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Homogeneous superstrings with retract $\mathcal{CP}^{1|4}$

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Abstract. Any complex-analytic supermanifold whose retract is diffeomorphic to the complex projective superline (superstring) $\mathcal{CP}^{1|4}$ is, up to a diffeomorphism, either a member of a 1-parameter family or one of 9 exceptional supermanifolds. I singled out the homogeneous of these supermanifolds and described Lie superalgebras of vector fields on them.

1 Preliminaries

Let $(M, \mathcal{F})$ be a complex-analytic manifold of dimension m. (More precisely, almost complex manifold, see [BGLS*], since the vanishing of the Nijenhuis tensor is never need; however, from the very beginning (see [Gr]) one requires the underlying manifold to be complex. *This comment and starred references are added by the editor of this Special Volume. D.Leites.*)

Let $\mathbf{E}$ be a vector bundle of rank n over M, and $\mathcal{E}$ the be locally free analytic sheaf of sections of $\mathbf{E}$. Set $\widetilde{\mathcal{O}} := \Lambda^{\bullet}_{\mathcal{F}}(\mathcal{E})$.

The supermanifold isomorphic to the one of the form $\mathcal{M} := (M, \widetilde{\mathcal{O}})$ is called *split*. The ringed space locally isomorphic to $(M, \Lambda^{\bullet}_{\mathcal{F}}(\mathcal{E}))$ is called a *supermanifold* of *superdimension* $m|n$. Physicists call supermanifolds of dimension $1|n$ *superstrings*, see [W*]. Let $\mathcal{O}$ be a structure sheaf of any supermanifold. Let $\mathcal{I} \subset \mathcal{O}$ be the subsheaf of ideals generated by subsheaf $\mathcal{O}_{\bar{1}}$ and let $\mathcal{O}_{\mathrm{rd}} := \mathcal{O}/\mathcal{I}$.

Consider the following filtration of $\mathcal{O}$ by powers of $\mathcal{I}$

$$\mathcal{O} = \mathcal{I}^0 \supset \mathcal{I} \supset \mathcal{I}^2 \supset \ldots \supset \mathcal{I}^n \supset \mathcal{I}^{n+1} = 0.$$

The graded sheaf $\mathrm{gr}\,\mathcal{O} = \oplus_{p=0}^{n} \mathrm{gr}_p\mathcal{O}$ with $\mathrm{gr}_p\mathcal{O} := \mathcal{I}^p/\mathcal{I}^{p+1}$ defines the split supermanifold $(M, \mathrm{gr}\,\mathcal{O})$ called the *retract* of $(M, \mathcal{O})$.

Let $\pi : \mathcal{I}^p \to \mathrm{gr}_p\mathcal{O}$ denote the natural projection. Then, we have the exact sequences of sheaves

$$0 \longrightarrow \mathcal{I}^{p+1} \longrightarrow \mathcal{I}^p \xrightarrow{\pi_p} \mathrm{gr}_p\mathcal{O} \longrightarrow 0. \tag{1}$$

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The supermanifold $(M,\mathcal{O})$ is split if and only if there exists an isomorphism of the superalgebra sheaves $h:\operatorname{gr}\mathcal{O}\to\mathcal{O}$ such that its restriction $h_p:\operatorname{gr}_p\mathcal{O}\to\mathcal{I}^p$ splits the sequence (1), i.e., satisfies $\pi_p\circ h_p=\mathrm{id}$. Such an isomorphism exists in a neighborhood of any point of M.

Let $(M,\mathcal{O})$ be a supermanifold and $\mathfrak{g}$ a complex finite-dimensional Lie superalgebra. An *action* of $\mathfrak{g}$ on $(M,\mathcal{O})$ is an arbitrary Lie superalgebra homomorphism $\varphi:\mathfrak{g}\to\mathfrak{v}(M,\mathcal{O})$. Then, a linear mapping $\varphi^x:\mathfrak{g}\to T_x(M,\mathcal{O})$ is associated with any $x\in M$. The action φ is called *transitive* if φ^x is surjective for any $x\in M$. By restricting the action $\varphi:\mathfrak{g}\to\mathfrak{v}(M,\mathcal{O})$ to the even component we get a homomorphism $\varphi_{\bar 0}:\mathfrak{g}_{\bar 0}\to\mathfrak{v}(M,\mathcal{O})_{\bar 0}$. If M is compact, then it is possible to integrate $\varphi_{\bar 0}$ getting a homomorphism $\Phi:G\to\operatorname{Aut}(M,\mathcal{O})$, where G is the simply connected complex Lie group whose Lie algebra is $\mathfrak{g}_{\bar 0}$. This homomorphism induces a homomorphism $\Phi_0:G\to\operatorname{Bih}M$ into the group of biholomorpic automorphisms of M, in other words — an action of G on M. The action φ is said to be $\bar 0$-*transitive* if Φ_0 is transitive.

If a Lie group G acts $\bar 0$-transitivity on M, then $\varphi^x:\mathfrak{g}_{\bar 0}\to T_x(M)$ is surjective for any $x\in M$. Conversely, if M is compact and $\varphi^x:\mathfrak{g}_{\bar 0}\to T_x(M)$ is surjective for any $x\in M$, we can integrate this action to a $\bar 0$-transitive action of a Lie group.

The supermanifold $(M,\mathcal{O})$ is called *homogeneous* (resp. $\bar 0$-*homogeneous*) if the natural action of the Lie superalgebra $\mathfrak{v}(M,\mathcal{O})$ on $(M,\mathcal{O})$ is transitive (resp. $\bar 0$-transitive), see [BO2], [O3]. This means that the evaluation mapping $ev_x:\mathfrak{v}(M,\mathcal{O})\to T_x(M,\mathcal{O})$ (resp. the restriction of ev_x to $\mathfrak{v}(M,\mathcal{O})_{\bar 0}$ is surjective for any $x\in M$.

Thanks to [O3] we know that φ is transitive if and only if it is $\bar 0$-transitive, M is compact, and the mapping $\varphi^x_{\bar 1}:\mathfrak{g}_{\bar 1}\to T_{x_0}(M,\mathcal{O})_{\bar 1}$ is surjective at a certain point $x_0\in M$. This implies that a $\bar 0$-homogeneous supermanifold is homogeneous if and only if the odd component of the mapping $ev_{x_0}:\mathfrak{v}(M,\mathcal{O})\to T_{x_0}(M,\mathcal{O})$ is surjective at a certain point $x_0\in M$.

One easily proves (see [BO2]) that the retract of a homogeneous supermanifold $(M,\mathcal{O})$ with compact M is homogeneous, too.

In what follows, I consider the problem of classification (up to a diffeomorphism) of supermanifolds with retract $\mathcal{CP}^{1|4}$ and describe which of the supermanifolds considered are homogeneous or at least $\bar 0$-*homogeneous*.

2 Superstring $\mathcal{CP}^{1|4}$. The first cohomology of the tangent sheaf

Over $\mathbb{CP}^1$, consider the holomorphic vector bundle

$$\mathbf{E}=\mathbf{L}_{-k_1}\oplus\mathbf{L}_{-k_2}\oplus\mathbf{L}_{-k_3}\oplus\mathbf{L}_{-k_4},\text{ where } k_1\geq k_2\geq k_3\geq k_4\geq 0.$$

Let $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}:=(\mathbb{CP}^1,\mathcal{O}_{\Lambda^{\bullet}_{\mathcal{F}}(\mathcal{E})})$ designate the split supermanifold determined by $\mathbf{E}$. As shown in [BO2], if $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}$ is homogeneous, then the k_i must be non-negative.

Let us cover $\mathbb{CP}^1$ by two charts U_0 and U_1 with local coordinates x and $y=\frac{1}{x}$, respectively. For $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}$, the transition functions in $U_0\cap U_1$ are $y=x^{-1}$ and $\eta_i=x^{-k_i}\xi_i$ for $i=1,\dots,4$, where ξ_i and η_i are basis sections of $\mathbf{E}$ over U_0 and U_1, respectively.

If $\mathcal{O}_{\rm gr}$ is the structure sheaf of $\mathcal{CP}^{1|4} := \mathcal{CP}^{1|4}_{1111}$, then $\mathcal{T}_{\rm gr} := \mathcal{D}er\,\mathcal{O}_{\rm gr}$ is the *tangent sheaf* (or the sheaf of vector fields). This is a sheaf of Lie superalgebras. The sections of the tangent sheaf are *holomorphic vector fields* on the supermanifold. Their sections are elements of the Lie superalgebra $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\rm gr}) := \Gamma(\mathbb{CP}^1, \mathcal{T}_{\rm gr})$ of vector fields on $\mathcal{CP}^{1|4}$.

The sheaf $\mathcal{T}_{\rm gr}$ has a $\mathbb{Z}$-grading $\mathcal{T}_{\rm gr} = \bigoplus_{-1\le p\le 4} (\mathcal{T}_{\rm gr})_p$, where

$$(\mathcal{T}_{\rm gr})_p := \mathcal{D}er_p\mathcal{O}_{\rm gr} = \{v \in \mathcal{T}_{\rm gr} \mid v((\mathcal{O}_{\rm gr})_q) \subset (\mathcal{O}_{\rm gr})_{p+q} \ \text{ for any } \ q\}.$$

The Lie superalgebra $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\rm gr})$ becomes also $\mathbb{Z}$-graded with the induced grading compatible with $\mathbb{Z}/2$-grading by parity.

We can regard $\mathcal{T}_{\rm gr}$ as a locally free analytic sheaf on $\mathbb{CP}^1$. From [O3] we have the following exact sequence of locally free analytic sheaves on $\mathbb{CP}^1$:

$$0 \longrightarrow \mathcal{E}^* \otimes \Lambda^{\bullet}\mathcal{E} \ \xrightarrow{i} \mathcal{T}_{\rm gr} \xrightarrow{\alpha} \Theta \otimes \Lambda^{\bullet}\mathcal{E} \longrightarrow 0,$$

where $\Theta = \mathcal{D}er\ \mathcal{F}$ is the tangent sheaf of the manifold $\mathbb{CP}^1$, and $\mathcal{F}$ is the sheaf of functions on $\mathbb{CP}^1$. The mapping α is the restriction of a derivation of $\mathcal{O}$ to $\mathcal{F}$, and i identifies any sheaf homomorphism $\mathcal{E} \to \Lambda^{\bullet}\mathcal{E}$ with its prolongation to a derivation that vanishes on $\mathcal{F}$. Hence, the analytic sheaf $\mathcal{T}_{\rm gr}$ is locally free. Therefore, $\mathcal{T}_{\rm gr}$ is the sheaf of holomorphic sections of a holomorphic vector bundle over $\mathbb{CP}^1$. We call it the *supertangent bundle* and denote **ST**.

Thanks to the Bott-Borel-Weil theorem the following theorem holds.

2.1 Theorem ([BO1]). *For* $\dim H^p(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_q)$*, see the following table*

q	-1	0	1	2	3	4
$p=0$	8	19	8	0	0	0
$p=1$	0	0	0	10	8	1

The group $\mathrm{SL}_2(\mathbb{C})$ *trivially acts on* $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2)$ *and* $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)$*.*

Set

$$\begin{aligned} &\delta := \xi_1\xi_2\xi_3\xi_4, \quad \delta_l := \frac{\partial\delta}{\partial\xi_l}, \quad \nabla := \sum_{1\le i\le 4} \xi_i\partial_{\xi_i};\\ &\delta' := \eta_1\eta_2\eta_3\eta_4, \quad \delta'_l := \frac{\partial\delta'}{\partial\eta_l}, \quad \nabla' := \sum_{1\le i\le 4} \eta_i\partial_{\eta_i}. \end{aligned} \tag{2}$$

The next Theorem expounds the result of Theorem 2.1 by giving the Čzech cocycles of the covering $\{U_0, U_1\}$; these cocycles determine the bases of non-zero spaces H^1.

2.2. Theorem. *The basis of* $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_p)$*, where* $p = 2, 3, 4$*, can be represented by the following cocycles* z_{01}*:*

p	z_{01}
2	$x^{-1}\delta_l\partial_{\xi_k} \sim x^{-1}\delta_k\partial_{\xi_l}$ *for* $l < k$ *and* $k, l = 1, \dots, 4$,
	$x^{-1}\delta_r\partial_{\xi_r}$ *for* $r = 1, \dots, 4$;
3	$x^{-1}\delta_l\partial_x \sim x^{-2}\delta\partial_{\xi_l}$ *and* $x^{-1}\delta\partial_{\xi_l}$ *for* $l = 1, \dots, 4$;
4	$x^{-1}\delta\partial_x$.

Proof. Let us find a basis in $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2)$. For this, consider the part $\mathbf{ST}(\mathbf{E})_2$ of the tangent space $\mathbf{ST}(\mathbf{E}) := \oplus \mathbf{ST}(\mathbf{E})_i$. In [BO2], the following decomposition was considered:

$$\mathbf{ST}(\mathbf{E})_2 = \bigoplus_{i<j} \mathbf{ST}(\mathbf{E})_2^{(ij)} \oplus \bigoplus_{i<j<k,\quad r\neq i,j,k} \mathbf{ST}(\mathbf{E})_2^{(ijk,r)}, \text{where}$$

$$\begin{aligned}\mathbf{ST}(\mathbf{E})_2^{(ij)} &=< \xi_i\xi_j\partial_x \quad \text{and} \quad \xi_i\xi_j\xi_k\partial_{\xi_k}, \quad \text{where } i<j, \text{ and } k\neq i,j>,\\ \mathbf{ST}(\mathbf{E})_2^{(ijk,r)} &=< \xi_i\xi_j\xi_k\partial_{\xi_r}, \quad \text{where } i\langle j<k, \text{ and } r\neq i,j,k.\rangle\end{aligned}$$

Moreover, it was shown in [BO2] that $\mathbf{ST}(\mathbf{E})_2^{(ij)} \simeq \mathbf{L}_{-2} \oplus 2\mathbf{L}_{-1}$ and $\mathbf{ST}(\mathbf{E})_2^{(ijk,r)} \simeq \mathbf{L}_{-2}$.

Consider the bundles $\mathbf{ST}(\mathbf{E})_2^{(ij)}$ and $\mathbf{ST}(\mathbf{E})_2^{(ijk,r)}$ separately. Let $(\mathcal{T}_{\rm gr})_p^{i_1\dots i_k}$ designate the sheaf of holomorphic sections of $\mathbf{ST}(\mathbf{E})_p^{(i_1\dots i_k)}$. We see that (recall notation (2))

$$\begin{aligned}&\delta_l\partial_{\xi_k} = y^{-2}\delta'_l\partial_{\eta_k}, \text{ where } \ l<k, \quad k\neq i,j, \quad l\neq i,j,\\ &\xi_i\xi_j\partial_x = -\eta_i\eta_j\partial_y - y^{-1}\eta_i\eta_j\nabla' = -y^{-1}(\eta_i\eta_j\nabla' + y\eta_i\eta_j\partial_y),\\ &\xi_i\xi_j\nabla + x\xi_i\xi_j\partial_x = -y^{-1}\xi_i\xi_j\partial_x.\end{aligned}$$

Hence, for the basis sections of $\mathbf{L}_{-2}$ (resp. $\mathbf{L}_{-1}$) we can take

$$\delta_l\partial_{\xi_k}, \quad \xi_i\xi_j\partial_x, \text{ (resp. } \xi_i\xi_j\nabla + x\xi_i\xi_j\partial_x) \text{ for all } i,j,k,l.$$

Then, $\mathbf{ST}(\mathbf{E})_2^{(ij)} \simeq \mathbf{L}_{-2} \oplus 2\mathbf{L}_{-1}$ and $(\mathcal{T}_{\rm gr})_2^{ij} \simeq \mathcal{F}(-2) \oplus 2\mathcal{F}(-1)$, where $\mathcal{F}$ is the sheaf of holomorphic functions on $\mathbb{CP}^1$, is the corresponding isomorphism of sheaves.

The results of [BO2] imply that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2^{ij}) \simeq H^1(\mathbb{CP}^1, \mathcal{F}(-2)) \quad \text{and} \quad \dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2^{ij}) = 1,$$

so the cocycle desired is of the form

$$x^{-1}\delta_l\partial_{\xi_k}.$$

Let us show that $\qquad x^{-1}\delta_l\partial_{\xi_k} \sim x^{-1}\delta_k\partial_{\xi_l}.$

Indeed,

$$\begin{aligned}x^{-1}\delta_l\partial_{\xi_k} &= y^{-1}\delta'_l\partial_{\eta_k} \sim y^{-1}\delta'_l\partial_{\eta_k} + \eta_s\eta_t\partial_y\\ &= x^{-1}\delta_l\partial_{\xi_k} - \xi_s\xi_t\partial_x - x^{-1}\delta_l\partial_{\xi_k} + x^{-1}\delta_k\partial_{\xi_l} \sim x^{-1}\delta_k\partial_{\xi_l},\end{aligned}$$

where $(l,k;s,t) \in \{(1,2;3,4),\ (1,4;2,3),\ (2,3;1,4),\ (3,4;1,2)\}$;

$$\begin{aligned}x^{-1}\delta_l\partial_{\xi_k} &= y^{-1}\delta'_l\partial_{\eta_k} \sim y^{-1}\delta'_l\partial_{\eta_k} - \eta_s\eta_t\partial_y\\ &= x^{-1}\delta_l\partial_{\xi_k} + \xi_s\xi_t\partial_x - x^{-1}\delta_l\partial_{\xi_k} + x^{-1}\delta_k\partial_{\xi_l} \sim x^{-1}\delta_k\partial_{\xi_l},\end{aligned}$$

where $(l,k;s,t) \in \{(1,3;2,4),\ (2,4;1,3)\}$.

Since $\delta_r\partial_{\xi_r} = y^{-2}\delta'_r\partial_{\eta_r}$, then take $\delta_r\partial_{\xi_r}$ for a basis section of $\mathbf{L}_{-2}$. We have

$$\mathbf{ST}(\mathbf{E})_2^{(ijk,r)} \simeq \mathbf{L}_{-2}$$

and the corresponding isomorphism of sheaves $(\mathcal{T}_{\rm gr})_2^{ijk,r} \simeq \mathcal{F}(-2)$.

The results of [BO2] imply that $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2^{ijk,r}) \simeq H^1(\mathbb{CP}^1, \mathcal{F}(-2))$,

$$\dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2^{ijk,r}) = 1 \text{ for any } i<j<k, \text{ and } r \neq i,j,k,$$

and the cocycle desired is of the form

$$x^{-1}\delta_r\partial_{\xi_r}.$$

Let us now find the basis of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_3)$. Consider $\mathbf{ST}(\mathbf{E})_3$ and again apply the results of [BO2]; we get

$$\mathbf{ST}(\mathbf{E})_3 = \bigoplus_{i<j<k} \mathbf{ST}(\mathbf{E})_3^{(ijk)},$$

where $\mathbf{ST}(\mathbf{E})_3^{(ijk)}$ is spanned by

$$\begin{array}{l}\xi_i\xi_j\xi_k\partial_x,\ \ \delta\partial_{\xi_l}, \text{ where } 1\leq i<j<k\leq 4,\ \ l\neq i,j,k,\ \ l\in\{1,\dots,4\};\\ \xi_i\xi_j\xi_k\partial_x == -y^{-2}\big(y\eta_i\eta_j\eta_k\partial_y + \eta_i\eta_j\eta_k\eta_l\partial_{\eta_l}\big),\\ \delta\partial_{\xi_l} - x\delta_l\partial_x = y^{-3}\delta'\partial_{\eta_l} + y^{-2}\delta_l'\partial_y - y^{-3}\delta'\partial_{\eta_l} = y^{-2}\delta_l'\partial_y.\end{array}$$

Hence, take $\xi_i\xi_j\xi_k\partial_x$ and $\delta\partial_{\xi_l} - x\delta_l\partial_x$ for basis sections of $\mathbf{L}_{-2}$ and $\mathbf{L}_{-1}$.

We see that $\mathbf{ST}(\mathbf{E})_3^{(ijk)} \simeq 2\mathbf{L}_{-2}$ and $(\mathcal{T}_{\rm gr})_3^{ijk} \simeq 2\mathcal{F}(-2)$.

The results of [BO2] imply that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_3^{ijk}) \simeq 2H^1(\mathbb{CP}^1, \mathcal{F}(-2)), \text{ and } \dim H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_3^{ijk}) = 2,$$

where $1\leq i<j<k\leq 4$, and the cocycles desired are of the form

$$x^{-1}\delta_l\partial_x \quad\text{and}\quad x^{-1}\delta\partial_{\xi_l} - \delta_l\partial_x \sim x^{-1}\delta\partial_{\xi_l}, \quad \text{where } l\neq i,j,k.$$

Let us show that

$$x^{-1}\delta_l\partial_x \sim x^{-2}\delta\partial_{\xi_l}.$$

Indeed, $\qquad x^{-1}\delta_l\partial_x = -\delta_l'\partial_y + y^{-1}\delta'\partial_{\eta_l} \sim y^{-1}\delta'\partial_{\eta_l} = x^{-2}\delta\partial_{\xi_l}$.

In [BO1], it is proved that the basis element of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)$ can be represented by the cocycle $x^{-1}\delta\partial_x$. □

3 Non-split supermanifolds with retract $\mathcal{CP}^{1|4}$

The structure sheaf of the split supermanifold $(M, \mathcal{O}_{\rm gr})$ is endowed with a $\mathbb{Z}$-grading

$$\mathcal{O}_{\rm gr} = \bigoplus_{0\leq p\leq n} (\mathcal{O}_{\rm gr})_p, \quad \text{where } (\mathcal{O}_{\rm gr})_p = \Lambda^p_{\mathcal{F}}(\mathcal{E}).$$

Clearly, $(\mathcal{O}_{\rm gr})_{\rm rd}$ is naturally isomorphic to the subsheaf $\mathcal{F}\subset\mathcal{O}_{\rm gr}$.

Observe that the natural filtration of the sheaf $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$ yields the following filtration

$$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}) = \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(-1)} \supset \ldots \supset \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(4)} \supset \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(5)} = 0,$$

where $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(p)} = \Gamma(\mathbb{CP}^1, \mathcal{T}_{(p)})$.

Thanks to results in [O1], we have the following exact sequence

$$0 \to \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(p+1)} \to \mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{(p)} \xrightarrow{\sigma_p} \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_p \quad \text{for any } p \geq -1.$$

We say that $u \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_p$ is *liftable* in $(\mathbb{CP}^1, \mathcal{O})$, if $u \in \operatorname{Im} \sigma_p$.

Consider

$$\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}} = \{a \in \mathcal{A}ut\, \mathcal{O}_{\mathrm{gr}} \mid a(f) - f \in \mathcal{J}^2 \text{ for any } f \in \mathcal{O}_{\mathrm{gr}}\}.$$

Let $\operatorname{Aut} \mathbf{E}$ be the group of fiber-preserving automorphisms of $\mathbf{E}$. Then, the following theorem holds.

3.1 Theorem ([Gr]). *There is a bijective correspondence between the isomorphism classes of supermanifolds $(M, \mathcal{O})$ such that $\operatorname{gr} \mathcal{O} \simeq \mathcal{O}_{\mathrm{gr}}$ and the orbits of the action of* $\operatorname{Aut} \mathbf{E}$ *on $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$, and $(M, \mathcal{O}_{\mathrm{gr}})$ corresponds to the unit class e in $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$.*

4 On $\bar{0}$-homogeneity of supermanifolds with retract $\mathcal{CP}^{1|4}$

Recall a fine resolution of the sheaf $\mathcal{T}_{gr} = \mathcal{D}er\mathcal{O}_{\mathrm{gr}}$ endowed with a bracket operation that extends the bracket given in $\mathcal{T}_{gr}$. Let us denote by $\Phi^{p,q}$ the sheaf of smooth complex-valued forms of type (p, q) on M. We form the standard Dolbeault-Serre resolution $\widehat{\Phi}$ of $\mathcal{O}_{\mathrm{gr}}$ by setting for any $\varphi \in \Phi^{0,q}$ and $u \in (\mathcal{O}_{\mathrm{gr}})_p$

$$\widehat{\Phi}^{p,q} := \Phi^{0,q} \otimes (\mathcal{O}_{\mathrm{gr}})_p, \quad \widehat{\Phi}^{\bullet,\bullet} := \oplus_{p,q \geq 0} \widehat{\Phi}^{p,q}, \quad \bar{\partial}(\varphi \otimes u) = \bar{\partial}(\varphi) \otimes u.$$

Then, regarding $\mathcal{S}$ as a sheaf of graded algebras with respect to the total degree, consider the sheaf of bigraded Lie superalgebras $\widehat{\mathcal{T}} = \mathcal{D}er\widehat{\Phi}$. Clearly, $\overline{D} = \mathrm{ad}_{\bar{\partial}}$ is a derivation of bidegree $(0, 1)$ of $\widehat{\mathcal{T}}$ satisfying $\overline{D}^2 = 0$. Set

$$\mathcal{S} := \{u \in \widehat{\mathcal{T}} \mid u(\bar{f}) = 0 \text{ and } u(d\bar{f}) = 0 \ \text{ for any } f \in \mathcal{F}\}.$$

It is easy to see that $\mathcal{S}$ is a $\overline{D}$-invariant subsheaf of bigraded subalgebras of $\widehat{\mathcal{T}}$. Moreover, $\widehat{\mathcal{T}}_{gr}$ is identified with the kernel of the mapping $\overline{D} : \mathcal{S}^{\bullet,0} \longrightarrow \mathcal{S}^{\bullet,1}$. Thus, we get the sequence

$$0 \longrightarrow \widehat{\mathcal{T}} \xrightarrow{\tau} \mathcal{S}^{*,0} \xrightarrow{\overline{D}} \mathcal{S}^{*,1} \xrightarrow{\overline{D}} \dots$$

Let us specify an explicit form of τ. Let $\mathcal{F}^\infty$ be a sheaf of differentiable complex-valued functions on M, then $\mathcal{O}_{\mathrm{gr}}^\infty = \mathcal{O}_{\mathrm{gr}} \otimes \mathcal{F}^\infty$ and

$$\mathcal{P}\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}^\infty = \{a \in \mathcal{A}ut\, \mathcal{O}_{\mathrm{gr}}^\infty \mid a(\overline{f}) = \overline{f} \text{ for any } f \in \mathcal{F};\ a(u) - u \in \bigoplus_{k \geq 2} (\mathcal{O}_{\mathrm{gr}}^\infty)_k \text{ for any } u \in \mathcal{O}_{\mathrm{gr}}^\infty\}.$$

If $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ is a cocycle in the covering $\mathfrak{U}$, then $z_{ij} = h_i^{-1}h_j$, where $h_i : \mathcal{O}_{\mathrm{gr}}|_{U_i} \to \mathcal{O}|_{U_i}$. On the other hand, $z_{ij} = a_i^{-1}a_j$, where $a_i \in \mathcal{P}\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}^\infty(U_i)$. Then, over $U_i \cap U_j$, we have $h_i^{-1}h_j = a_i^{-1}a_j$, and hence $\varrho := a_i h_i^{-1} = a_j h_j^{-1}$ is an injective sheaf homomorphism $\mathcal{O} \longrightarrow \mathcal{O}^\infty_{\text{gr}}$. Then, $\tau : \widehat{\mathcal{T}} \to \mathcal{S}^{*,0}$ is defined by the formula $\tau(v) := \varrho v \varrho^{-1}$.

Now, let the bundle $\mathbf{E}$ correspond to the supermanifold $\mathcal{CP}^{1|4}$; let the tangent bundle $\mathbf{ST}(\mathbf{E})$ be endowed with a smooth SU_2-invariant hermitian metric (see [O1]). Since $\mathbb{CP}^1$ is compact, we can apply the Hodge theory. In [O1], [O3], [O2] a complex $(\mathcal{S}, \overline{D})$ is constructed which can be considered as a complex of $(0,*)$-forms with values in the bundle $\mathbf{ST}(\mathbf{E})$.

Let $\mathbf{H} \subset \mathcal{S}$ denote the bigraded space of harmonic elements, H the orthogonal projection to $\mathbf{H}$. As is known,

$$\mathbf{H}_{p,q} \simeq H^{p,q}(\mathcal{S}, \overline{D}) \simeq H^q(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_p) \text{ for any } p, q \geq 0. \tag{3}$$

Set

$$\mathbf{H}_{(1)} = \bigoplus_{p\geq 1} \mathbf{H}_{2p,1} = \mathbf{H}_{2,1} \oplus \mathbf{H}_{4,1} \simeq H^1(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_2) \oplus H^1(\mathbb{CP}^1, (\widehat{\mathcal{T}}_{\text{gr}})_4).$$

The SU_2-invariance of the metric implies SU_2-equivariance of $\mathbf{H}$, and isomorphisms (3).

4.1. Theorem. *Let $(\mathbb{CP}^1, \mathcal{O})$ be any supermanifold with retract $\mathcal{CP}^{1|4}$. Then, the SU_2-action on $\mathbb{CP}^1$ can be lifted to $(\mathbb{CP}^1, \mathcal{O})$. In particular, $(\mathbb{CP}^1, \mathcal{O})$ is $\bar{0}$-homogeneous.*

Proof. Consider the non-linear complex $K = (K^0, K^1, K^2)$ (see [BO2]), where

$$K^0 = \Gamma(\mathbb{CP}^1, \mathcal{PA}ut_{(2)}\mathcal{O}^\infty_{\text{gr}}),$$
$$K^q = \bigoplus_{k\geq 1} S^{2k,q} \text{ for } q = 1, 2,$$

with the coboundary operators $\delta_q : K^q \to K^{q+1}$ for $q = 0, 1$, and the action ρ of the group K^0 on K^1, defined by the formulas

$$\delta_0(a) = \overline{\partial} - a\overline{\partial}a^{-1},$$
$$\delta_1(u) = \overline{\partial}u - \tfrac{1}{2}[u, u],$$
$$\rho(a)(u) = a(u - \overline{\partial})a^{-1} + \overline{\partial}.$$

By definition , the corresponding set of 1-cohomology has the form $Z^1(K)/\rho(K^0)$, where $Z^1(K) = \{u \in K^1 \mid \delta_1(u) = 0\}$.

Since $\dim \mathbb{CP}^1 = 1$, it follows that $\mathbf{H}_{(1)} \subset Z^1(K)$. Moreover, as is shown in [O2], the natural map $\mathbf{H}_{(1)} \longrightarrow H^1(K)$ is surjective.

Further, $H^0(\mathbb{CP}^1, (\mathcal{T}_{\text{gr}})_2) = 0$ by Theorem 2.1. Hence, Theorem 3.13 from [O2] is applicable implying that this map is bijective. Thus, the bijection $\mathbf{H}_{(1)} \longrightarrow H^1(K)$ is SU_2-invariant.

On the other hand, Theorem 2.1 implies that SU_2 acts on $\mathbf{H}_{(1)}$ trivially. Hence, it acts on $H^1(K)$ also trivially, and every cohomology class contains an invariant cocycle.

Applying the obtained in [O3] criterion for lifting the action of the compact groups on the non-split supermanifold we see that the SU_2-action on $\mathcal{CP}^{1|4}$ can be lifted to any supermanifold $(\mathbb{CP}^1, \mathcal{O})$ with $\mathcal{CP}^{1|4}$ as its retract. Since SU_2 transitively acts on $\mathbb{CP}^1$, all these supermanifolds are $\bar{0}$–homogeneous. □

5 Description of supermanifolds via cocycles

Thanks to Theorem 3.1, the classes of isomorphic supermanifolds $(M,\mathcal{O})$ are in bijective correspondence with the Aut $\mathbf{E}$-orbits on the set $H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. Let $M=\mathbb{CP}^1$ and the odd dimension of supermanifolds $(\mathbb{CP}^1,\mathcal{O})$ be ≤ 5.

As in [O2], define the *exponential map*

$$\exp:\ \mathcal{T}_{\mathrm{gr}}=(\mathcal{T}_{\mathrm{gr}})_2\oplus(\mathcal{T}_{\mathrm{gr}})_4\longrightarrow\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}},$$

its inverse $\log:=\exp^{-1}$, and the map

$$\lambda_{2p}:\ \mathcal{A}ut_{(2p)}\mathcal{O}_{\mathrm{gr}}\longrightarrow(\mathcal{T}_{\mathrm{gr}})_{2p}$$

sending any $a\in\mathcal{A}ut_{(2p)}\mathcal{O}_{\mathrm{gr}}$ to the $2p$-component $(\log a)_{2p}$ of $\log a$.

The map exp is an isomorphism of sheaves of sets, λ_{2p} is a surjective homomorphism of sheaves of groups. In what follows, we will represent the cocycle $g=\exp u$ by the cocycle $u=u_2+u_4$, where

$$u_2\in Z^1(\mathbb{CP}^1,(\mathcal{T}_{\mathrm{gr}})_2)\text{ and }u_4\in Z^1(\mathbb{CP}^1,(\mathcal{T}_{\mathrm{gr}})_4).\tag{4}$$

Since

$$(\mathcal{T}_{\mathrm{gr}})_2\cdot(\mathcal{T}_{\mathrm{gr}})_4=(\mathcal{T}_{\mathrm{gr}})_4\cdot(\mathcal{T}_{\mathrm{gr}})_4=0,$$

it follows that $(\mathcal{T}_{\mathrm{gr}})_4$ is a central ideal in $\mathcal{T}_{\mathrm{gr}}$.

The exact sequence (see [O2])

$$e\longrightarrow\mathcal{A}ut_{(2p+2)}\mathcal{O}\longrightarrow\mathcal{A}ut_{(2p)}\mathcal{O}\xrightarrow{\lambda_{2p}}(\mathcal{T}_{\mathrm{gr}})_{2p}\longrightarrow 0$$

yields — for $p=2$ — an isomorphism of the sheaves of groups

$$\exp:\ (\mathcal{T}_{\mathrm{gr}})_4\longrightarrow\mathcal{A}ut_{(4)}\mathcal{O}_{\mathrm{gr}}\subset\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}.$$

Hence, $\mathcal{A}ut_{(4)}\mathcal{O}_{\mathrm{gr}}$ belongs to the center of $\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}$.

Define the action of the sheaf of groups $(\mathcal{T}_{\mathrm{gr}})_4$ on $\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}$ by mean of right shifts

$$\Psi\colon\ v\mapsto t_v:z\mapsto z(\exp v),\text{ where }v\in(\mathcal{T}_{\mathrm{gr}})_4,\text{ and }z\in\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}.$$

Let us translate this action to Čzech cocycles of the covering U_0,U_1. Let us check if it is also well defined on cohomology. Let $v'\sim v$ and $z'\sim z$, where

$$v',v\in Z^1(\mathbb{CP}^1,(\mathcal{T}_{\mathrm{gr}})_4),\quad z',z\in Z^1(\mathbb{CP}^1,\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}).$$

Then, $v'=b^{(0)}+v-b^{(1)}$ and $z'=c^{(0)}z(c^{(1)})^{-1}$, where $b^{(i)},c^{(i)}$ are holomorphic sections over U_i for $i=0,1$ of the sheaves $(\mathcal{T}_{\mathrm{gr}})_4$ and $\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}$, respectively. We see that

$$\begin{aligned}&z'\exp v'=(c^{(0)}z(c^{(1)})^{-1})(\exp v)(\exp b^{(0)})(\exp b^{(1)})^{-1}\\&=(c^{(0)}\exp b^{(0)})(z\exp v)(c^{(1)}\exp b^{(1)})^{-1}\sim z\exp v.\end{aligned}$$

Therefore, the action Ψ on cohomology is well defined.

5.1. Theorem. *Let* $\dim(\mathbb{CP}^1, \mathcal{O}) = 1|n$ *with* $n \leq 5$. *Then, the action* Ψ *defines a free action of the group* $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)$ *on* $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$*; the orbits of this action are the fibers of the map*

$$\lambda_2^*: \ H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr}) \longrightarrow H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2).$$

Proof. To show that $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)$ freely acts on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$, i.e., the stabilizer of any element $z \in Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ is trivial, let $z(\exp v) \sim z$. By the above, there exists an element

$$u \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2 \oplus (\mathcal{T}_{\rm gr})_4)$$

such that $z = \exp u$.

Let $u = u_2 + u_4$, see (4). Since

$$v = v_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4),$$

then

$$z(\exp v) = \exp(u_2 + u_4)\exp v = \exp(u_2 + u_4 + v_4).$$

Hence,

$$\exp(u_2 + u_4 + v_4) \sim \exp(u_2 + u_4).$$

Therefore,

$$\exp(u_2 + u_4 + v_4) = c^{(0)}\exp(u_2 + u_4)(c^{(1)})^{-1},$$

where $c^{(i)} \in \Gamma(U_i, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ are holomorphic sections for $i = 0, 1$.

Let us represent $c^{(i)} = \exp(a_2^{(i)} + a_4^{(i)})$, where $a_2^{(i)}$ and $a_4^{(i)}$ are holomorphic sections of the sheaves $(\mathcal{T}_{\rm gr})_2$ and $(\mathcal{T}_{\rm gr})_4$, respectively, over U_i for $i = 0, 1$. Then, applying the Campbell-Hausdorff decomposition twice, we see that

$$\begin{aligned}&\exp(a_2^{(0)} + a_4^{(0)}) \exp(u_2 + u_4) \exp(-a_2^{(1)} - a_4^{(1)})\\ &= \exp(a_2^{(0)} + a_4^{(0)} + u_2 + u_4 - a_2^{(1)} - a_4^{(1)} + \tfrac{1}{2}[a_2^{(0)}, u_2] - \tfrac{1}{2}[u_2, a_2^{(1)}]-\\ &-\tfrac{1}{2}[a_2^{(0)}, a_2^{(1)}]).\end{aligned}$$

Hence,

$$\begin{aligned}&u_2 + u_4 + v_4 = a_2^{(0)} + a_4^{(0)} + u_2 + u_4 - a_2^{(1)} - a_4^{(1)} + \tfrac{1}{2}[a_2^{(0)}, u_2]\\ &-\tfrac{1}{2}[u_2, a_2^{(1)}] - \tfrac{1}{2}[a_2^{(0)}, a_2^{(1)}].\end{aligned}$$

Therefore,

$$v_4 = a_2^{(0)} + a_4^{(0)} - a_2^{(1)} - a_4^{(1)} + \tfrac{1}{2}[a_2^{(0)}, u_2] - \tfrac{1}{2}[u_2, a_2^{(1)}] - \tfrac{1}{2}[a_2^{(0)}, a_2^{(1)}].$$

With respect to the degrees this equality breaks into two:

$$\begin{aligned}&0 = a_2^{(0)} - a_2^{(1)},\\ &v_4 = a_4^{(0)} - a_4^{(1)} + \tfrac{1}{2}[a_2^{(0)}, u_2] - \tfrac{1}{2}[u_2, a_2^{(1)}] - \tfrac{1}{2}[a_2^{(0)}, a_2^{(1)}].\end{aligned}$$

Hence, we see that $v_4 = a_4^{(0)} - a_4^{(1)}$, i.e., $v \sim 0$.

Let us show now that the orbits of the $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)$-action on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ are fibers of the map λ_2^*. Indeed, let $z(\exp v) \sim y$, where

$$y, z \in Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr}), \quad v = v_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4).$$

Then, there exist $u, w \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2 \oplus (\mathcal{T}_{\rm gr})_4)$ such that $z = \exp u$ and $y = \exp w$.

Let

$$u = u_2 + u_4 \text{ and } w = w_2 + w_4,$$

where $u_2, w_2 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2)$ and $u_4, w_4 \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)$. We have

$$\exp(u_2 + u_4 + v_4) \sim \exp(w_2 + w_4)$$

or, as in the first part of the proof,

$$\exp(u_2 + u_4 + v_4) = c^{(0)}\exp(w_2 + w_4)(c^{(1)})^{-1},$$

where $c^{(i)} = \exp(a_2^{(i)} + a_4^{(i)})$. We similarly obtain

$$\exp(u_2 + u_4 + v_4) = \exp(a_2^{(0)} + a_4^{(0)} + w_2 + w_4 - a_2^{(1)} - a_4^{(1)}).$$

Having applied λ_2 to both sides of this equality we get

$$u_2 = a_2^{(0)} + w_2 - a_2^{(1)}.$$

Hence, $u_2 \sim w_2$, i.e., $z(\exp v)$ and y determine the same class in $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2)$ with respect to λ_2^*. □

5.2. Corollary. *The classes in $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ are determined by pairs (u_2, u_4), see (4).*

Proof. Theorem 2.1 implies that $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ can be viewed as the bundle with base $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2)$ and fiber $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)$. Thus, any class in $H^1(\mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ is determined by (u_2, u_4), see (4). □

5.3. Theorem. *Any supermanifold with retract $\mathcal{CP}^{1|4}$ coincides, up to an isomorphism, with one of the following supermanifolds $\mathcal{M}$ determined by the cocycle $u^{(01)} = u_2 + u_4$, see (4).*

For homogeneous of these supermanifolds — the cases marked by $*$, the Lie superalgebra $\mathfrak{v}(\mathcal{M})$ is described in § 7.

1*) $u_2 = 0$, *and* $u_4 = 0$;

2*) $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, *and* $u_4 = 0$;

3*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, *and* $u_4 = 0$;

4*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, *and* $u_4 = 0$;

5*) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}$, *and* $u_4 = 0$;

6) $u_2 = 0$, *and* $u_4 = x^{-1}\delta\partial_x$;
7) $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, *and* $u_4 = x^{-1}\delta\partial_x$;
8) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, *and* $u_4 = x^{-1}\delta\partial_x$;
9) $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, *and* $u_4 = x^{-1}\delta\partial_x$;
10*) $u_2 = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4})$ *with* $t \in \mathbb{C}^\times$, *and* $u_4 = x^{-1}\delta\partial_x$.

Proof. Theorem 2.1 implies that

$$H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2) \simeq \mathbb{C}^{10}, \text{ and } H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4) \simeq \mathbb{C}.$$

Thus, by Corollary 5.2, $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ is the bundle with base $\mathbb{C}^{10}$ and fiber $\mathbb{C}$.

Theorem 2.2 provides us with the system of generating cocycles for the basis and the fiber of this bundle. Thus, any supermanifold with retract $\mathcal{CP}^{1|4}$ is determined, up to an isomorphism, by the cocycle which in U_0 is of the form

$$u = \sum_{i,j=1,\dots,4} c_{ij}x^{-1}\delta_i\partial_{\xi_j} + cx^{-1}\delta\partial_x, \quad \text{where } c_{ij} = c_{ji}, \text{ and } c \in \mathbb{C}. \tag{5}$$

Let $\alpha \in \operatorname{Aut}\mathbf{E}$. Since the automorphism α is a linear function in sections $\xi_1, \dots, \xi_4$, then in U_0 we see that

$$\alpha(\xi_i) = \sum_{1\le j\le 4} a_{ji}(x)\xi_j, \text{ where } a_{ji}(x) \text{ are holomorphic functions in } x.$$

Therefore, in U_1 this equality takes the form

$$y^{-1}\alpha(\eta_i) = \sum_{1\le j\le 4} a_{ji}(y^{-1})y^{-1}\eta_j, \qquad \text{or} \qquad \alpha(\eta_i) = \sum_{1\le j\le 4} a_{ji}(y^{-1})\eta_j.$$

This is how the action of the group $\operatorname{Aut}\mathbf{E}$ on sections $\eta_1, \dots, \eta_4$ on U_1 is defined. Hence, $a_{ji}(y^{-1})$ should be holomorphic functions in y on U_1. Hence, $a_{ji} = const$ for any i, j.

Therefore, any automorphism $\alpha \in \operatorname{Aut}\mathbf{E}$ is given by a complex matrix $A = (a_{ij})$, i.e., $\operatorname{Aut}\mathbf{E} \simeq \mathrm{GL}_4(\mathbb{C})$.

Let $B = (b_{ij})$ be the inverse of $A = (a_{ij})$. Then,

$$\alpha(\partial_{\xi_i})\xi_j = (\alpha\partial_{\xi_i}\alpha^{-1})\xi_j = \alpha\partial_{\xi_i}(\sum_{1\le k\le 4} b_{kj}\xi_k) = \alpha(b_{ij}) = b_{ij},$$

i.e.,

$$\alpha(\partial_{\xi_i}) = \sum_{1\le j\le 4} b_{ij}\partial_{\xi_j}.$$

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix} \in S_4$. Further,

$$\begin{aligned}
&\alpha(\delta) = \alpha(\xi_1\xi_2\xi_3\xi_4) = \alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3)\alpha(\xi_4)\\
&= \Big(\sum_{1\le i\le 4} a_{i1}\xi_i\Big)\Big(\sum_{1\le j\le 4} a_{j2}\xi_j\Big)\Big(\sum_{1\le k\le 4} a_{k3}\xi_k\Big)\Big(\sum_{1\le l\le 4} a_{l4}\xi_l\Big)\\
&= \Big(\sum_{\sigma\in S_4} \operatorname{sign}\sigma \cdot a_{i1}a_{j2}a_{k3}a_{l4}\Big)\delta = \det A \cdot \delta;\\
&\alpha(\delta_i) = \alpha\Big(\partial_{\xi_i}\delta\Big) = \alpha(\partial_{\xi_i})\alpha(\delta) = \det A \cdot \sum_{1\le k\le 4} b_{ik}\partial_{\xi_k}\delta = \det A \cdot \sum_{1\le k\le 4} b_{ik}\delta_k.
\end{aligned}$$

We see that

$$\begin{aligned}
&\alpha\Big(x^{-1}\delta\partial_x\Big) = x^{-1}\alpha(\delta)\partial_x\\
&= x^{-1}\Big(\sum_{1\le i\le 4} a_{i1}\xi_i\Big)\Big(\sum_{1\le j\le 4} a_{j2}\xi_j\Big)\Big(\sum_{1\le k\le 4} a_{k3}\xi_k\Big)\Big(\sum_{1\le l\le 4} a_{l4}\xi_l\Big)\partial_x\\
&= x^{-1}\Big(\sum_{\sigma\in S_4} \operatorname{sign}\sigma\cdot a_{i1}a_{j2}a_{k3}a_{l4}\Big)\delta\partial_x = \det A\cdot x^{-1}\delta\partial_x;\\
&\alpha\Big(x^{-1}\delta_i\partial_{\xi_j}\Big) = x^{-1}\alpha(\delta_i)\alpha\Big(\partial_{\xi_j}\Big)\\
&= \det A\cdot \sum_{1\le k\le 4} b_{ik}\delta_k \sum_{1\le l\le 4} b_{jl}\partial_{\xi_l} = \det A\cdot \sum_{1\le k,l\le 4} b_{ik}b_{jl}\delta_k\partial_{\xi_l}.
\end{aligned}$$

Then, having applied α to the cocycle (5), we get

$$\alpha(u) = \det A\cdot \sum_{1\le i,j,k,l\le 4} x^{-1}c_{ij}b_{ik}b_{jl}\delta_k\partial_{\xi_l} + \det A\cdot cx^{-1}\delta\partial_x.$$

This implies, in particular, that the matrix $C = (c_{ij})$ transforms into $(\det A)B^tCB$.

Since every cocycle of the form (5) is uniquely determined by the matrix C and the number c, and the $(\operatorname{Aut}\mathbf{E})$-action is known, it suffices to consider the following cases.

1. In this case, $C = 0$ and $c = 0$.

2. Let $\operatorname{rk} C = 1$ and $c = 0$. Then, as follows from Algebra course, the group $\mathrm{GL}_4(\mathbb{C})$ does not change the rank of C and there is an invertible operator reducing C to the 4×4 matrix E_{11}. To this matrix the cocycle $x^{-1}\delta_1\partial_{\xi_1}$ corresponds; we will consider this cocycle as a representative of the corresponding $(\operatorname{Aut}\mathbf{E})$-orbit on $Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$.

3, 4 and 5. Let $\operatorname{rk} C = 2, 3$ and 4, respectively and $c = 0$. As in case 2, we get, respectively, the following representatives:

$$\begin{aligned}
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2},\\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3},\\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}.
\end{aligned}$$

6. Let $C = 0$ and $c \ne 0$. Then, the transformation determined by the matrix A with $\det A = \frac{1}{c}$ leads to the cocycle $x^{-1}\delta\partial_x$. Let us take it for the representative of the corresponding $(\operatorname{Aut}\mathbf{E})$-orbit on $Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$.

7. Let $\operatorname{rk} C = 1$ and $c \ne 0$. Since C is not invertible, it is possible (elementary linear algebra) to find a matrix A with $\det A = \dfrac{1}{c}$ so that C becomes the matrix unit E_{11}. The cocycle corresponding to this matrix and the number 1 is $x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta\partial_x$. Let us take it for the representative of the corresponding $(\operatorname{Aut}\mathbf{E})$-orbit on $Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$.

8 and 9. Let $\operatorname{rk} C = 2$ and 3, respectively, and $c \ne 0$. In analogy with case 7, the representatives are, respectively,

$$\begin{aligned}
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta\partial_x,\\
&x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta\partial_x.
\end{aligned}$$

10. Let $\operatorname{rk} C = 4$ and $c \ne 0$. In this case, by an invertible transformation with matrix A such that $\det A = \frac{1}{c}$ the matrix C can be reduced to the diagonal form $t\,1_4$, where $t \in \mathbb{C}^\times$.

As a result, we get a 1-parameter family of cocycles

$$c_t := t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}) + x^{-1}\delta\partial_x,$$

Every cocycle c_t defines an orbit in $Z^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$. □

6 Which of the supermanifolds with retract $\mathcal{CP}^{1|4}$ are homogeneous

Consider an open covering $\mathcal{U} = (U_i)_{i\in I}$ of a topological space M. Let the supermanifold $(M, \mathcal{O})$ be determined by a cocycle $g \in Z^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$. Recall the definition of a liftable field, see § 2.

6.1 Theorem ([BO1]). *The vector field $v \in \mathfrak{v}(M, \mathcal{O}_{\rm gr})_p$ can be lifted to $(M, \mathcal{O})$ if and only if there exists a $v^{(i)} \in C^0(M, (\mathcal{T}_{\rm gr})_{(p)})$ such that*

$$\begin{array}{l} v^{(i)} \equiv v \mod(\mathcal{T}_{\rm gr})_{(p+1)}(U_i),\\ g^{(ij)}v^{(j)} = v^{(i)}g^{(ij)} \quad in \quad U_i \cap U_j \neq \varnothing. \end{array}$$

Consider the representation

$$g^{(01)} = \exp u^{(01)}, \text{ where } u^{(01)} \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2 \oplus (\mathcal{T}_{\rm gr})_4).$$

Let $u^{(01)} = u_2 + u_4$, see (4).

6.2. Corollary. *The vector field $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\rm gr})_p$ can be lifted if and only if the following conditions hold*

$$\begin{array}{lll} p = -1 & [v, u_2] = v_1^{(1)} - v_1^{(0)}, & (2)\\ & [v, u_4] = v_3^{(1)} - v_3^{(0)} + [u_2, v_1^{(1)}] + \frac{1}{2}[u_2, [u_2, v]], & (3)\\ p = 0 & [v, u_2] = v_2^{(1)} - v_2^{(0)}, & \\ & [v, u_4] = v_4^{(1)} - v_4^{(0)} + [u_2, v_2^{(1)}] + \frac{1}{2}[u_2, [u_2, v]], & \\ p = 1 & [v, u_2] = v_3^{(1)} - v_3^{(0)}. & \end{array}$$

Proof. Let $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\rm gr})_p$, where $p = -1, 0$. Let us seek $v^{(i)}$ in the form $v + v_{p+2}^{(i)} + v_{p+4}^{(i)}$ for $v_{p+2}^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\rm gr})_{p+2}$, and $v_{p+4}^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\rm gr})_{p+4}$.

By Theorem 6.1 we have a condition

$$g^{(01)}v^{(1)} = v^{(0)}g^{(01)}.$$

Then,

$$\begin{array}{l} v^{(0)} = g^{(01)}v^{(1)}(g^{(01)})^{-1} = (\exp \mathrm{ad}_{u^{(01)}})v^{(1)}\\ = v^{(1)} + [u^{(01)}, v^{(1)}] + \frac{1}{2}[u^{(01)}, [u^{(01)}, v^{(1)}]]. \end{array}$$

Therefore,

$$v + v_{p+2}^{(0)} + v_{p+4}^{(0)} = v + v_{p+2}^{(1)} + v_{p+4}^{(1)} + [u_2 + u_4, v + v_{p+2}^{(1)}] + \frac{1}{2}[u_2, [u_2, v]].$$

Hence, we get the conditions

$$[v, u_2] = v^{(1)}_{p+2} - v^{(0)}_{p+2},$$
$$[v, u_4] = v^{(1)}_{p+4} - v^{(0)}_{p+4} + [u_2, v^{(1)}_{p+2}] + \tfrac{1}{2}[u_2, [u_2, v]].$$

Let $v \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1$. We seek $v^{(i)}$ in the form $v + v_3^{(i)}$, where $v_3^{(i)} \in \mathfrak{v}(U_i, \mathcal{O}_{\mathrm{gr}})_3$. By Theorem 6.1 we get the condition

$$g^{(01)}v^{(1)} = v^{(0)}g^{(01)}.$$

Then,

$$v^{(0)} = g^{(01)}v^{(1)}(g^{(01)})^{-1} = (\exp \mathrm{ad}_{u^{(01)}})v^{(1)} = v^{(1)} + [u^{(01)}, v^{(1)}].$$

Therefore,

$$v + v_3^{(0)} = v + v_3^{(1)} + [u_2 + u_4, v + v_3^{(1)}].$$

Hence, we get the condition

$$[v, u_2] = v_3^{(1)} - v_3^{(0)}.$$

□

The definition of homogeneous supermanifold implies that the supermanifold is homogeneous if and only if the following map is surjective

$$\mathrm{ev}_x : \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}) \longrightarrow T_x(\mathbb{CP}^1, \mathcal{O}) \quad \text{for any } x \in \mathbb{CP}^1.$$

Since $\bar{0}$-homogeneity takes place by the proved, it is necessary and sufficient to prove that all fields $\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$ can be lifted to $(\mathbb{CP}^1, \mathcal{O})$.

Let us use the conditions obtained to verify the homogeneity of the supermanifolds in the 10 cases of Theorem 6.1. For every homogeneous $(\mathbb{CP}^1, \mathcal{O})$, I compute the Lie superalgebra of vector fields on it.

6.2 Case 1. It corresponds to the supermanifold $\mathcal{CP}^{1|4}$. It is well-known that this supermanifold is homogeneous. The Lie superalgebra of vector fields on $\mathcal{CP}^{1|4}$ is known (see [O3]):

$$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1} =< v_{5i} = -\partial_{\xi_i}, v_{6i} = -x\partial_{\xi_i}, \quad \text{where} \ \ i = 1, \dots, 4 >,$$
$$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_0 =< v_{ij} = -\xi_i\partial_{\xi_j}, \quad \text{where} \ \ i, j = 1, \dots, 4,\ v_{55} = x\partial_x + \nabla,$$
$$v_{66} = -x\partial_x,\ v_{56} = -\partial_x,\ v_{65} = x^2\partial_x + x\nabla, \quad \text{where} \sum_{1 \le i \le 6} v_{ii} = 0 >;$$
$$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1 =< v_{i5} = -\xi_i(x\partial_x + \nabla), v_{i6} = -\xi_i\partial_x \ \ \text{for} \ \ i = 1, \dots, 4 > .$$

6.3 Case 2. Then, $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$, where $i = 1, \dots, 4$.

We will repeatedly use the following "formula"

Since $[v_i, u_2] \in Z^1(\mathbb{CP}^1, (\mathcal{T}_{\mathrm{gr}})_1)$, then $[v_i, u_2] \sim 0$ by Theorem 2.1, and hence condition (2) of Corollary 6.2 holds. (6)

Since

$$[v_i, u_2] = [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1}] = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}$$

is holomorphic in U_1, then for each v_i we set

$$v_i^{(0)} = 0,\ v_i^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}.$$

We have

$$[x^{-1}\delta_1\partial_{\xi_1}, -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}] = -x^{-2}\delta_1\partial_{\xi_1}(\frac{\partial\delta_1}{\partial\xi_i})\partial_{\xi_1} + x^{-2}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1}(\delta_1)\partial_{\xi_1} = 0.$$

The first summand vanishes since δ_1 does not contain ξ_1 by definition and $\partial_{\xi_1}(\frac{\partial\delta_1}{\partial\xi_i}) = 0$.
The second summand also vanishes since $\partial_{\xi_1}(\delta_1) = 0$.
Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Therefore, all fields $v_i = -\partial_{\xi_i}$, where $i = 1, \dots, 4$, can be lifted; moreover, in U_0 they have the same form. Hence, the supermanifolds corresponding to the cocycle of the 2nd case is homogeneous.

6.3. Theorem. *Let supermanifold* $(\mathbb{CP}^1, \mathcal{O})$ *be isomorphic to the supermanifold determined by the cocycle* $u^{(01)} = x^{-1}\delta_1\partial_{\xi_1}$. *Then,* (recall notation (2))

$$\begin{aligned}
&\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar 0} = < -\partial_x,\quad x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1},\quad 2x\partial_x + \nabla,\\
&-x\partial_x - \xi_2\partial_{\xi_2},\quad -x\partial_x - \xi_3\partial_{\xi_3},\quad -x\partial_x - \xi_4\partial_{\xi_4},\quad -\xi_2\partial_{\xi_1},\quad -\xi_2\partial_{\xi_3},\\
&-\xi_2\partial_{\xi_4},\quad -\xi_3\partial_{\xi_1},\quad -\xi_3\partial_{\xi_2},\quad -\xi_3\partial_{\xi_4},\quad -\xi_4\partial_{\xi_1},\quad -\xi_4\partial_{\xi_2},\quad -\xi_4\partial_{\xi_3} >;\\
&\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar 1} = < -\partial_{\xi_i}\ \ \textit{for}\ \ i = 1, \dots, 4,\ -x\partial_{\xi_1},\ -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1},\ -\xi_j\partial_x,\\
&-x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1},\ -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1},\ -\xi_j(x\partial_x + \nabla),\quad \textit{for}\ \ j = 2, 3, 4 > .
\end{aligned}$$

6.4 Case 3. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = 0$.
Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$, where $i = 1, \dots, 4$.
Thanks to (6) and since

$$[v_i,\ u_2] = [-\partial_{\xi_i},\quad x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}$$

is holomorphic in U_1, then for each v_i we set

$$v_1^{(0)} = 0,\quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}.$$

In the same way as in Case 2 we see that

$$[x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2},\ -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2}] = 0.$$

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Thus, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of Case 3 is homogeneous.

6.4. Theorem. *Let supermanifold* $(\mathbb{CP}^1, \mathcal{O})$ *be isomorphic to the supermanifold determined by the cocycle* $u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$. *Then,* (recall notation (2))

$$\begin{aligned}
&\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar 0} =< -\partial_x,\ x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2},\ 2x\partial_x + \nabla,\\
&\xi_2\partial_{\xi_2},\ -x\partial_x - \xi_3\partial_{\xi_3},\ \ -x\partial_x - \xi_4\partial_{\xi_4},\ \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1},\ -\xi_3\partial_{\xi_1},\\
&-\xi_3\partial_{\xi_2},\ -\xi_3\partial_{\xi_4},\ -\xi_4\partial_{\xi_1},\ -\xi_4\partial_{\xi_2},\ -\xi_4\partial_{\xi_3} >;\\
&\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar 1} =< -\partial_{\xi_i},\ i = 1, \dots, 4,\ -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2},\ -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1},\\
&-x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2},\ -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2},\\
&-\xi_j(x\partial_x + \nabla),\ -\xi_j\partial_x,\ j = 3, 4 > .
\end{aligned}$$

6.5 Case 4. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}$, and $u_4 = 0$.

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$\begin{aligned}
[v_i, u_2] &= [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}]\\
&= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3}
\end{aligned}$$

is holomorphic in U_1, then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3}.$$

In the same way as in Case 2 we see that $[X, Y] = 0$, where

$$\begin{aligned}
X &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3},\\
Y &:= -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3}.
\end{aligned}$$

Hence, condition (3) of Corollary 6.2 is of the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Thus, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of the 4th case is homogeneous.

6.5. Theorem. *Let supermanifold* $(\mathbb{CP}^1, \mathcal{O})$ *be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}.$$

Then, (recall notation (2))

$$\begin{array}{l}\mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar 0} =< -\partial_x,\ x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2} - \delta_3\partial_{\xi_3},\ 2x\partial_x + \nabla,\\ \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1},\ \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1},\ \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2},\ -x\partial_x - \xi_4\partial_{\xi_4},\\ -\xi_4\partial_{\xi_1},\ -\xi_4\partial_{\xi_2},\ -\xi_4\partial_{\xi_3} >;\\ \mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar 1} =< -\partial_{\xi_i}\ \ \textit{for}\ \ i=1,\dots,4,\ -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3},\\ -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3},\ -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2},\\ -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3},\ -\xi_4(x\partial_x + \nabla),\ -\xi_4\partial_x > .\end{array}$$

6.6 Case 5. Let

$$u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}, \text{ and } u_4 = 0.$$

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\rm gr})_{-1}$, where $i = 1, \dots, 4$.

Thanks to (6) and since

$$\begin{array}{l}[v_i, u_2] = [-\partial_{\xi_i}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] =\\ = -x^{-1}\dfrac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\dfrac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\dfrac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\dfrac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}\end{array}$$

is holomorphic in U_1, then for each v_i we set

$$v_1^{(0)} = 0, \quad v_1^{(1)} = -x^{-1}\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}.$$

In the same way as in Case 2 we see that $[A, B] = 0$, where

$$\begin{array}{l}A := x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4},\\ B := -x^{-1}\dfrac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} - x^{-1}\dfrac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} - x^{-1}\dfrac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} - x^{-1}\dfrac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}.\end{array}$$

Hence, condition (3) of Corollary 6.2 takes the form

$$0 = v_3^{(1)} - v_3^{(0)}.$$

Set $v_3^{(0)} = v_3^{(1)} = 0$.

Therefore, all fields $v_i = -\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted and in U_0 they have the same form. Therefore, the supermanifold corresponding to the cocycle of the 5th case is homogeneous.

6.6. Theorem. *Let supermanifold* $(\mathbb{CP}^1, \mathcal{O})$ *be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}.$$

Then, (recall notation (2))

$$\begin{array}{l}\mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar 0} =< -\partial_x,\ x^2\partial_x + x\nabla + \nabla,\ 2x\partial_x + \nabla,\ \ \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1},\\ \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1},\ \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2},\ \xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1},\ \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2},\ \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3} >,\\ \mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar 1} =< -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4},\\ -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4},\ -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4},\\ -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3},\ -\partial_{\xi_i}\ \ \textit{for}\ \ i=1,\dots,4\ \ > .\end{array}$$

6.7 Case 6. Let $u_2 = 0$, and $u_4 = x^{-1}\delta\partial_x$.

Consider the fields $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$, where $i = 1, \dots, 4$. Since $[v_i, u_2] = 0$, then condition (2) of Corollary 6.2 is satisfied; it takes the form

$$[v_i, u_4] = v_3^{(1)} - v_3^{(0)},$$

where $v_3^{(j)}$ is holomorphic in U_j for $j = 0, 1$. Then, $[v_i, u_4]$ should be cohomologous to 0. Substituting the values of v_i and u_4, we see that $[\partial_{\xi_i}, x^{-1}\delta\partial_x] = x^{-1}\delta_i\partial_x$ which is a basis cocycle (see Theorem 2.2). Hence, condition (3) of Corollary 6.2 is not satisfied.

Therefore, none of the fields $-\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted. Therefore, the supermanifold corresponding to the cocycle of the 6th case is not homogeneous.

6.8 Case 7. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1}$, and $u_4 = x^{-1}\delta\partial_x$.

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$. Thanks to (6) and since

$$[v, u_2] = [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1}] = -x^{-1}\xi_2\xi_3\partial_{\xi_1}$$

is holomorphic in U_1, set $v_1^{(0)} = v'$, and $v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + v'$, where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1$.

Since $[x^{-1}\delta_1\partial_{\xi_1}, x^{-1}\xi_2\xi_3\partial_{\xi_1}] = 0$, then condition (3) of Corollary 6.2 is of the form

$$[-\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1$, let us consider (recall notation (2))

$$v' = \sum_{1\le k\le 4} \Big(A_k\xi_k(x\partial_x + \nabla) + B_k\xi_k\partial_x\Big), \text{ where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned}
&[v', x^{-1}\delta_1\partial_{\xi_1}] = -A_1x^{-1}\delta\partial_{\xi_1} + 3A_1\delta\partial_{\xi_1} - B_1x^{-2}\delta\partial_{\xi_1} - \\
&-(A_1\delta_1\partial_x - 2A_1x^{-1}\delta\partial_{\xi_1} + B_1x^{-1}\delta_1\partial_x]) \\
&= A_1x^{-1}\delta\partial_{\xi_1} - B_1x^{-2}\delta\partial_{\xi_1} - B_1x^{-1}\delta_1\partial_x + 3A_1\delta\partial_{\xi_1} - A_1\delta_1\partial_x \\
&\sim A_1x^{-1}\delta\partial_{\xi_1} - 2B_1x^{-1}\delta_1\partial_x.
\end{aligned}$$

But then, for any $A_1, B_1 \in \mathbb{C}$, we have

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1}] \sim x^{-1}\delta_4\partial_x + A_1x^{-1}\delta\partial_{\xi_1} - 2B_1x^{-1}\delta_1\partial_x \nsim 0.$$

Therefore, the field $-\partial_{\xi_4}$ can not be lifted. This suffices to conclude that the supermanifold corresponding to the cocycle of the 7th case is not homogeneous.

6.9 Case 8. Let $u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}$, and $u_4 = x^{-1}\delta\partial_x$.

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$. Thanks to (6) and since

$$[v, u_2] = [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2}$$

is holomorphic in U_1, we set

$$v_1^{(0)} = v', \quad v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} + v', \quad \text{where } v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1.$$

Since

$$[x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2}] = 0,$$

then condition (3) of Corollary 6.2 is of the form

$$[\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1$, let us consider (recall notation (2))

$$v' = \sum_{1\le k\le 4} \Big(A_k\xi_k(x\partial_x + \nabla) + B_k\xi_k\partial_x\Big), \text{where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned}&[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}]\\ &\sim A_1x^{-1}\delta\partial_{\xi_1} + A_2x^{-1}\delta\partial_{\xi_2} - 2B_1x^{-1}\delta_1\partial_x - 2B_2x^{-1}\delta_2\partial_x.\end{aligned}$$

But then, for any $A_1, A_2, B_1, B_2 \in \mathbb{C}$, we have

$$\begin{aligned}&x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2}]\\ &\sim x^{-1}\delta_4\partial_x + A_1x^{-1}\delta\partial_{\xi_1} + A_2x^{-1}\delta\partial_{\xi_2} - 2B_1x^{-1}\delta_1\partial_x - 2B_2x^{-1}\delta_2\partial_x \not\sim 0.\end{aligned}$$

Therefore, the field $-\partial_{\xi_4}$ can not be lifted. This suffices to conclude that the supermanifold corresponding to the cocycle of the 8th case is not homogeneous.

6.10 Case 9. Let

$$u_2 = x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \text{ and } u_4 = x^{-1}\delta\partial_x.$$

Consider $v = -\partial_{\xi_4} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$. Thanks to (6) and since

$$\begin{aligned}&[v, u_2] = [-\partial_{\xi_4}, x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}]\\ &= -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3}\end{aligned}$$

is holomorphic in U_1, let us set $v_1^{(0)} = v'$, and

$$v_1^{(1)} = -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3} + v',$$

where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1$.

Since $[X, Y] = 0$, where

$$\begin{aligned} X &:= x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, \\ Y &:= -x^{-1}\xi_2\xi_3\partial_{\xi_1} + x^{-1}\xi_1\xi_3\partial_{\xi_2} - x^{-1}\xi_1\xi_2\partial_{\xi_3}, \end{aligned}$$

then condition (3) of Corollary 6.2 is of the form

$$[\partial_{\xi_4}, x^{-1}\delta\partial_x] = v_3^{(1)} - v_3^{(0)} + [x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}, v'].$$

Therefore,

$$x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1$, then we can consider (recall notation (2))

$$v' = \sum_{1\le k\le 4} \Big(A_k\xi_k(x\partial_x + \nabla) + B_k\xi_k\partial_x\Big), \quad \text{where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned} &[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &\sim A_1x^{-1}\delta\partial_{\xi_1} + A_2x^{-1}\delta\partial_{\xi_2} + A_3x^{-1}\delta\partial_{\xi_3} \\ &-2B_1x^{-1}\delta_1\partial_x - 2B_2x^{-1}\delta_2\partial_x - 2B_3x^{-1}\delta_3\partial_x. \end{aligned}$$

But then, for any $A_1, A_2, A_3, B_1, B_2, B_3 \in \mathbb{C}$, we have

$$\begin{aligned} &x^{-1}\delta_4\partial_x + [v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3}] \\ &\sim x^{-1}\delta_4\partial_x + A_1x^{-1}\delta\partial_{\xi_1} + A_2x^{-1}\delta\partial_{\xi_2} + A_3x^{-1}\delta\partial_{\xi_3} - 2B_1x^{-1}\delta_1\partial_x \\ &-2B_2x^{-1}\delta_2\partial_x - 2B_3x^{-1}\delta_3\partial_x \not\sim 0. \end{aligned}$$

Hence, the field $-\partial_{\xi_4}$ can not be lifted. This shows that the supermanifold of the 9th case is not homogeneous.

6.11 Case 10.

Let $t \in \mathbb{C}^\times$. Let

$$u_2 = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), \text{ and } u_4 = x^{-1}\delta\partial_x\ .$$

Let $v_i = -\partial_{\xi_i} \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_{-1}$, where $i = 1, \ldots, 4$. Thanks to (6) and since

$$\begin{aligned} [v_i, u_2] &= [-\partial_{\xi_i}, t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4})] \\ &= -tx^{-1}\left(\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}\right) \end{aligned}$$

is holomorphic in U_1, then set $v_1^{(0)} = v'$, where $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\mathrm{gr}})_1$, and

$$v_1^{(1)} = v' - tx^{-1}\left(\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}\right).$$

We have $[A, B] = 0$, where

$$A := t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}),$$
$$B := -tx^{-1}\left(\frac{\partial\delta_1}{\partial\xi_i}\partial_{\xi_1} + \frac{\partial\delta_2}{\partial\xi_i}\partial_{\xi_2} + \frac{\partial\delta_3}{\partial\xi_i}\partial_{\xi_3} + \frac{\partial\delta_4}{\partial\xi_i}\partial_{\xi_4}\right).$$

Hence, condition (3) of Corollary 6.2 is of the form

$$\begin{aligned}&[-\partial_{\xi_i}, x^{-1}\delta\partial_x]\\ &= v_3^{(1)} - v_3^{(0)} + [t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}), v'].\end{aligned}$$

Therefore,

$$-x^{-1}\delta_i\partial_x + t[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}] \sim 0.$$

Since $v' \in \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{\text{gr}})_1$, take

$$v' = \sum_{1\le k\le 4}\Big(A_k\xi_k(x\partial_x + \nabla) + B_k\xi_k\partial_x\Big), \text{ where } A_k, B_k \in \mathbb{C}.$$

Then,

$$\begin{aligned}&[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}]\\ &= \sum_{1\le k\le 4}\Big(2A_kx^{-1}\delta\partial_{\xi_k} - B_kx^{-2}\delta\partial_{\xi_k} - B_kx^{-1}\delta_k\partial_x - A_k\delta_k\partial_x\Big)\\ &\sim \sum_{1\le k\le 4}\Big(2A_kx^{-1}\delta\partial_{\xi_k} - 2B_kx^{-1}\delta_k\partial_x\Big).\end{aligned}$$

But then, for $B_i = -\frac{1}{2t}$, and $B_j = 0$ for $j \in \{1, \dots, 4 \mid j \neq i\}$, and $A_k = 0$ for $k = 1, \dots, 4$, we have

$$\begin{aligned}&-x^{-1}\delta_i\partial_x + t[v', x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}]\\ &\sim -x^{-1}\delta_i\partial_x + t\sum_{1\le k\le 4}\Big(2A_kx^{-1}\delta\partial_{\xi_k} - 2B_kx^{-1}\delta_k\partial_x\Big) \sim 0.\end{aligned}$$

Thus, all fields $-\partial_{\xi_i}$ for $i = 1, \dots, 4$ can be lifted. Hence, the supermanifold corresponding to the cocycle of the 4th case is homogeneous.

6.7. Theorem. *Let supermanifold* $(\mathbb{CP}^1, \mathcal{O})$ *be isomorphic to the supermanifold determined by the cocycle*

$$u^{(01)} = t(x^{-1}\delta_1\partial_{\xi_1} + x^{-1}\delta_2\partial_{\xi_2} + x^{-1}\delta_3\partial_{\xi_3} + x^{-1}\delta_4\partial_{\xi_4}) + x^{-1}\delta\partial_x.$$

Then (recall notation (2)),

$$\begin{aligned}&\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar 0} = < -\partial_x,\ x^2\partial_x + x\nabla - t\omega - \delta\partial_x,\ 2x\partial_x + \nabla,\\ &\xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1},\ \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1},\ \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2},\\ &\xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1},\ \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2},\ \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3} >, \quad \textit{where } \omega = \sum_{1\le i\le 4}\delta_i\partial_{\xi_i};\end{aligned}$$

$$\begin{aligned}&\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar 1} = < -\partial_{\xi_i} - \tfrac{1}{2t}\xi_i\partial_x,\ i = 1, \dots, 4,\\ &-x\partial_{\xi_1} + t(-\xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_4} - \xi_2\xi_3\partial_{\xi_4}) - \tfrac{1}{2t}\xi_1(x\partial_x + \nabla) + \delta_1\partial_x,\\ &-x\partial_{\xi_2} + t(\xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}) - \tfrac{1}{2t}\xi_2(x\partial_x + \nabla) + \delta_2\partial_x,\\ &-x\partial_{\xi_3} + t(-\xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}) - \tfrac{1}{2t}\xi_3(x\partial_x + \nabla) + \delta_3\partial_x,\\ &-x\partial_{\xi_4} + t(\xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}) - \tfrac{1}{2t}\xi_4(x\partial_x + \nabla) + \delta_4\partial_x > .\end{aligned}$$

6.8 Theorem (Summary). *The supermanifold isomorphic to one of the supermanifolds of cases* $1-5$ *or* 10 *is homogeneous; it is not homogeneous in cases* $6-9$.

7 Description of $\mathfrak{v}(\mathcal{M})$ for homogeneous superstrings $\mathcal{M}$ with retract $\mathcal{CP}^{1|4}$

The cases are numbered as in Theorem 5.3. We reproduce here the bases found in §2.3, and multiplication tables. In all cases (recall notation (2)),

$$h := [e, f] = 2x\partial_x + \nabla \text{ and } [h, e] = 2e,\ [h, f] = -2f.$$

1. Fact (well-known): $\mathfrak{v}(\mathcal{CP}^{1|4}) \simeq \mathfrak{pgl}_{\mathbb{C}}(4|2)$.
2. $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 12\rangle$, where

$$e = x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1},\ f = -\partial_x,\quad a_1 = -x\partial_x - \xi_2\partial_{\xi_2},$$
$$a_2 = -x\partial_x - \xi_3\partial_{\xi_3},\ a_3 = -x\partial_x - \xi_4\partial_{\xi_4},\ a_4 = -\xi_2\partial_{\xi_1},\ a_5 = -\xi_2\partial_{\xi_3},$$
$$a_6 = -\xi_2\partial_{\xi_4},\ a_7 = -\xi_3\partial_{\xi_2},\ a_8 = -\xi_3\partial_{\xi_1},\ a_9 = -\xi_3\partial_{\xi_4},\ a_{10} = -\xi_4\partial_{\xi_2},$$
$$a_{11} = -\xi_4\partial_{\xi_3},\ a_{12} = -\xi_4\partial_{\xi_1};$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i,\ i = 1, \dots, 14\rangle$, where

$$z_1 = -\partial_{\xi_1},\ z_2 = -\partial_{\xi_2},\ z_3 = -\partial_{\xi_3},\ z_4 = -\partial_{\xi_4},\ z_5 = -\xi_2\partial_x,$$
$$z_6 = -\xi_3\partial_x,\ z_7 = -\xi_4\partial_x,\ z_8 = -x\partial_{\xi_1},\ z_9 = -\xi_2(x\partial_x + \nabla),$$
$$z_{10} = -\xi_3(x\partial_x + \nabla),\ z_{11} = -\xi_4(x\partial_x + \nabla),\ z_{12} = -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1},$$
$$z_{13} = -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1},\ z_{14} = -x\partial_{\xi_3} + \xi_2\xi_3\partial_{\xi_1}.$$

$$[h, a_i] = [e, a_i] = [f, a_i] = 0 \text{ for all } \quad i = 1, \dots, 12.$$

[,]	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_1	0	0	0	$-a_4$	$-a_5$	$-a_6$	a_7	0	0	a_{10}	0	0
a_2	0	0	0	0	a_5	0	$-a_7$	$-a_8$	$-a_9$	0	a_{11}	0
a_3	0	0	0	0	0	a_6	0	0	a_9	$-a_{10}$	$-a_{11}$	$-a_{12}$
a_4	a_4	0	0	0	0	0	a_8	0	0	a_{12}	0	0
a_5	a_5	$-a_5$	0	0	0	0	a_2-a_1	$-a_4$	$-a_6$	a_{11}	0	0
a_6	a_6	0	$-a_6$	0	0	0	a_9	0	0	a_3-a_1	$-a_5$	$-a_4$
a_7	$-a_7$	a_7	0	$-a_8$	a_1-a_2	$-a_9$	0	0	0	0	a_{12}	0
a_8	0	a_8	0	0	a_4	0	0	0	0	0	a_{10}	0
a_9	0	a_9	$-a_9$	0	a_6	0	0	0	0	$-a_7$	a_3-a_2	$-a_8$
a_{10}	$-a_{10}$	0	$-a_{10}$	$-a_{12}$	$-a_{11}$	a_1-a_3	0	0	a_7	0	0	0
a_{11}	0	$-a_{11}$	a_{11}	0	0	a_5	$-a_{12}$	$-a_{10}$	a_2-a_3	0	0	0
a_{12}	0	0	a_{12}	0	0	a_4	0	0	a_8	0	0	0

$[\ ,\ ]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
e	$-z_8$	$-z_{12}$	$-z_{13}$	$-z_{14}$	$-z_9$	$-z_{10}$	$-z_{11}$	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	$-z_1$	$-z_5$	$-z_6$	$-z_7$	$-z_2$	$-z_3$	$-z_4$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$	$-z_7$	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
a_1	0	z_2	0	0	0	z_6	z_7	$-z_8$	$-z_9$	0	0	0	$-z_{13}$	$-z_{14}$
a_2	0	0	z_3	0	z_5	0	z_7	$-z_8$	0	$-z_{10}$	0	$-z_{12}$	0	$-z_{14}$
a_3	0	0	0	z_4	z_5	z_6	0	$-z_8$	0	0	$-z_{11}$	$-z_{12}$	$-z_{13}$	0
a_4	0	z_1	0	0	0	0	0	0	0	0	0	z_8	0	0
a_5	0	z_3	0	0	0	$-z_5$	0	0	0	$-z_9$	0	$-z_{12}$	0	0
a_6	0	z_4	0	0	0	0	$-z_5$	0	0	0	$-z_9$	z_{14}	0	0
a_7	0	0	z_2	0	z_6	0	0	0	$-z_{10}$	0	0	0	z_{12}	0
a_8	0	0	z_1	0	0	0	0	0	0	0	0	0	z_8	0
a_9	0	0	z_4	0	0	0	$-z_6$	0	0	0	$-z_{10}$	0	z_4	0
a_{10}	0	0	0	z_2	$-z_7$	0	0	0	$-z_{11}$	0	0	0	0	z_{12}
a_{11}	0	0	0	z_3	0	$-z_7$	0	0	0	$-z_{11}$	0	0	0	z_{13}
a_{12}	0	0	0	z_1	0	0	0	0	0	0	0	0	0	z_8

$[\ ,\ ]$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}
z_1	0	0	0	0	0	0	0	0	a_4	a_8	a_{12}	0	0	0
z_2	0	0	0	0	$-f$	0	0	0	a_1+h	a_7	a_{10}	0	$-a_{12}$	a_8
z_3	0	0	0	0	0	$-f$	0	0	a_5	a_2+h	a_{11}	a_{12}	0	$-a_4$
z_4	0	0	0	0	0	0	$-f$	0	a_8	a_9	a_3+h	$-a_8$	a_4	0
z_5	0	$-f$	0	0	0	0	0	$-a_4$	0	0	0	$-a_1$	$-a_5$	$-a_6$
z_6	0	0	$-f$	0	0	0	0	$-a_8$	0	0	0	$-a_7$	$-a_2$	$-a_9$
z_7	0	0	0	$-f$	0	0	0	$-a_{12}$	0	0	0	$-a_{10}$	$-a_{11}$	$-a_3$
z_8	0	0	0	0	$-a_4$	$-a_8$	$-a_{12}$	0	0	0	0	0	0	0
z_9	a_4	a_1+h	a_5	a_6	0	0	0	0	0	0	0	e	0	0
z_{10}	a_8	a_7	a_2+h	a_9	0	0	0	0	0	0	0	0	e	0
z_{11}	a_{12}	a_{10}	a_{11}	a_3+h	0	0	0	0	0	0	0	0	0	e
z_{12}	0	0	a_{12}	$-a_8$	$-a_1$	$-a_7$	$-a_{10}$	0	e	0	0	0	0	0
z_{13}	0	$-a_{12}$	0	a_4	$-a_5$	$-a_2$	$-a_{11}$	0	0	e	0	0	0	0
z_{14}	0	a_8	$-a_4$	0	$-a_6$	$-a_9$	$-a_3$	0	0	0	e	0	0	0

3. We have $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 9\rangle$, where

$$e = x^2\partial_x + x\nabla - \delta_1\partial_{\xi_1} - \delta_2\partial_{\xi_2},\ f = -\partial_x,\ a_1 = -x\partial_x - \xi_3\partial_{\xi_3},$$
$$a_2 = -x\partial_x - \xi_4\partial_{\xi_4},\ a_3 = \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1},\ a_4 = -\xi_3\partial_{\xi_1},\ a_5 = -\xi_3\partial_{\xi_2},$$
$$a_6 = -\xi_3\partial_{\xi_4},\ \ a_7 = -\xi_4\partial_{\xi_1},\ a_8 = -\xi_4\partial_{\xi_2},\ a_9 = -\xi_4\partial_{\xi_3};$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 12\rangle$, where

$$z_1 = -\partial_{\xi_1},\ z_2 = -\partial_{\xi_2},\ z_3 = -\partial_{\xi_3},\ z_4 = -\partial_{\xi_4},\ z_5 = -\xi_3\partial_x,$$
$$z_6 = -\xi_4\partial_x,\ z_7 = -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2},\ z_8 = -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1},$$
$$z_9 = -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2},\ z_{10} = -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2},$$
$$z_{11} = -\xi_3(x\partial_x + \nabla),\ \ z_{12} = -\xi_4(x\partial_x + \nabla).$$

$$[h,a_i]=[e,a_i]=[f,a_i]=0,\ i=1,\dots,9.$$

[,]	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	0	0	0	$-a_4$	$-a_5$	$-a_6$	0	0	a_9
a_2	0	0	0	0	0	a_6	$-a_7$	$-a_8$	$-a_9$
a_3	0	0	0	$-a_5$	a_4	0	$-a_8$	a_7	0
a_4	a_4	0	a_5	0	0	0	0	0	a_7
a_5	a_5	0	$-a_4$	0	0	0	0	0	a_8
a_6	a_6	$-a_6$	0	0	0	0	$-a_4$	$-a_5$	a_2-a_1
a_7	0	a_7	a_8	0	0	a_4	0	0	0
a_8	0	a_8	$-a_7$	0	0	a_5	0	0	0
a_9	$-a_9$	a_9	0	$-a_7$	$-a_8$	a_1-a_2	0	0	0

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
e	$-z_7$	$-z_8$	$-z_9$	$-z_{10}$	$-z_{11}$	$-z_{12}$	0	0	0	0	0	0
f	0	0	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	$-z_6$	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
a_1	0	0	z_3	0	0	z_6	$-z_7$	$-z_8$	0	$-z_{10}$	$-z_{11}$	0
a_2	0	0	0	z_4	z_5	0	$-z_7$	$-z_8$	$-z_9$	0	0	z_{12}
a_3	$-z_2$	z_1	0	0	0	0	$-z_8$	z_7	0	0	0	0
a_4	0	0	z_1	0	0	0	0	0	z_7	0	0	0
a_5	0	0	z_2	0	0	0	0	0	z_8	0	0	0
a_6	0	0	z_4	0	0	$-z_5$	0	0	z_{10}	0	0	$-z_{11}$
a_7	0	0	0	z_1	0	0	0	0	0	z_7	0	0
a_8	0	0	0	z_2	0	0	0	0	0	z_8	0	0
a_9	0	0	0	z_3	$-z_6$	0	0	0	0	z_9	$-z_{12}$	0

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
z_1	0	0	0	0	0	0	0	0	a_8	$-a_5$	a_4	a_7
z_2	0	0	0	0	0	0	0	0	$-a_7$	a_4	a_5	a_8
z_3	0	0	0	0	$-f$	0	$-a_8$	a_7	0	$-a_3$	a_1+h	a_9
z_4	0	0	0	0	0	$-f$	a_5	$-a_4$	a_3	0	a_6	a_2+h
z_5	0	0	$-f$	0	0	0	$-a_4$	$-a_5$	$-a_1$	$-a_6$	0	0
z_6	0	0	0	$-f$	0	0	$-a_7$	$-a_8$	$-a_9$	$-a_2$	0	0
z_7	0	0	$-a_8$	a_7	$-a_4$	$-a_7$	0	0	0	0	0	0
z_8	0	0	a_7	$-a_4$	$-a_5$	$-a_8$	0	0	0	0	0	0
z_9	a_8	$-a_7$	0	a_3	$-a_1$	$-a_9$	0	0	0	0	e	0
z_{10}	$-a_5$	a_4	$-a_3$	0	$-a_6$	$-a_2$	0	0	0	0	0	e
z_{11}	a_4	a_5	a_1+h	a_6	0	0	0	0	e	0	0	0
z_{12}	a_7	a_8	a_9	a_2+h	0	0	0	0	0	e	0	0

4. $\mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar 0}=\langle e,f,h,a,a_i \mid i=1,\dots,6\rangle$, where

$$e=x^2\partial_x+x\nabla-\delta_1\partial_{\xi_1}-\delta_2\partial_{\xi_2}-\delta_3\partial_{\xi_3},\ f=-\partial_x,\ a=-x\partial_x-\xi_4\partial_{\xi_4},$$
$$a_1=\xi_1\partial_{\xi_2}-\xi_2\partial_{\xi_1},\ a_2=\xi_1\partial_{\xi_3}-\xi_3\partial_{\xi_1},\ a_3=\xi_2\partial_{\xi_3}-\xi_3\partial_{\xi_2},$$
$$a_4=-\xi_4\partial_{\xi_1},\ a_5=-\xi_4\partial_{\xi_2},\ a_6=-\xi_4\partial_{\xi_3};$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \dots, 10\rangle$, where

$$z_1 = -\partial_{\xi_1},\ z_2 = -\partial_{\xi_2},\ z_3 = -\partial_{\xi_3},\ z_4 = -\xi_4\partial_x,\ z_5 = -\partial_{\xi_4},$$
$$z_6 = -x\partial_{\xi_1} - \xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3},\ z_7 = -x\partial_{\xi_2} + \xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3},$$
$$z_8 = -x\partial_{\xi_3} - \xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2},\ z_9 = -\xi_4(x\partial_x + \nabla),$$
$$z_{10} = -x\partial_{\xi_4} + \xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}.$$

$$[h, v] = [e, v] = [f, v] = 0, \text{ where } v \in \{a, a_i \,|\, i = 1, \dots, 6\}.$$

[,]	a	a_1	a_2	a_3	a_4	a_5	a_6
a	0	0	0	0	$-a_4$	$-a_5$	$-a_6$
a_1	0	0	$-a_3$	a_2	$-a_5$	a_4	0
a_2	0	a_3	0	$-a_1$	$-a_6$	0	a_4
a_3	0	$-a_2$	a_1	0	0	$-a_6$	a_5
a_4	a_4	a_5	a_6	0	0	0	0
a_5	a_5	$-a_4$	0	a_6	0	0	0
a_6	a_6	0	$-a_4$	$-a_5$	0	0	0

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}
e	0	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$
f	$-z_6$	$-z_7$	$-z_8$	$-z_9$	$-z_{10}$	0	0	0	0	0
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	$-z_5$	z_6	z_7	z_8	z_9	z_{10}
a	0	0	0	0	z_5	$-z_6$	$-z_7$	$-z_8$	$-z_9$	0
a_1	$-z_2$	z_1	0	0	0	$-z_7$	z_6	0	0	0
a_2	$-z_3$	0	z_1	0	0	$-z_8$	0	z_6	0	0
a_3	0	$-z_3$	z_2	0	0	0	$-z_8$	z_7	0	0
a_4	0	0	0	0	z_1	0	0	0	0	z_6
a_5	0	0	0	0	z_2	0	0	0	0	z_7
a_6	0	0	0	0	z_3	0	0	0	0	z_8

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}
z_1	0	0	0	0	0	0	$-a_6$	a_5	a_4	$-a_3$
z_2	0	0	0	0	0	a_6	0	$-a_4$	a_5	a_2
z_3	0	0	0	0	0	$-a_5$	a_4	0	a_6	$-a_1$
z_4	0	0	0	0	$-f$	$-a_4$	$-a_5$	$-a_6$	0	$-a$
z_5	0	0	0	$-f$	0	a_3	$-a_2$	a_1	$a+h$	0
z_6	0	a_6	$-a_5$	$-a_4$	a_3	0	0	0	0	0
z_7	$-a_6$	0	a_4	$-a_5$	$-a_2$	0	0	0	0	0
z_8	a_5	$-a_4$	0	$-a_6$	a_1	0	0	0	0	0
z_9	a_4	a_5	a_6	0	$a+h$	0	0	0	0	e
z_{10}	$-a_3$	a_2	$-a_1$	$-a$	0	0	0	0	e	0

5. $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} = \langle e, f, h, a_i \mid i = 1, \dots, 6\rangle$, where

$$e = x^2\partial_x + x\nabla - \omega,\ f = -\partial_x, \quad a_1 = \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1},\ a_2 = \xi_1\partial_{\xi_3} - \xi_3\partial_{\xi_1},$$
$$a_3 = \xi_1\partial_{\xi_4} - \xi_4\partial_{\xi_1},\ a_4 = \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2},\ a_5 = \xi_2\partial_{\xi_4} - \xi_4\partial_{\xi_2},$$
$$a_6 = \xi_3\partial_{\xi_4} - \xi_4\partial_{\xi_3};$$

$\mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar{1}}=\langle z_i\mid i=1,\dots,8\rangle$, where

$$\begin{aligned}&z_1=-\partial_{\xi_1},\ z_2=-\partial_{\xi_2},\ z_3=-\partial_{\xi_3},\ z_4=-\partial_{\xi_4},\\&z_5=-x\partial_{\xi_1}-\xi_3\xi_4\partial_{\xi_2}+\xi_2\xi_4\partial_{\xi_3}-\xi_2\xi_3\partial_{\xi_4},\\&z_6=-x\partial_{\xi_2}+\xi_3\xi_4\partial_{\xi_1}-\xi_1\xi_4\partial_{\xi_3}+\xi_1\xi_3\partial_{\xi_4},\\&z_7=-x\partial_{\xi_3}-\xi_2\xi_4\partial_{\xi_1}+\xi_1\xi_4\partial_{\xi_2}-\xi_1\xi_2\partial_{\xi_4},\\&z_8=-x\partial_{\xi_4}+\xi_2\xi_3\partial_{\xi_1}--\xi_1\xi_3\partial_{\xi_2}+\xi_1\xi_2\partial_{\xi_3}.\end{aligned}$$

$$[h,a_i]=[e,a_i]=[f,a_i]=0,\ i=1,\dots,6.$$

Direct computations show that $\mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar{0}}\simeq\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2$.

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
e	$-z_5$	$-z_6$	$-z_7$	$-z_8$	0	0	0	0
f	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$
h	$-z_1$	$-z_2$	$-z_3$	$-z_4$	z_5	z_6	z_7	z_8
a_1	$-z_2$	z_1	0	0	$-z_6$	z_5	0	0
a_2	$-z_3$	0	z_1	0	$-z_7$	0	z_5	0
a_3	$-z_4$	0	0	z_1	$-z_8$	0	0	z_5
a_4	0	$-z_3$	z_2	0	0	$-z_7$	z_6	0
a_5	0	$-z_4$	0	z_2	0	$-z_8$	0	z_6
a_6	0	0	$-z_4$	z_3	0	0	$-z_8$	z_7

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	0	0	0	0	0	$-a_6$	a_5	$-a_3$
z_2	0	0	0	0	a_6	0	$-a_4$	a_2
z_3	0	0	0	0	$-a_5$	a_4	0	$-a_1$
z_4	0	0	0	0	a_3	$-a_2$	a_1	0
z_5	0	a_6	$-a_5$	a_3	0	0	0	0
z_6	$-a_6$	0	a_4	$-a_2$	0	0	0	0
z_7	a_5	$-a_4$	0	a_1	0	0	0	0
z_8	$-a_3$	a_2	$-a_1$	0	0	0	0	0

Let $(\sigma_1,\sigma_2,\sigma_3)\neq(0,0,0)$ and $\sigma_1+\sigma_2+\sigma_3=0$. In [BO1], the family of Lie superalgebras $\Gamma(\sigma_1,\sigma_2,\sigma_3)$, discovered by Kaplansky [Kapp*], [Kap*], is described. Observe that the Lie superalgebra $\mathfrak{osp}(4|2;\alpha)$, where $\alpha=\frac{\sigma_i}{\sigma_j}$ for $\sigma_j\neq0$, is simple except for $\alpha=0$ or -1.

Since $\Gamma(\sigma_1,\sigma_2,\sigma_3)\simeq\Gamma(\sigma'_1,\sigma'_2,\sigma'_3)$ if and only if $(\sigma'_1,\sigma'_2,\sigma'_3)=a(\sigma_1,\sigma_2,\sigma_3)$, where $a\in\mathbb{C}^\times$ and the triple σ differs from σ' by a permutation of its components, see [BGL*], it follows that $\mathfrak{v}(\mathcal{M})$ is isomorphic to $\Gamma(1,-1,0)=\mathfrak{osp}(4|2;-1)$.

10. We have $\mathfrak{v}(\mathbb{CP}^1,\mathcal{O})_{\bar{0}}=\langle e,f,h,a_i\mid i=1,\dots,6\rangle$, where

$$\begin{aligned}&e=x^2\partial_x+x\nabla-t\omega-\delta\partial_x,\ f=-\partial_x,\\&a_1=\xi_1\partial_{\xi_2}-\xi_2\partial_{\xi_1},\ a_2=\xi_1\partial_{\xi_3}-\xi_3\partial_{\xi_1},\ a_3=\xi_1\partial_{\xi_4}-\xi_4\partial_{\xi_1},\\&a_4=\xi_2\partial_{\xi_3}-\xi_3\partial_{\xi_2},\ a_5=\xi_2\partial_{\xi_4}-\xi_4\partial_{\xi_2},\ a_6=\xi_3\partial_{\xi_4}-\xi_4\partial_{\xi_3};\end{aligned}$$

$\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{1}} = \langle z_i \mid i = 1, \ldots, 8\rangle$, where

$$\begin{aligned}
&z_1 = -\partial_{\xi_1} - \tfrac{1}{2t}\xi_1\partial_x, \quad z_2 = -\partial_{\xi_2} - \tfrac{1}{2t}\xi_2\partial_x,\\
&z_3 = -\partial_{\xi_3} - \tfrac{1}{2t}\xi_3\partial_x, \quad z_4 = -\partial_{\xi_4} - \tfrac{1}{2t}\xi_4\partial_x,\\
&z_5 = -x\partial_{\xi_1} + t(-\xi_3\xi_4\partial_{\xi_2} + \xi_2\xi_4\partial_{\xi_3} - \xi_2\xi_3\partial_{\xi_4}) - \tfrac{1}{2t}\xi_1(x\partial_x + \nabla) + \delta_1\partial_x,\\
&z_6 = -x\partial_{\xi_2} + t(\xi_3\xi_4\partial_{\xi_1} - \xi_1\xi_4\partial_{\xi_3} + \xi_1\xi_3\partial_{\xi_4}) - \tfrac{1}{2t}\xi_2(x\partial_x + \nabla) + \delta_2\partial_x,\\
&z_7 = -x\partial_{\xi_3} + t(-\xi_2\xi_4\partial_{\xi_1} + \xi_1\xi_4\partial_{\xi_2} - \xi_1\xi_2\partial_{\xi_4}) - \tfrac{1}{2t}\xi_3(x\partial_x + \nabla) + \delta_3\partial_x,\\
&z_8 = -x\partial_{\xi_4} + t(\xi_2\xi_3\partial_{\xi_1} - \xi_1\xi_3\partial_{\xi_2} + \xi_1\xi_2\partial_{\xi_3}) - \tfrac{1}{2t}\xi_4(x\partial_x + \nabla) + \delta_4\partial_x.
\end{aligned}$$

$$[e, a_i] = [f, a_i] = [h, a_i] = 0, \ i = 1, \ldots, 6.$$

Direct computations show that $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O})_{\bar{0}} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \times \mathfrak{sl}_2$.

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
f	0	0	0	0	$-z_1$	$-z_2$	$-z_3$	$-z_4$
e	z_5	z_6	z_7	z_8	0	0	0	0
h	z_1	z_2	z_3	z_4	$-z_5$	$-z_6$	$-z_7$	$-z_8$
a_1	$-z_2$	z_1	0	0	$-z_6$	z_5	0	0
a_2	$-z_3$	0	z_1	0	$-z_7$	0	z_5	0
a_3	$-z_4$	0	0	z_1	$-z_8$	0	0	z_5
a_4	0	$-z_3$	z_2	0	0	$-z_7$	z_6	0
a_5	0	$-z_4$	0	z_2	0	$-z_8$	0	z_6
a_6	0	0	$-z_4$	z_3	0	0	$-z_8$	z_7

[,]	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	$-\frac{1}{t}f$	0	0	0	$\frac{1}{2t}h$	$-ta_6+\frac{1}{2t}a_1$	$ta_5+\frac{1}{2t}a_2$	$-ta_4+\frac{1}{2t}a_3$
z_2	0	$-\frac{1}{t}f$	0	0	$ta_6-\frac{1}{2t}a_1$	$\frac{1}{2t}h$	$-ta_3+\frac{1}{2t}a_4$	$ta_2+\frac{1}{2t}a_5$
z_3	0	0	$-\frac{1}{t}f$	0	$-ta_5-\frac{1}{2t}a_2$	$ta_3-\frac{1}{2t}a_4$	$\frac{1}{2t}h$	$-ta_1+\frac{1}{2t}a_6$
z_4	0	0	0	$-\frac{1}{t}f$	$ta_4-\frac{1}{2t}a_3$	$-ta_2-\frac{1}{2t}a_5$	$ta_1-\frac{1}{2t}a_6$	$\frac{1}{2t}h$
z_5	$\frac{1}{2t}h$	$-ta_6+\frac{1}{2t}a_1$	$ta_5+\frac{1}{2t}a_2$	$-ta_4+\frac{1}{2t}a_3$	$\frac{1}{t}e$	0	0	0
z_6	$ta_6-\frac{1}{2t}a_1$	$\frac{1}{2t}h$	$-ta_3+\frac{1}{2t}a_4$	$ta_2+\frac{1}{2t}a_5$	0	$\frac{1}{t}e$	0	0
z_7	$-ta_5-\frac{1}{2t}a_2$	$ta_3-\frac{1}{2t}a_4$	$\frac{1}{2t}h$	$-ta_1+\frac{1}{2t}a_6$	0	0	$\frac{1}{t}e$	0
z_8	$ta_4-\frac{1}{2t}a_3$	$-ta_2-\frac{1}{2t}a_5$	$ta_1-\frac{1}{2t}a_6$	$\frac{1}{2t}h$	0	0	0	$\frac{1}{t}e$

Comparing the above table with the tables in [BO1] we deduce that this Lie superalgebra is

$$\Gamma(\frac{1}{2t}, -\frac{1}{2}(\frac{1}{2t}+t), -\frac{1}{2}(\frac{1}{2t}-t)) \simeq \Gamma(2, -(2t^2+1), 2t^2-1) \simeq \mathfrak{osp}(4|2; \frac{2t^2+1}{1-2t^2}).$$

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