Communications in Mathematics **30** (2022), no. 1, 175–190
DOI: https://doi.org/10.46298/cm.9740
©2022 A.S. Dzhumadil'daev, B.K. Zhakhayev, S.A. Abdykassymova
This is an open access article licensed under the CC BY-SA 4.0

Assosymmetric operad

A.S. Dzhumadil'daev, B.K. Zhakhayev, S.A. Abdykassymova

Abstract. An algebra with identities $(a, b, c) = (a, c, b) = (b, a, c)$ is called *assosymmetric*, where $(x, y, z) = x(yz) - (xy)z$ is associator. We establish that operad of assosymmetric algebras is not Koszul. We study S_n-module, A_n-module and GL_n-module structures on multilinear parts of assosymmetric operad.

1 Introduction

A variety of algebras is a class of algebras with polynomial identities. Operads of algebas are constructed by multilinear parts of free algebras. These multilinear parts have module structures over symmetric groups and have composition rules that depends from polynomial identities. If polynomial identities have dimension no more than 3, then they generate so called quadratic operads. One of important problems on quadratic operads concern Koszulity problem of operads. For details on algebraic operads see [16] and [7].

For example, operad of associative algebras $\mathcal{A}s$ is defined by identity

$$(a, b, c) = a(bc) - (ab)c = 0,$$

where $(x, y, z) = x(yz) - (xy)z$ is associator. A multilinear part of free asociative algebras, $\mathcal{A}s(n)$ has dimension $n!$ and is isomorphic to regular S_n-module. The Koszul dual of associative operad is isomorphic to itself and associative operad is Koszul.

MSC 2020: 17A50; 20C30
Keywords: Free assosymmetric algebras, Specht module, Weyl module, cocharacter, codimension, colength.
Affiliation:
A.S. Dzhumadil'daev, S.A. Abdykassymova – Institute of Mathematics and Mathematical Modeling, Pushkin str., 125, Almaty, Kazakhstan
E-mail: dzhuma@hotmail.com
E-mail: saule_asan@hotmail.com
B.K. Zhakhayev – Faculty of Engineering and Natural Sciences, Suleyman Demirel University, Abylai Khan Str., 1/1, Kaskelen, Kazakhstan
Institute of Mathematics and Mathematical Modeling, Pushkin str., 125, Almaty, Kazakhstan
E-mail: bekzat22@hotmail.com

Operad of left-symmetric algebras is defined by left-symmetric identity

$$(a, b, c) = (b, a, c).$$

Similarly, right-symmetric operad is constructed by right-symmetric identity

$$(a, b, c) = (a, c, b).$$

Operads of left- or right-symmetric algebras are Koszul. Their Koszul duals are defined by perm identities, i.e. by associative identity and one-sided commutative identities.

An algebra is called *assosymmetric* if it satisfies two-sided symmetric identities

$$(a, b, c) = (b, a, c), \qquad (a, b, c) = (a, c, b).$$

In our paper we prove that assosymmetric operad is not Koszul. We study the multilinear part of free assosymmetric algebra as S_n-module, A_n-module and GL_n-module. We find dimension of homogeneous component, sequence of dimensions of multilinear parts or codimension sequence, colength sequence, cocharacter sequence in S_n-case, cocharacter sequence in A_n-case for assosymmetric algebras.

To formulate our results in exact form we need to introduce some definitions and notations. Let K be an algebraically closed field of characteristic 0. All algebras, vector spaces, modules and tensor products we consider will be over field K.

Let $X = \{x_1, x_2, \dots\}$ be a set of generators and $K\{X\}$ be the absolutely free nonassociative algebra. A polynomial $f(x_1, x_2, \dots, x_n) \in K\{X\}$ is called *polynomial identity* or *identity* for the K-algebra R if $f(r_1, r_2, \dots, r_n)$=0 for all $r_1, r_2, \dots, r_n \in R$.

Let $\{f_i \in K\{X\} | i \in I\}$ be a set of elements in $K\{X\}$. The class $\mathfrak{V}$ of all algebras satisfying the polynomial identities $f_i = 0, i \in I$ is called the *variety* defined by the system of polynomial identities $\{f_i | i \in I\}$. The set $T(\mathfrak{V})$ of all polynomial identities satisfied by the variety $\mathfrak{V}$ is called the *T-ideal* or *verbal ideal* of $\mathfrak{V}$.

Assosymmetric algebras was studied in [1], [3], [11], [13], [18]. Basis of free assosymmetric algebras was constructed in [11]. Moreover, this paper contains multiplication rule of base elements that allows to present an element of free assosymmetric algebra as a linear combination by base elements. In [3] it was proved that assosymmetric algebras under Jordan product satisfy Lie triple and Glennie identities.

In polynomial identities theory there are two main questions: 1) describe algebras with identities; 2) describe identities in algebras. The language of varieties allows one to freely pass from identity to algebra and from algebra to identity. Therefore studying varieties of algebras is one of the important problem in modern algebras. In 1950, A.I. Malcev [17] and W.Specht [20] first time and independently used the representation theory of symmetric group to classify polynomial identities of algebraic structures. If $char K = 0$, then every polynomial is equivalent to a finite set of multilinear polynomials.

For several classes of algebras S_n-, GL_n-module structures on multilinear parts of free algebras are studied. Some cases these structures can be easy described. For example, multilinear parts of free associative, free Zinbiel and free Leibniz algebras of degree n as

S_n-module are isomorphic to regular module KS_n [22]. But as operads these varieties are different, since composition rules are different. In case of Lie algebras module structures are slightly complicated. In [14] it was found list of irreducible S_n-representations that are involved in decomposition of multilinear parts of free Lie algebras. Description of multiplicities of irreducible S_n-representations in decomposition of multilinear part of free Lie algebra by language of major indices of standard Young tableaux is given in [15] .

2 Statement of main result

In [2] an algebra with identity

$$[a, b]c + [b, c]a + [c, a]b = 0$$

is called left-Alia.

Theorem 2.1. *The Koszul dual to the assosymmetric operad is left-Alia and associative. Assosymmetric operad is not Koszul.*

Let n be a positive integer. The sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called *partition* of n, if

1. $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$,
2. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$,

and denoted by $\lambda \vdash n$. *Length* of partition $\lambda \vdash n$ is the number of parts in λ and denoted by $\ell(\lambda)$. It is known that between partitions of n and Young diagrams with n boxes exist one-to-one correspondence. We denote Young diagram with λ-shape by Y_λ. Let $\lambda, \mu \vdash n$. Partition λ is *conjugate* to partition μ, if Y_μ is obtained from Y_λ by turning the rows into columns and denoted by $\mu = \lambda'$. A partition that is conjugate to itself is said to be a *self-conjugate* partition, that is $\lambda = \lambda'$.

Let S_n be symmetric group on set $\{1, 2, \ldots, n\}$ and A_n be alternating subgroup of S_n. The symmetric group S_n and alternating group A_n acts on multilinear part of free assosymmetric algebra in natural way (left action or variable action).

Let R be an algebra with T-ideal $T(R)$ and let V_n be a multilinear part of $K\{X\}$ of degree n. For $n \geq 1$, the S_n-character of $V_n/(V_n \cap T(R))$ is called the *n-th cocharacter* of R and denoted by $\chi_n(R)$, and

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity.

Let $T(R)$ be T-ideal of R. Then the non-negative integer

$$c_n(R) = dim(V_n/V_n \cap T(R)),$$

is called the *n-th codimension* of the algebra R.

Let R be an algebra and

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda.$$

Then the non-negative integer

$$l_n(R) = \sum_{\lambda \vdash n} m_\lambda$$

is called the *n-th colength* of R.

For more information about codimension sequence, cocharacter sequence, colength sequence see [6].

We denote the irreducible S_n-module or Specht module associated to partition $\lambda \vdash n$ by S^λ and dimension of S^λ by d_λ, the irreducible A_n-module associated to non-self-conjugate partition $\lambda \vdash n$ by S_A^λ and to self-conjugate partition $\lambda \vdash n$ by $S_A^{\lambda\pm}$, the irreducible GL_n-module or Weyl module associated to partition $\lambda \vdash n$ by W^λ in S_n-case and by W_A^λ in A_n-case.

For more information about the theory of representations of S_n, A_n and GL_n see [4], [5], [8], [9], [10], [12] and [19].

Free base of assosymmetric algebras was found in [11]. We use this result to find formulas for dimensions of free assosymmetric algebras. Let $F(r)$ be free assosymmetric algebra generated by r elements $a_1, \dots, a_r$. Let $F^{l_1,\dots,l_r}(r)$ be a subspace of free assosymmetric algebra generated by l_i elements a_i, where $i = 1, \dots, r$, and $F_n(r)$ be a subspace of free assosymmetric algebra $F(r)$ of degree n and $P_n = F^{1,\dots,1}(n)$ be multilinear part of $F_n(n)$.

Theorem 2.2. *Let $p = char K \neq 2, 3$. Then*

$$\dim F^{l_1,\dots,l_r}(r) = \binom{l_1 + \cdots + l_r}{l_1 \cdots l_r} + (l_1+1)\cdots(l_r+1) - \binom{r+1}{2} - r - 1 + w,$$

where $w = w(l_1, \dots, l_r)$ is a number of 1's in the sequence $l_1 \dots l_r$,

$$\dim F_n(r) = \\ r^n + \binom{n+2r-1}{n} - \binom{r+1}{2}\binom{n+r-3}{n-2} - r\binom{n+r-2}{n-1} - \binom{n+r-1}{n},$$

and

$$\dim P_n = n! + 2^n - \binom{n+1}{2} - 1.$$

By Stirling formula $n! \sim \sqrt{2\pi n}(n/e)^n$, and therefore,

$$\dim P_n^{1/n} \sim n/e.$$

We divide the set of multilinear base elements into two types.

First type

$$T_n = \{(\dots((x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)})\dots)x_{\sigma(n)} \mid \sigma \in S_n\},$$

Second type

$$T_{k,n-k}=\Big\{x_{\sigma(1)}\big(x_{\sigma(2)}(\dots x_{\sigma(k)}[\dots[(x_{\sigma(k+1)}x_{\sigma(k+2)},x_{\sigma(k+3)}),x_{\sigma(k+4)}],\dots,x_{\sigma(n)}]\dots)\big)\ \Big| \\ \sigma(1)<\sigma(2)<\cdots<\sigma(k),\ \sigma(k+1)<\sigma(k+2)<\cdots<\sigma(n),\ \sigma\in S_n\Big\}.$$

Theorem 2.3. *The group S_n acts transitively on the sets T_n and $T_{k,n-k}$, $k=0,1,\dots,n-3$.*

Let KT_n and $KT_{k,n-k}$ be subspaces of P_n spanned by the sets T_n and $T_{k,n-k}$, for $k=0,1,\dots,n-3$, respectively.

Corollary 2.4. *As an S_n-module*

$$P_n\cong KT_n\oplus\bigoplus_{k=0,1,\dots,n-3}KT_{k,n-k}.$$

Theorem 2.5. *As an S_n-module*

$$P_n\cong\bigoplus_{\lambda\vdash n}d_\lambda S^\lambda\oplus\bigoplus_{(\lambda_1,\lambda_2)\vdash n}m(\lambda_1,\lambda_2)S^{(\lambda_1,\lambda_2)},$$

where

$$m(\lambda_1,\lambda_2)=\begin{cases}n-2-\lambda_2, & \lambda_2\le 3,\\ n+1-2\lambda_2, & \lambda_2\ge 4.\end{cases}$$

Example 2.6.

$$P_1\cong S^{(1)};$$
$$P_2\cong S^{(2)}\oplus S^{(1,1)};$$
$$P_3\cong 2S^{(3)}\oplus 2S^{(2,1)}\oplus S^{(1,1,1)};$$
$$P_4\cong 3S^{(4)}\oplus 4S^{(3,1)}\oplus 2S^{(2,2)}\oplus 3S^{(2,1,1)}\oplus S^{(1,1,1,1)};$$
$$P_5\cong 4S^{(5)}\oplus 6S^{(4,1)}\oplus 6S^{(3,2)}\oplus 6S^{(3,1,1)}\oplus 5S^{(2,2,1,1)}\oplus 4S^{(2,1,1,1,1)}\oplus S^{(1,1,1,1,1)}.$$

Let V be a vector space with dimension m. Let $F(V)$ be free assosymmetric algebra generated by base elements of V and $H_n(V)$ be homogeneous part of $F(V)$ of degree n.

Definition 2.7. Let $\operatorname{inv}(S_n)$ be number of involutions,

$$\operatorname{inv}(S_n)=\#\{\sigma\in S_n\mid\sigma^2=e\}$$

Corollary 2.8. *a.*

$$\chi_{S_n}(P_n)\cong\sum_{\lambda\vdash n}d_\lambda\chi_{S_n}(\lambda)+\sum_{(\lambda_1,\lambda_2)\vdash n}m(\lambda_1,\lambda_2)\chi_{S_n}(\lambda_1,\lambda_2),$$

where

$$m(\lambda_1,\lambda_2)=\begin{cases}n-2-\lambda_2, & \lambda_2\le 3,\\ n+1-2\lambda_2, & \lambda_2\ge 4.\end{cases}$$

b.

$$\chi_{A_n}(P_n) = 2\chi_{A_n}(KA_n) + \sum_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2)\chi_{A_n}(\lambda_1,\lambda_2),$$

where

$$m(\lambda_1,\lambda_2) = \begin{cases} n-2-\lambda_2, & \lambda_2 \leq 3, \\ n+1-2\lambda_2, & \lambda_2 \geq 4. \end{cases}$$

c. (S_n-case)

$$H_n(V) \cong \bigoplus_{\lambda\vdash n} d_\lambda W^\lambda \oplus \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) W^\lambda,$$

where

$$d_\lambda > 0,\ m(\lambda_1,\lambda_2) > 0$$

and

$$m(\lambda_1,\lambda_2) = \begin{cases} n-2-\lambda_2, & \lambda_2 \leq 3, \\ n+1-2\lambda_2, & \lambda_2 \geq 4, \end{cases}$$

if $\dim V \geq \ell(\lambda), \ell((\lambda_1,\lambda_2))$, *and*

$$d_\lambda = 0,\ m(\lambda_1,\lambda_2) = 0,$$

if $\dim V < \ell(\lambda), \ell((\lambda_1,\lambda_2))$.

d. (A_n-case)

$$H_n(V) \cong \left[\bigoplus_{\lambda\neq\lambda'} 2d_\lambda W_A^\lambda\right] \oplus \left[\bigoplus_{\lambda=\lambda'} 2\left(\frac{d_\lambda}{2} W_A^{\lambda+} \oplus \frac{d_\lambda}{2} W_A^{\lambda-}\right)\right] \oplus \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) W_A^\lambda,$$

where

$$m(\lambda_1,\lambda_2) = \begin{cases} n-2-\lambda_2, & \lambda_2 \leq 3, \\ n+1-2\lambda_2, & \lambda_2 \geq 4. \end{cases}$$

e. For $1 \leq n \leq 3$

$$l_n(P_n) = \delta_{n,3} + \mathrm{inv}(S_n),$$

where $\delta_{i,j}$ *is Kronecker delta. For* $n \geq 4$

$$l_n(P_n) = \begin{cases} k^2+2k-5+\mathrm{inv}(S_n), & n = 2k, \\ k^2+3k-4+\mathrm{inv}(S_n), & n = 2k+1. \end{cases}$$

Example 2.9.

$$\begin{aligned}
\chi_{S_1}(P_1) &= \chi_{S_1}(1);\\
\chi_{S_2}(P_2) &= \chi_{S_2}(2) + \chi_{S_2}(1,1);\\
\chi_{S_3}(P_3) &= 2\chi_{S_3}(3) + 2\chi_{S_3}(2,1) + \chi_{S_3}(1,1,1);\\
\chi_{S_4}(P_4) &= 3\chi_{S_4}(4) + 4\chi_{S_4}(3,1) + 2\chi_{S_4}(2,2) + 3\chi_{S_4}(2,1,1) + \chi_{S_4}(1,1,1,1);\\
\chi_{S_5}(P_5) &= 4\chi_{S_5}(5) + 6\chi_{S_5}(4,1) + 6\chi_{S_5}(3,2) + 6\chi_{S_5}(3,1,1) + 5\chi_{S_5}(2,2,1)\\
&\quad + 4\chi_{S_5}(2,1,1,1,1) + \chi_{S_5}(1,1,1,1,1).
\end{aligned}$$

Example 2.10.

$$\begin{aligned}
\chi_{A_1}(P_1) &= \chi_{A_1}(1);\\
\chi_{A_2}(P_2) &= \chi_{A_2}(2) + \chi_{A_2}(1,1);\\
\chi_{A_3}(P_3) &= 3\chi_{A_3}(3) + \chi^{+}_{A_3}(2,1) + \chi^{-}_{A_3}(2,1);\\
\chi_{A_4}(P_4) &= 4\chi_{A_4}(4) + 7\chi_{A_4}(3,1) + \chi^{+}_{A_4}(2,2) + \chi^{-}_{A_4}(2,2);\\
\chi_{A_5}(P_5) &= 5\chi_{A_5}(5) + 10\chi_{A_5}(4,1) + 11\chi_{A_5}(3,2) + 3\chi^{+}_{A_5}(3,1,1) + 3\chi^{-}_{A_5}(3,1,1).
\end{aligned}$$

Example 2.11.

$$l_1(P_1) = 1; \quad l_2(P_2) = 2; \quad l_3(P_3) = 5; \quad l_4(P_4) = 13; \quad l_5(P_5) = 32.$$

3 Proof of Theorem 2.3

Let I be the T-ideal in $K\{X\}$ determined by identities

$$(x,y,z) = (x,z,y) = (y,x,z).$$

Let R be assosymmetric algebra and $J = (R,R,R) + (R,R,R)R$ is the ideal generated by associators. Proof of Theorem 2.3 is based on the following two results of [11].

Lemma 3.1 ([11], Lemma 1)**.** *The expression* $\big[[\ldots[[(a_1,a_2,a_3),a_4],a_5]\ldots]a_n\big]$ *is invariant, modulo* I*, under all permutations of the arguments.*

Lemma 3.2 ([11], Lemma 2)**.** *If* $x \in J$*, the expression* $a_1(a_2(a_3(\ldots a_n x)\ldots))$ *is invariant, modulo* I*, under all permutations of the* a_i*'s.*

Now we give proof of Theorem 2.3.

Proof of Theorem 2.3. First type. Let

$$(\cdots(a_{i_1}a_{i_2})\cdots)a_{i_n} \in T_n, \quad i_j \in \{1,2,\ldots,n\}.$$

Let $\sigma = (i_k, i_{k+1})$ be a transposition in S_n. Then

$$(i_k, i_{k+1})\colon (\cdots(((\cdots(a_{i_1}a_{i_2})\cdots)a_{i_k})a_{i_{k+1}})\cdots)a_{i_n} \mapsto \\ (\cdots(((\cdots(a_{i_1}a_{i_2})\cdots)a_{i_{k+1}})a_{i_k})\cdots)a_{i_n}.$$

By definition of first type this element $(\cdots(((\cdots(a_{i_1}a_{i_2})\cdots)a_{i_{k+1}})a_{i_k})\cdots)a_{i_n}$ is multilinear base element in T_n.

Second type. Let

$$v = a_{i_1}(\cdots(a_{i_k}[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots, a_{i_n}])\cdots) \in T_{k,n-k}, \quad i_j \in \{1, 2, \dots, n\}.$$

We present it in form

$$\underbrace{a_{i_1}(\cdots(a_{i_k}}_{A\text{-part}}\underbrace{[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots, a_{i_n}])}_{B\text{-part}}\cdots).$$

It suffices to consider the action of transposition $\sigma = (i_j, i_{j+1}) \in S_n$ in three cases:

Case 1 (σ acts on A-part):

$$\sigma : a_{i_1}(\cdots(a_{i_j}(a_{i_{j+1}}(\cdots(a_{i_k}[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots, a_{i_n}])\cdots)))\cdots) \mapsto$$

$$a_{i_1}(\cdots(a_{i_{j+1}}(a_{i_j}(\cdots(a_{i_k}[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots, a_{i_n}])\cdots)))\cdots).$$

By Lemma 3.2 $\sigma v \in T_{k,n-k}$ and $\sigma v = v$.

Case 2 (σ acts on B-part):

$$\sigma : a_{i_1}(\cdots(a_{i_k}[\cdots[[[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots], a_{i_j}], a_{i_{j+1}}], \dots, a_{i_n}])\cdots) \mapsto$$

$$a_{i_1}(\cdots(a_{i_k}[\cdots[[[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots], a_{i_{j+1}}], a_{i_j}], \dots, a_{i_n}])\cdots).$$

By Lemma 3.1 $\sigma v \in T_{k,n-k}$ and $\sigma v = v$.

Case 3 (σ acts on A-part and B-part simultaneously): Let

$$v = a_{i_1}(\cdots(a_{i_k}[\dots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots, a_{i_n}])\cdots) \in T_{k,n-k}, \quad i_j \in \{1, 2, \dots, n\}.$$

Assume that a_{i_j} belongs to A-part and $a_{i_{j+1}}$ belongs to B-part, i.e.

$$v = a_{i_1}(\cdots(a_{i_j}(\cdots(a_{i_k}[\cdots[[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots], a_{i_{j+1}}], \dots, a_{i_n}])\cdots))\cdots).$$

Then

$$\sigma : a_{i_1}(\cdots(a_{i_j}(\cdots(a_{i_k}[\cdots[[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots], a_{i_{j+1}}], \dots, a_{i_n}])\cdots))\cdots) \mapsto$$

$$\mapsto a_{i_1}(\cdots(a_{i_{j+1}}(\cdots(a_{i_k}[\cdots[[\cdots[(a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), a_{i_{k+4}}], \dots], a_{i_j}], \dots, a_{i_n}])\cdots))\cdots)$$

$$= (\text{by Lemma 3.2 and Lemma 3.1})$$

$$= a_{p_1}(\cdots(a_{p_l}(\cdots(a_{p_k}[\cdots[[\cdots[(a_{p_{k+1}}, a_{p_{k+2}}, a_{p_{k+3}}), a_{p_{k+4}}], \dots], a_{p_m}], \dots, a_{p_n}])\cdots))\cdots),$$

where

$$\{i_1, i_2, \ldots, i_{j+1}, \ldots, i_k\} = \{p_1, p_2, \ldots, p_l, \ldots, p_k \mid p_1 < p_2 < \cdots < p_l < \cdots < p_k\}$$

and

$$\{i_{k+1}, i_{k+2}, i_{k+3}, i_{k+4} \ldots, i_j, \ldots, i_n\} \\ = \{p_{k+1}, p_{k+2}, \ldots, p_m, \ldots p_n \mid p_{k+1} < p_{k+2} < \cdots < p_m < \cdots < p_n\}.$$

As we have noticed $\sigma v \neq v$. □

4 Proof of Theorem 2.5

Let V be a vector space with dimension 1. By Theorem 2.3 KT_n is isomorphic to $\underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$ as S_n-module. Therefore

$$KT_n \cong Ind^{S_n}_{S_1 \times S_1 \times \cdots \times S_1}(\mathbf{1}_{S_1} \otimes \mathbf{1}_{S_1} \otimes \cdots \otimes \mathbf{1}_{S_1}) \cong \bigoplus_{\lambda \vdash n} d_\lambda S^\lambda,$$

where $\mathbf{1}_{S_1}$ is one-dimensional trivial representation of S_1.

By Theorem 2.3 group of automorphisms of A-part of $T_{k,n-k}$ is S_k and group of automorphisms of B-part of $T_{k,n-k}$ is S_{n-k}. Therefore $S_k \times S_{n-k}$ is group of automorphisms of $T_{k,n-k}$.

Let

$$g_A = \sum_{\sigma \in S_k} \sigma \in KS_k, \quad g_B = \sum_{\tau \in S_{n-k}} \tau \in KS_{n-k}.$$

be elements of group algebras KS_k and KS_{n-k}, respectivley. Then by Theorem 2.3 $g_{T_{k,n-k}} = g_A \otimes g_B$ is generator of all base elements of $KT_{k,n-k}$ and g_A, g_B are one-dimensional trivial representations of S_k and S_{n-k}, respectively, and $KT_{k,n-k}$ is $S_k \times S_{n-k}$-module. Therefore $KT_{k,n-k}$ as S_n-module is isomorphic to

$$Ind^{S_n}_{S_k \times S_{n-k}}(\mathbf{1}_{S_k} \otimes \mathbf{1}_{S_{n-k}}) \cong \bigoplus_{(\lambda_1,\lambda_2) \vdash n} S^{(\lambda_1,\lambda_2)}, \quad \lambda_2 \leq min\{k, n-k\},$$

where $\mathbf{1}_{S_k} = g_A$, $\mathbf{1}_{S_{n-k}} = g_B$.

By Corollary 2.4

$$P_n \cong KS_n \oplus \bigoplus_{k=0,1,\ldots,n-3} KT_{k,n-k} \cong \bigoplus_{\lambda \vdash n} d_\lambda S^\lambda \oplus \bigoplus_{(\lambda_1,\lambda_2) \vdash n} m(\lambda_1, \lambda_2) S^{(\lambda_1,\lambda_2)},$$

where

$$m(\lambda_1, \lambda_2) = \begin{cases} n - 2 - \lambda_2, & \lambda_2 \leq 3, \\ n + 1 - 2\lambda_2, & \lambda_2 \geq 4. \end{cases}$$

5 Proof of Corollary 2.8

a. Follows from Theorem 2.5.

b. KT_n as S_n-module is isomorphic to

$$KT_n \cong \bigoplus_{\lambda \vdash n} d_\lambda S^\lambda.$$

KA_n as A_n-module is isomorphic to

$$KA_n \cong \left[\bigoplus_{\lambda \neq \lambda'} d_\lambda S_A^\lambda\right] \oplus \left[\bigoplus_{\lambda = \lambda'} \left(\frac{d_\lambda}{2} S_A^{\lambda+} \oplus \frac{d_\lambda}{2} S_A^{\lambda-}\right)\right],$$

where S_A^λ is irreducible A_n-module.

If $\lambda \vdash n$ is non-self-conjugate partition, then S^λ and $S^{\lambda'}$ as A_n-modules are isomorphic to

$$Res_{A_n}^{S_n}(S^\lambda) \cong S_A^\lambda, \quad Res_{A_n}^{S_n}(S^{\lambda'}) \cong S_A^{\lambda'}$$

and

$$S_A^\lambda \cong S_A^{\lambda'},$$

where $dim(S_A^\lambda) = dim(S_A^{\lambda'}) = d_\lambda$.

If $\lambda \vdash n$ is self-conjugate partition, then S^λ as A_n-module is isomorphic to

$$Res_{A_n}^{S_n}(S^\lambda) \cong (S_A^{\lambda+} \oplus S_A^{\lambda-}),$$

where $dim(S_A^{\lambda+}) = dim(S_A^{\lambda-}) = \frac{d_\lambda}{2}$. For details see [10].

Therefore

$$KT_n \cong 2KA_n.$$

Note that $KT_{k,n-k}$, $k = 0, 1, \ldots, n-3$, as S_n-module is isomorphic to

$$KT_{k,n-k} \cong \bigoplus_{(\lambda_1,\lambda_2) \vdash n} S^{(\lambda_1,\lambda_2)}, \quad \lambda_2 \leq min\{k, n-k\}.$$

Therefore $KT_{k,n-k}$ as A_n-module is isomorphic to

$$\begin{aligned} Res_{A_n}^{S_n}(KT_{k,n-k}) &\cong Res_{A_n}^{S_n}\Big(\bigoplus_{(\lambda_1,\lambda_2) \vdash n} S^{(\lambda_1,\lambda_2)}\Big) \\ &\cong \bigoplus_{(\lambda_1,\lambda_2) \vdash n} Res_{A_n}^{S_n} S^{(\lambda_1,\lambda_2)} \cong \bigoplus_{(\lambda_1,\lambda_2) \vdash n} S_A^{(\lambda_1,\lambda_2)}, \quad \lambda_2 \leq min\{k, n-k\}. \end{aligned}$$

c. (S_n-case) It is well known, that

$$W^\lambda \cong V^{\otimes n} \otimes_{KS_n} S^\lambda.$$

Then

$$\begin{aligned}
H_n(V) &\cong V^{\otimes n} \otimes_{KS_n} P_n \\
&\cong V^{\otimes n} \otimes_{KS_n} \left(\bigoplus_{\lambda \vdash n} d_\lambda S^\lambda \oplus \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) S^{(\lambda_1,\lambda_2)} \right) \\
&\cong \left(V^{\otimes n} \otimes_{KS_n} \bigoplus_{\lambda \vdash n} d_\lambda S^\lambda \right) \oplus \left(V^{\otimes n} \otimes_{KS_n} \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) S^{(\lambda_1,\lambda_2)} \right) \\
&\cong \left(\bigoplus_{\lambda \vdash n} d_\lambda (V^{\otimes n} \otimes_{KS_n} S^\lambda) \right) \oplus \left(\bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) (V^{\otimes n} \otimes_{KS_n} S^{(\lambda_1,\lambda_2)}) \right) \\
&\cong \left(\bigoplus_{\lambda \vdash n} d_\lambda W^\lambda \right) \oplus \left(\bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) W^{(\lambda_1,\lambda_2)} \right).
\end{aligned}$$

d. (A_n-case) As in case **c** (S_n-case)

e. Follows from **a** and Corollary 7.13.9 in [21]

6 Proof of Theorem 2.2

In calculation of dimensions we need the following easily proved combinatorial results.

Lemma 6.1. *For non-negative integers α, β and n takes place the following formula*

$$\sum_{i=0}^{n} \binom{i+\alpha}{i} \binom{n-i+\beta}{n-i} = \binom{n+\alpha+\beta+1}{n}$$

In particular,

$$\sum_{i=0}^{n} \binom{i+\alpha}{i} \binom{n-i+\alpha}{n-i} = \binom{n+2\alpha+1}{n}$$

Lemma 6.2. *The number of non-decreasing sequences of length m with components in the set $S = \{1, 2, \ldots, r\}$ is $\binom{m+r-1}{m}$.*

Lemma 6.3. *The number of non-decreasing sequences of length m with components in the set $S = \{1, 2, \ldots, r\}$ such that each $i \in S$ appears no more than l_i times is $(l_1+1)\cdots(l_r+1)$.*

In [11] is proved that a base of free assosymmetric algebras can be constructed by elements of two kinds. If $X = \{a_1, \dots, a_r\}$ is a set of generators, then in degree n the base consists elements of a form

$$(\cdots((a_{i_1}a_{i_2})a_{i_3})\cdots)a_{i_n}, \quad a_{i_s} \in X,$$
$$a_{i_1}(\cdots(a_{i_m}[\cdots[(a_{j_1}, a_{j_2}, a_{j_3}), a_{j_4}], \dots, a_{j_k}])\cdots), \quad a_{i_s}, a_{j_t} \in X,$$
$$i_1 \le i_2 \le \cdots \le i_m, \;\; j_1 \le j_2 \le \cdots \le j_k, \quad m \ge 0, \; k \ge 3.$$

Number of elements of first kind is r^n. By Lemma 6.2 number of elements of second kind L is equal to

$$\begin{aligned} L &= \sum_{m+k=n, m\ge 0, k\ge 3} \binom{m+r-1}{m}\binom{k+r-1}{k} \\ &= \sum_{m+k=n, m\ge 0, k\ge 0} \binom{m+r-1}{m}\binom{k+r-1}{k} \\ &\quad - \binom{n+r-3}{n-2}\binom{r+1}{2} - \binom{n+r-2}{n-1}\binom{r}{1} - \binom{n+r-1}{n}\binom{r-1}{0}. \end{aligned}$$

By Lemma 6.1

$$L = \binom{n+2r-1}{n} - \binom{r+1}{2}\binom{n+r-3}{n-2} - r\binom{n+r-2}{n-1} - \binom{n+r-1}{n}.$$

Therefore,

$$\dim F_n(r) = \\ r^n + \binom{n+2r-1}{n} - \binom{r+1}{2}\binom{n+r-3}{n-2} - r\binom{n+r-2}{n-1} - \binom{n+r-1}{n}.$$

Now suppose that any generator a_s, $s = 1, 2, \dots, r$, in each base element should enter l_s times. Then the number of base elements of first kind is

$$\binom{l_1 + \cdots + l_n}{l_1 \cdots l_n} = \frac{(l_1 + \cdots + l_r)!}{l_1! \cdots l_r!}.$$

Let M be set of sequences $\alpha = i_1 \dots i_m j_1 j_2 \dots j_k$ with components in $S = \{1, 2, \dots, r\}$ such that each $i \in S$ appears exactly l_i times and $i_1 \le \cdots \le i_m$, $j_1 \le \cdots \le j_k$. For $\alpha \in M$ call its subsequence of first m components $i_1 \dots i_m$ as *head* and denote $\tilde{\alpha}$. Note that each $\alpha \in M$ is uniquely defined by head $\tilde{\alpha}$. Denote set of heads by $\tilde{M}$. Note also that in the sequence $\tilde{\alpha} = i_1 \dots i_m$ each $i \in S$ enters no more than l_i times. Therefore by Lemma 6.3 the number of heads is

$$|\tilde{M}| = (l_1 + 1) \cdots (l_r + 1).$$

Let N be a subset of M consisting of sequences with the following heads

$$\underbrace{1\dots1}_{l_1}\dots\underbrace{i\dots i}_{l_i}\dots\underbrace{r\dots r}_{l_r},$$

(number of such sequences is 1)

$$\underbrace{1\dots1}_{l_1}\dots\underbrace{i\dots i}_{l_i-1}\dots\underbrace{r\dots r}_{l_r},\quad i\in S,$$

(number of such sequences is r)

$$\underbrace{1\dots1}_{l_1}\dots\underbrace{i\dots i}_{l_i-2}\dots\underbrace{r\dots r}_{l_r},\quad l_i>1,\quad i\in S,$$

(number of such sequences is $r-w$, where w is a number of 1's in the sequence $l_1\dots l_r$)

$$\underbrace{1\dots1}_{l_1}\dots\underbrace{i\dots i}_{l_i-1}\dots\underbrace{j\dots j}_{l_j-1}\dots\underbrace{r\dots r}_{l_r},\quad i<j,\quad i,j\in S$$

(number of such sequences is $r(r-1)/2$).

Let $M_1=M\setminus N$ be a supplement of N in the set M. Then any

$$\alpha=i_1\dots i_m j_1\dots j_k\in M_1$$

has the property $k\geq 3$ and any such sequence generates base element of free assosymmetric algebra of second kind. Hence the number of base elements of second kind is

$$\dim F^{l_1,\dots,l_r}(r)=|M_1|=\binom{l_1+\dots+l_r}{l_1\cdots l_r}+(l_1+1)\cdots(l_r+1)-\binom{r+1}{2}-r-1+w.$$

Dimension for the multilinear part is an easy consequence of this formula.

7 Proof of Theorem 2.1

Lemma 7.1. *Dual operad to assosymmetric operad is generated by identities*

$$[a,b]c+[b,c]a+[c,a]b=0,$$

$$(a,b,c)=0.$$

Let $d^!_n$ are dimensions of multilinear parts of free algebra with such identities. Then

$$d^!_1=1,d^!_2=2,d^!_3=5,$$

Proof. By [11, Theorem 1] the following elements form base of the multilinear part of free assosymmetric algebra in degree 3 $(ab)c$, $a(bc)$, $a(cb)$, $b(ac)$, $b(ca)$, $c(ab)$, $c(ba)$ and other 5 elements can be presented as a linear combination of these elements. By assosymmetric identities,

$$(ba)c = (ab)c - a(bc) + b(ac), \qquad (ac)b = (ab)c - a(bc) + a(cb),$$

$$(ca)b = (ab)c - a(bc) + c(ab), \qquad (bc)a = (ab)c + b(ca) - a(bc),$$

$$(cb)a = (ab)c - a(bc) + c(ba).$$

Let U be an algebra such that $A \otimes U$ is Lie-admissible. Then

$$\begin{aligned}[][a \otimes u, b \otimes v], c \otimes w] = \\ &(ab)c \otimes (uv)w - c(ab) \otimes w(uv) + c(ba) \otimes w(vu) \\ &\qquad - (ab)c \otimes (vu)w + a(bc) \otimes (vu)w - b(ac) \otimes (vu)w.\end{aligned}$$

In a similar way one calculates $[[b \otimes v, c \otimes w], a \otimes u]$, $[[c \otimes w, a \otimes u], b \otimes v]$ and obtain that

$$\begin{aligned}&[[a \otimes u, b \otimes v], c \otimes w] + [[b \otimes v, c \otimes w], a \otimes u] + [[c \otimes w, a \otimes u], b \otimes v] = \\ &\quad (ab)c \otimes \{(uv)w - (vu)w + (vw)u - (wv)u + (wu)v - (uw)v\} \\ &\quad + a(bc) \otimes \{(vu)w - (vw)u + (wv)u - u(vw) - (wu)v + (uw)v\} \\ &\quad + a(cb) \otimes \{u(wv) - (uw)v\} + b(ac) \otimes \{v(uw) - (vu)w\} + b(ca) \otimes \{(vw)u - v(wu)\} \\ &\quad + c(ab) \otimes \{(wu)v - w(uv)\} + c(ba) \otimes \{w(vu) - (wv)u\}\end{aligned}$$

Therefore, $A \otimes U$ is Lie-admissible iff

$$(uv)w - (vu)w + (vw)u - (wv)u + (wu)v - (uw)v = 0,$$

$$(vu)w - (vw)u + (wv)u - u(vw) - (wu)v + (uw)v = 0,$$

$$u(wv) - (uw)v = 0, \qquad v(uw) - (vu)w = 0,$$

$$(vw)u - v(wu) = 0, \qquad (wu)v - w(uv) = 0,$$

$$w(vu) - (wv)u = 0,$$

for any $u, v, w \in U$. Note that these conditions are equivalent to the following identities

$$[u, v]w + [v, w]u + [w, u]v = 0, \tag{1}$$

$$(uv)w = u(vw).$$

In [2] algebras with identity (1) are called *left-alia*. So, dual operad to assosymmetric operad is generated by left-alia and associativity identities.

It is easy to see that multilinear part of free dual assosymmetric algebras has the following base and dimensions for small degrees

n	$base$	dim
1	$\{a\}$	1
2	$\{ab, ba\}$	2
3	$\{(bc)a, (ca)b, (ac)b, (ba)c, (ab)c\}$	5

Hence, $d^!_1 = 1, d^!_2 = 2, d^!_3 = 5$,

Let d_n be dimension of multilinear part of free assosymmetric algebra in degree n. By Theorem 2.2 $d_1 = 1, d_2 = 2, d_3 = 7, d_4 = 29, d_5 = 136$.

By Lemma 7.1 Poincare series of assosymmetric and dual assosymmetric operads are as follows

$$G_{assym}(x) = -x + 2x^2/2! - 7x^3/3! + 29x^4/4! - 136x^5/5! + O(x)^6,$$

$$G^!_{assym}(x) = -x + 2x^2/2 - 5x^3/3! + d^!_4 x^4/4! - d^!_5 x^5/5! + O(x)^6.$$

We have

$$G_{assym}(G^!_{assym}(x)) = x + (3/8 - d^!_4/24)x^4 + 1/120(126 - 10d^!_4 + d^!_5)x^5 + O(x)^6.$$

Suppose that assosymmetric operad is Koszul. Then by Koszulity criterium ([7] Proposition 4.14(b))

$$G_{assym}(G^!_{assym}(x)) = x.$$

Hence

$$d^!_4 = 9, d^!_5 = -36.$$

But dimension $d^!_5$ can not be negative. Obtained contradiction shows that assosymmetric operad is not Koszul. By Lemma 7.1 Theorem 2.1 is proved completely. □

Acknowledgements

This project partially was carried out when the second named author visited the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences. He is very grateful for the creative atmosphere and the warm hospitality during his visit.

Authors are grateful to V. Drensky for his kind interest in our results and for essential comments.

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (grant number AP08855944).

References

[1] Boers, A.H.: On assosymmetric rings. *Indag. Math. (N.S.)* 5 (1994) 9–27.

[2] Dzhumadil'daev, A.S.: Algebras with skew-symmetric identity of degree 3. *J. Math. Sci* 161 (1) (2009) 11–30.

[3] Dzhumadil'daev, A.S.: Assosymmetric algebras under Jordan product. *Comm. Algebra* 46 (1) (2018) 129–142.

[4] Fulton, W.: *Young tableaux with Applications to Representation theory and Geometry*. Cambridge University Press (1997).

[5] Fulton, W.; Harris, J.: *Representation Theory. A first course*. Springer-Verlag New York Inc. (1991).

[6] Giambruno,A.; Zaicev, M.: *Polynomial Identities and Asymptotic Methods*. (2005).

[7] Ginzburg, V.; Kapranov, M.: Koszul duality for operads. *Duke Math. J.* 76 (1994) 203–272. Erratum: 80 (1995), 293; arXiv: 0709.1228.

[8] Henke, A.; Regev, A.: Weyl Modules for the Schur Algebra of the Alternating Group. *J. Algebra* 257 (1) (2002) 168–196.

[9] Henke, A.;Regev, A.: A-Codimensions and A-Cocharacters. *Isr. J. Math.* 133 (1) (2003) 339–355.

[10] Henke, A.; Regev, A.: Explicit decompositions of the group algebras FS_n and FA_n. *Lect. notes in pure and appl. math.* 235 (2003) 329–357.

[11] Hentzel, I.R.; Jacobs, D.P.; Peresi, L.A.: A basis for Free Assosymmetric Algebras. *J. Algebra* 183 (1996) 306–318.

[12] James, G.; Kerber, A.: *The representation Theory of the Symmetric Group*. Addison-Wesley Publishing Company, Inc. (1981).

[13] Kleinfeld, E.: Assosymmetric rings. *Proc. Amer. Math. Soc.* 8 (1957) 983–986.

[14] Klyachko, A.A.: Lie elements in the tensor algebra. *Siberian Mathematical Journal* 15 (1974) 914–920.

[15] Kraśkiewicz, W.; Weynman, J.: Algebra of coinvariants and the action of a Coxeter element. *Bayreuth. Math. Schr.* 63 (2001) 265–284.

[16] Loday, J.-L.; Vallette, B.: *Algebraic operads*. Grundlehren Math. Wiss. 346, Springer, Heidelberg (2012).

[17] Malcev, A.I.: On algebras defined by identities. *Mat. Sbornik N.S.* 26 (1950) 19–33. (in Russian)

[18] Pokrass, D.J.;Rodabaugh, D.: Solvable assosymmetric ring are nilpotent. *Proc. Amer. Math. Soc.* 64 (1977) 30–34.

[19] Sagan, B.E.: *The symmetric group: Representations, Combinatorial Algorithms, and Symmetric Functions, Graduate Texts in Mathematics*. Springer Verlag, New York (2001).

[20] Specht, W.: Gesetze in Ringen I. *Math. Z.* 52 (1950) 557–589.

[21] Stanley, R.P.: *Enumerative Combinatorics*. Cambridge University Press (1999).

[22] G.W. Zinbiel: Encyclopedia of types of algebras 2010. In: Proceedings of the International Conference on Operads and Universal Algebra. Tianjin, China, 5 – 9 July 2010, 217–297.

Received: 17 February, 2021
Accepted for publication: 24 June, 2021
Communicated by: Pasha Zusmanovich