_{i\alpha}}\\ &\quad + \sum_{i,j=1}^{r}\sum_{\alpha,\beta=1}^{s} v_{\beta j}(\xi_{i\beta}x_{j\alpha} + x_{i\beta}\xi_{j\alpha})\frac{\partial}{\partial \xi_{i\alpha}}. \end{aligned}$$

Let $\mathcal{O}$ denote the structure sheaf of the supermanifold $\Pi\operatorname{Gr}_{s|s}^{n|n}$. Clearly, the action of G on $(M,\mathcal{O})$ determines a linear representation of the group P by automorphisms of the superalgebra $\mathcal{O}_o$, which gives a linear representation $\chi = \chi_{\bar 0}+\chi_{\bar 1}$ of this group in $T_o(M,\mathcal{O})$, called the *isotropy representation*. Proposition 6.1 easily implies its explicit expression.

Indeed, denote the tautological representations of $\mathrm{GL}_r(\mathbb{C})$ and $\mathrm{GL}_s(\mathbb{C})$ by ρ_1 and ρ_2, respectively. Let $\widetilde{m}_o$ be the linear span of germs at o of all coordinate functions $x_{i\alpha},\ \xi_{i\alpha}$ in m_o. Then, $m_o = \widetilde{m}_o \oplus m_o^2$. As Proposition 6.1 shows, $v^*(\widetilde{m}_o) \subset m_o^2$ for all $v \in \mathfrak{n}_-$, and hence $\mathfrak{n}_-$ trivially acts on m_o/m_o^2. The same proposition implies that $\widetilde{m}_o$ is invariant under $\mathfrak{r}$ (or R), inducing in both components $(\widetilde{m}_o)_{\bar 0}$ and $(\widetilde{m}_o)_{\bar 1}$ the representation $\rho_1^*\otimes\rho_2$ of R.

As in Example 5.11, we consider the maximal algebraic torus T of R and G_0 consisting of all diagonal matrices. We will write the matrices of the corresponding Cartan subalgebra $\mathfrak{t}$ in the form

$$H = \operatorname{diag}(\lambda_1,\dots,\lambda_t,\lambda_{t+1},\dots,\lambda_n),\ \lambda_i\in\mathbb{C}.$$

Proposition 6.1 also implies that the germs of $x_{i\alpha},\ \xi_{\alpha i},\ \eta_{i\alpha}$ form a weight basis for the representation χ^* in $\widetilde{m}_o \simeq m_o/m_o^2$ with respect to T, the corresponding weights being $-\lambda_i+\lambda_{t+\alpha}$, where $1\le i\le r,\ 1\le\alpha\le s$ (with multiplicity 2). Thus, we got

6.2. Proposition. (1) *The isotropy representation χ is completely reducible, and the restrictions of its even and odd components onto R are as follows:*

$$\chi_{\bar 0}|_R \simeq \chi_{\bar 1}|_R \simeq \rho_1\otimes\rho_2^*.$$

(2) *The germs of* $x_{i\alpha}$, $\xi_{i\alpha}$ *form a weight basis with respect to* T *in their linear span* $\widetilde{m}_o$, *the corresponding weights being in both cases* $-\lambda_i + \lambda_{t+\alpha}$, *where* $1 \leq i \leq r$ *and* $1 \leq \alpha \leq s$.

Note that $\chi_{\bar{0}}$ coincides with the isotropy representation τ of the homogeneous space $\mathrm{Gr}_s^n(\mathbb{C})$ (see (5.70)).

Clearly, the action of G_0 on the sheaf $\mathcal{O}$ leaves invariant the filtration (2.12) and induces an action of this group on the locally free sheaf $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$, and hence on the corresponding vector bundle $\mathbf{E}$, covering its standard action on M. Thus, $\mathbf{E}$ is a homogeneous vector bundle over M.

6.3. Proposition. *The vector bundle* $\mathbf{E}$ *is isomorphic to the cotangent bundle* $\mathbf{T}(M)^*$. *The retract of the super-Grassmannian* $(M, \mathcal{O})$ *is isomorphic to the supermanifold* (M, Ω) *from Example 2.7.*

Proof. By Proposition 6.2, the representation of P in $E_o = T_o(M, \mathcal{O})^*_{\bar{1}}$ is isomorphic to τ^*. Hence, $\mathbf{E} \simeq \mathbf{E}_{\tau^*} = (\mathbf{E}_\tau)^* \simeq \mathbf{T}(M)^*$. □

Next, I want to prove that our super-Grassmannian is, as a rule, non-split. Note that the canonical action of G_0 on $(M, \mathcal{O})$ gives rise to a natural linear action of this groups on the tangent sheaf $\mathcal{T}$ leaving invariant the $\mathbb{Z}_2$-grading. As a result, we get a linear representation of G_0 in the cohomology groups of $\mathcal{T}$ and, in particular, in the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$. The corresponding linear representation of the Lie algebra $\mathfrak{g}_0$ is given by the formula $u \mapsto \mathrm{ad}_{u^*}$.

6.4. Proposition. *If* $r \geq 2$ *or* $s \geq 2$, *then* $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}^{G_0} = 0$.

Proof. Any $\delta \in \mathfrak{v}(M, \mathcal{O})_{\bar{0}}^{G_0}$ determines a P-invariant even derivation of the superalgebra $\mathcal{O}_o$ (we denote it by the same character δ), Clearly, δ preserves the maximal ideal m_o. Consider the vector subspace $\widetilde{m}_o \subset m_o$, spanned by the germs of local coordinates at o. By Proposition 6.2, R preserves the even and the odd parts of this subspace, inducing in each part an irreducible representation, and the germs of local coordinates constitute a weight basis of $\widetilde{m}_o$ with respect to T with the weights $-\lambda_i + \lambda_{r+\alpha}$, where $1 \leq i \leq r$ and $1 \leq \alpha \leq s$. Note that the remaining weights of the representation of R in the whole m_o are certain sums of these weights, and hence we see that the weight subspace of m_o corresponding to any of these weights is two-dimensional (and lies in $\widetilde{m}_o$). Since δ is even and P-invariant, the germs of local coordinates are eigenvectors for δ. Moreover, the Schur lemma implies that

$$\delta(x_{i\alpha}) = a x_{i\alpha},\ \delta(\xi_{i\alpha}) = b \xi_{i\alpha},$$

where $a, b \in \mathbb{C}$. We have $a = 0$. Indeed, consider the vector field $\widetilde{\delta} = \sigma_0(\delta) \in \mathfrak{v}(M, \mathrm{gr}\,\mathcal{O})_0$ (see Subsection 3.1). Clearly, $\widetilde{\delta}$ is G_0-invariant, too, and hence determines the G_0-invariant vector field $\alpha(\widetilde{\delta})$ (see (3.21). But it is well known (see, e.g., [38]) that the standard action of $\mathrm{GL}_n(\mathbb{C})$ on M is *asystatic*, i.e., M has no non-zero holomorphic G_0-invariant vector fields (for the origin of the term *asystatic*, see [15*, 16*] and interesting references therein). This implies that $\widetilde{\delta}(x_{i\alpha} + \mathcal{J}) = 0$. Therefore, $\delta(x_{i\alpha}) \in \mathcal{J}^2$, whence $a = 0$. Now we prove that $b = 0$, using the relation $[\delta, v^*] = 0$ for all $v \in \mathfrak{n}_-$. Proposition 6.1 implies that

$$0 = [\delta, E^*_{r+1,1}](x_{12}) = \delta(E^*_{r+1,1}(x_{12})) = \delta(\xi_{11}\xi_{12}) = 2b\xi_{11}\xi_{12}.$$

This implies our assertion whenever $s \geq 2$. To prove the assertion for $r \geq 2$, one takes x_{21} instead of x_{12}. □

This result makes it possible to solve the splittness question concerning the super-Grassmannians studied here.

6.5 Theorem (On splitness of $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$). *The super-Grassmannian $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ is split if and only if $n = 2$ and $s = 1$.*

Proof. Consider the grading derivation ε of the $\mathbb{Z}$-graded sheaf $\mathrm{gr}\,\mathcal{O}$ defined in Subsection 3.2 and the natural homomorphism of Lie superalgebras $\sigma_0 : H^0(M, \mathcal{T}_{\bar{0}}) \longrightarrow H^0(M, \widetilde{\mathcal{T}}_0)$ defined in Subsection 3.1. Proposition 6.4 implies that $\varepsilon \notin \mathrm{Im}\,\sigma$ whenever $s \geq 2$ or $r \geq 2$. Indeed, if $\varepsilon = \sigma_0(\delta)$, where $\delta \in H^0(M, \mathcal{T}_{\bar{0}})$, then the complete reducibility of the representation of G_0 in $H^0(M, \mathcal{T}_{\bar{0}})$ implies that δ can be chosen to be G_0-invariant. But then $\delta = 0$, whence $\varepsilon = 0$, which gives a contradiction. If $(M, \mathcal{O})$ is split, then σ is an isomorphism, but this is false whenever $s \geq 2$ or $r \geq 2$. In the case $n = 2$, $s = 1$, we can see that the super-Grassmannian is split, e.g., by calculating its transition functions. □

An important property of $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ is the homogeneity, which we are going to prove now.

6.6 Proposition ($\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ is homogeneous). (1) *The canonical action of $\mathfrak{q}_n(\mathbb{C})$ on the supermanifold $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ is transitive.*

(2) *The kernel of this action is $\langle I_{n|n}\rangle$.*

Proof. To prove (1), we have to calculate the vector fields y^* corresponding to certain odd elements of $\mathfrak{q}_n(\mathbb{C})$. More precisely, take the matrix $y = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \in \mathfrak{q}_n(\mathbb{C})_{\bar{1}}$, where

$$B = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix},$$

$Y = (y_{i\alpha})$ being an $(r \times s)$-matrix. Denoting by τ an odd parameter, we get

$$\begin{pmatrix} I_n & \tau B \\ \tau B & I_n \end{pmatrix} \begin{pmatrix} X & \Xi \\ I_s & 0 \\ \Xi & X \\ 0 & I_s \end{pmatrix} = \begin{pmatrix} X & \Xi + \tau Y \\ I_s & 0 \\ \Xi + \tau Y & X \\ 0 & I_s \end{pmatrix}.$$

It follows that

$$y^* = -\sum_{i,\alpha} y_{i\alpha} \frac{\partial}{\partial \xi_{i\alpha}}.$$

Clearly, $\mathrm{ev}_o(y^*)$ span the vector space $T_o((M, \mathcal{O}))_{\bar{1}}$. Since our action is $\bar{0}$-transitive, its transitivity follows from Proposition 3.4(2).

Let us denote by $\mathfrak{q}$ the kernel of our action. We see from Proposition 6.1 that $I_{n|n} \in \mathfrak{q}$. Since $\mathfrak{g}_0 = \mathfrak{gl}_n(\mathbb{C})$ acts on M in the standard way, it follows that $\mathfrak{q} \cap \mathfrak{g}_0 = \langle I_{n|n} \rangle$. But it is known (see, e.g., [24]) that the only ideal of $\mathfrak{q}_n(\mathbb{C})$ containing $\langle I_{n|n} \rangle$ is

$$\mathfrak{sq}_n(\mathbb{C}) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| \operatorname{tr} B = 0 \right\}.$$

As we have seen above, $\mathfrak{q} \neq \mathfrak{sq}_n(\mathbb{C})$. Hence, $\mathfrak{q} = \langle I_{n|n} \rangle$. □

Now we are able to prove our main result concerning Π-symmetric super-Grassmannians.

6.7. Theorem. *Let $(M, \mathcal{O}) = \Pi\,\mathrm{Gr}^{n|n}_{s|s}$ and $n \geq 3$.*

1) *In the classification of non-split supermanifolds with retract $(\mathrm{Gr}^n_{,s}, \Omega)$ given by Theorem 5.29, $(M, \mathcal{O})$ corresponds to the invariant $(2,1)$-form η.*

2) *The natural action of the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$ on $(M, \mathcal{O})$ determines an isomorphism of Lie superalgebras*

$$\mathfrak{v}(M, \mathcal{O}) \simeq \mathfrak{pq}_n(\mathbb{C}) := \mathfrak{q}_n(\mathbb{C})/\langle I_{n|n} \rangle.$$

3) *If $2 \geq s \geq n-2$, then*

$$H^1(M, \mathcal{T}_{\bar{0}}) \simeq \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C},$$
$$H^1(M, \mathcal{T}_{\bar{1}}) \simeq \mathfrak{sl}_n(\mathbb{C}).$$

4) *If $s = 1$ or $n-1$, then*

$$H^1(M, \mathcal{T}) = H^1(M, \mathcal{T}_{\bar{1}}) \simeq \begin{cases} 0 \text{ for } n \geq 4 \\ \mathbb{C} \text{ for } n = 3 \end{cases}.$$

Proof. By Corollary 4.2, the supermanifold corresponding to the form $a\theta_2 + b\eta$ in the case II cannot be homogeneous if $a \neq 0$. By Proposition 6.6(1), this implies (1).

By Proposition 6.6(2), the natural action of $\mathfrak{q}_n(\mathbb{C})$ on $(M, \mathcal{O})$ induces an injective homomorphism $\mathfrak{q}_n(\mathbb{C})/\langle I_{n|n} \rangle \longrightarrow \mathfrak{v}(M, \mathcal{O})$. Comparing this with Theorem 5.33(4), we see that this homomorphism is surjective. Thus, (2) is proved.

The assertions (2) and (3) follow from (1) and Theorem 5.33. □

Theorems 6.7(1) and 5.29 imply

6.8 Corollary (A family of deformations of $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$). *The supermanifold $\Pi\,\mathrm{Gr}^{n|n}_{s|s}$ for $2 \le s \le n-2$ is included in a 1-parameter family of mutually non-isomorphic supermanifolds with the same retract. In particular, it is not rigid.*

To conclude, we note that these properties of the Π-symmetric super-Grassmannians contrast with the rigidity of certain other series of super-Grassmannians (see Examples 2.7, 2.8, 2.9). Let us denote by $(M, \mathcal{O})$ one of these super-Grassmannians, by $(M, \mathcal{O}_{\mathrm{gr}})$ its retract and by $\mathcal{T}$, $\mathcal{T}_{\mathrm{gr}}$ the corresponding tangent sheaves.

6.9 Theorem (Rigid super-Grassmannians). *Suppose that $(M, \mathcal{O})$ is one of the following supermanifolds:*

$$\begin{aligned} &\mathrm{Gr}^{n|m}_{k|l} \ \textit{with}\ 0<k<m,\ 0<l<m,\\ (k,l) \neq &(1,n-1),\ (m-1,1),\ (1,n-2),\ (m-2,1),\ (2,n-1),\ (m-1,2);\\ &\mathrm{I}\,\mathrm{Gr}^{2r|r}_{2s|s} \ \textit{with}\ r\ge 2,\ (r,s)\neq(2,1);\\ &\mathrm{I}_{\mathrm{odd}}\,\mathrm{Gr}^{n|n}_{s|n-s} \ \textit{with}\ 4\le s\le n-3. \end{aligned}$$

Then, $(M, \mathcal{O})$ is the only non-split supermanifold with retract $(M, \mathcal{O}_{\mathrm{gr}})$ and, moreover, $(M, \mathcal{O})$ is rigid.

Proof. It is known that in all the cases listed above we have

$$H^1(M, (\mathcal{T}_{\mathrm{gr}})_p) = \begin{cases} \mathbb{C} & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

(see [37, Theorem 1] for $(M, \mathcal{O}) = \mathrm{Gr}^{n|m}_{k|l}$, [43, Theorem 1] for $(M, \mathcal{O}) = \mathrm{I}\,\mathrm{Gr}^{2r|r}_{2s|s}$, [44, Theorem 1] for $(M, \mathcal{O}) = \mathrm{I}_{\mathrm{odd}}\,\mathrm{Gr}^{n|n}_{s|n-s}$. Moreover, it was proved in these papers that the supermanifolds $(M, \mathcal{O})$ are non-split. By Proposition 4.8, $(M, \mathcal{O})$ is the only non-split supermanifold with retract $(M, \mathcal{O}_{\mathrm{gr}})$, and the corresponding class $\lambda_2^*(\gamma)$ is a basic element of $H^1(M, (\mathcal{T}_{\mathrm{gr}})_2)$. As in the proof of Theorem 5.33, we have $d_2(\varepsilon) = -2\lambda_2^*(\gamma)$ in the spectral sequence (E_r). Theorem 4.20 implies that $H^1(M, \mathcal{T}) = 0$, and hence $(M, \mathcal{O})$ is rigid. (The vanishing of $H^1(M, \mathcal{T})$ was proved in the cited papers as well.) □

Note that the super-Grassmannians listed in Theorem 6.9, together with the Π-symmetric super-Grassmannians, are just the supermanifolds of flags that can be called symmetric superspaces (see [49]).

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