Homogeneous non-split superstrings of odd dimension 4

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Abstract. Let $L_k$ be the holomorphic line bundle of degree $k \in \mathbb{Z}$ on the projective line. The tuples $(k_1k_2k_3k_4)$ for which there exists no homogeneous non-split supermanifolds $\mathbb{C}P_{k_1k_2k_3k_4}$ associated with the vector bundle $L_{-k_1} \oplus L_{-k_2} \oplus L_{-k_3} \oplus L_{-k_4}$ are classified.

For many types of the remaining tuples, there are listed cocycles that determine homogeneous non-split supermanifolds.


1 Introduction

In this paper, I summarize the results of classification (up to a diffeomorphism) of homogeneous complex (more precisely, almost complex, see [BGLS*], since the vanishing of the Nijenhuis tensor is never required) supermanifolds $\mathcal{M} := (M, \mathcal{O})$, where $M = \mathbb{C}P^1$ and $\dim \mathcal{M} = 1|n$. (Comments with starred references are added by the editor of this Special Volume. D.L.)

For the case where $\mathcal{M}$ is split, the classification is known, see [BuO1]: the non-diffeomorphic supermanifolds are in one-to-one correspondence with $n$-tuples of non-negative integers.

If $\mathcal{M}$ is non-split, the classification is considerably more complicated and reduces to computation of cohomology of split homogeneous supermanifolds with coefficients in the tangent sheaf.

For $n = 2$ and 3, V. A. Bunegina and A. L. Onishchik completely investigated the case, see [BuO1], [BuO2].

For $n = 4$, see below (summary of the results of [B1] – [B4], [BaO1], [BaO2]). For the method of the proof, see [BuO1], [BuO2].
2 Results

As is known, any holomorphic bundle \( E \) over \( \mathbb{CP}^1 \) can be uniquely decomposed into a direct sum of line bundles: Grothendieck’s theorem, see [HM*]. (For interesting applications of Linear Superalgebra (with elements of category theory) to the description of vector bundles over projective spaces, see the review [BG*]. For the latest results on non-splitness of supermanifolds whose retract is the Grassmann manifold, see [Vi*], [Vi1*], [Vi2*].) Let \( L_k \) be the holomorphic line bundle of degree \( k \in \mathbb{Z} \).

Consider a holomorphic bundle

\[
E = L_{-k_1} \oplus L_{-k_2} \oplus L_{-k_3} \oplus L_{-k_4},
\]

where \( k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0 \).

If \( \mathcal{M} \) is homogeneous, then the \( k_i \) must be non-negative, see [BuO1].

Let \( \mathcal{CP}_{k_1,k_2,k_3,k_4} \) designate the split supermanifold determined by \( E \).

Let us cover \( \mathbb{CP}^1 \) by two affine charts \( U_0 \) and \( U_1 \) with local coordinates \( x \) and \( y = \frac{1}{x} \), respectively. Then, the transition functions on \( \mathcal{CP}_{k_1,k_2,k_3,k_4} \) in \( U_0 \cap U_1 \) are of the form

\[
y = x^{-1},
\]

\[
\eta_i = x^{-k_i} \xi_i \quad \text{for} \quad i = 1, \ldots, 4,
\]

where \( \xi_i \) and \( \eta_i \) are basis sections of \( E \) over \( U_0 \) and \( U_1 \), respectively.

Let \( M \) be a compact complex manifold. We will sometimes need general statements about the \( m|n \)-dimensional supermanifold \( \mathcal{M} = (M, \mathcal{O}) \). Let \( \mathcal{I} \subset \mathcal{O} \) be the subsheaf of ideals generated by the subsheaf \( \mathcal{O}_1 \). Consider the filtration of \( \mathcal{O} \) by powers of \( \mathcal{I} \):

\[
\mathcal{O} = \mathcal{I}^0 \supset \mathcal{I} \supset \mathcal{I}^2 \supset \cdots \supset \mathcal{I}^n \supset \mathcal{I}^{n+1} = 0.
\]

The graded sheaf \( \text{gr} \mathcal{O} = \oplus_{0 \leq i \leq n} \text{gr} \mathcal{O} \) with \( \text{gr} \mathcal{O} := \mathcal{I}^i/\mathcal{I}^{i+1} \) defines the split supermanifold \( (M, \text{gr} \mathcal{O}) \) called the retract of \( (M, \mathcal{O}) \). Let \( \mathcal{T}_{\text{gr}} := \oplus_{-1 \leq p \leq 4} (\mathcal{T}_{\text{gr}})_p \) denote the graded tangent sheaf of the split supermanifold \( (M, \mathcal{O}_{\text{gr}}) \). Consider the subsheaf

\[
\text{Aut}_{(2)} \mathcal{O}_{\text{gr}} = \exp((\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4)
\]

of the sheaf \( \text{Aut} \mathcal{O}_{\text{gr}} \). Thanks to a theorem due to Green ([G]), the set of supermanifolds with a given retract \( (M, \mathcal{O}_{\text{gr}}) \) is isomorphic to the set of orbits of the group \( \text{Aut} \mathcal{E} \) in \( H^1(M, \text{Aut}_{(2)} \mathcal{O}_{\text{gr}}) \).

In what follows we often assume that \( H^0(M, (\mathcal{T}_{\text{gr}})_2) = 0 \). This is needed for existence of a bijection (see [O3, Section 3.2 pp.23–37])

\[
H^1(M, \text{Aut}_{(2)} \mathcal{O}_{\text{gr}}) \leftrightarrow H^1(M, (\mathcal{T}_{\text{gr}})_2) \oplus H^1(M, (\mathcal{T}_{\text{gr}})_4).
\]

2.1 Statement ([BaO2]). For \( n \leq 5 \), let \( H^0(M, (\mathcal{T}_{\text{gr}})_2) = 0 \) and let there be given subspaces \( Q_{2p} \subset Z^1((\mathcal{O}, (\mathcal{T}_{\text{gr}})_{2p})) \), where \( p = 1, 2 \), such that every cohomology class in \( H^1(M, (\mathcal{T}_{\text{gr}})_{2p}) \) contains precisely one cocycle in \( Q_{2p} \).

Then, every cohomology class in \( H^1(M, \text{Aut}_{(2)} \mathcal{O}_{\text{gr}}) \) can be represented by a unique cocycle of the form \( z = \exp(u^2 + u^4) \), where \( u^2 \in Q_2 \) and \( u^4 \in Q_4 \).
Hereafter, the expression \( (M, \mathcal{O}) \) is determined by the cocycle \( u^2 + u^4 \) means that 
\( (M, \mathcal{O}) \) actually corresponds to the cocycle \( z = \exp(u^2 + u^4) \).

In [BuO1, Prop. 12], there is given a description of the algebra \( \text{End } E \). Let us reformulate this Proposition in terms more convenient for us here. Any endomorphism \( a \in \text{End } E \) can be considered as an endomorphism of the sheaf \( \mathcal{E} \) of \( \mathcal{F} \)-modules. In \( U_0 \), we have

\[
a(\xi_i) = \sum_{1 \leq j \leq n} a_{ji} \xi_j \quad \text{for } i = 1, \ldots, n,
\]

where \( A = (a_{ji}) \) is a matrix with elements \( a_{ji} \in \mathcal{F}(U_0) \). The matrix \( A \) completely determines the endomorphism \( a \), and \( a \in \text{Aut } E \) if and only if \( A \) is invertible.

The following statement is needed in the classification of homogeneous non-split supermanifolds up to a diffeomorphism.

**2.2 Statement ([BaO2])**. The matrix \( A = (a_{ji}) \) over \( \mathcal{F}(U_0) \) corresponds to an element in \( \text{End } E \) if and only if

\[
a_{ij} = \begin{cases} 
0 & \text{if } k_i < k_j, \\
\text{is a polynomial of degree } \leq k_i - k_j & \text{if } k_i \geq k_j.
\end{cases}
\]

First, consider the tuples \( (k; k, 2, 0) \) for \( k \geq 2 \). Let \( \mathfrak{v}(\mathbb{C}P^1, \mathcal{O}_{gr}) \) be the Lie superalgebra of vector fields on \( \mathbb{C}P^1 \). Theorem 14 in [BaO1] implies the following

**2.3. Lemma.** For a basis of \( H^1(\mathbb{C}P^1, (\mathcal{T}_{gr})_q) \), where \( q = 1, 2, 4 \), we can take (the classes of) the following cocycles defining \( \mathbb{C}P^1 \).

1) \( q = 1 \) for \( k = 2 \)

\[
x^{-1}\xi_1 \xi_2 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_2 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1},
\]

for \( k = 3 \)

\[
x^{-1}\xi_1 \xi_2 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1},
\]

for \( k \geq 4 \)

\[
x^{-1}\xi_1 \xi_2 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_3 \partial_{\xi_1}, \quad x^{-1}\xi_1 \xi_4 \partial_{\xi_1},
\]
2) $q = 2$

for $k = 2$

\[ x^{-1} \xi_1 \xi_2 \partial_x, \ x^{-r} \xi_1 \xi_2 \xi_j \partial_{\xi_j} \ (j = 3, 4, \ r = 1, 2, 3), \ x^{-1} \xi_1 \xi_4 \xi_2 \partial_{\xi_2}, \ x^{-1} \xi_2 \xi_4 \xi_1 \partial_{\xi_1}, \]

\[ x^{-1} \xi_1 \xi_3 \partial_x, \ x^{-r} \xi_1 \xi_3 \xi_j \partial_{\xi_j} \ (j = 2, 4, \ r = 1, 2, 3), \ x^{-1} \xi_3 \xi_4 \xi_1 \partial_{\xi_1}, \ x^{-1} \xi_2 \xi_3 \xi_4 \partial_{\xi_3}, \]

\[ x^{-1} \xi_2 \xi_3 \partial_x, \ x^{-r} \xi_2 \xi_3 \xi_j \partial_{\xi_j} \ (j = 1, 4, \ r = 1, 2, 3), \ x^{-1} \xi_3 \xi_4 \xi_2 \partial_{\xi_2}, \ x^{-1} \xi_1 \xi_2 \xi_4 \partial_{\xi_3}, \]

\[ x^{-r} \xi_1 \xi_2 \xi_3 \partial_{\xi_4} \ (r = 1, 2, 3, 4, 5); \]

for $k = 3$

\[ x^{-r} \xi_1 \xi_2 \partial_x \ (r = 1, 2, 3), \ x^{-r} \xi_1 \xi_2 \xi_j \partial_{\xi_j} \ (j = 3, 4, \ r = 1, 2, 3, 4, 5), \ x^{-1} \xi_3 \xi_4 \xi_1 \partial_{\xi_1}, \]

\[ x^{-r} \xi_1 \xi_3 \partial_x \ (r = 1, 2), \ x^{-r} \xi_1 \xi_3 \xi_j \partial_{\xi_j} \ (j = 2, 4, \ r = 1, 2, 3, 4), \ x^{-1} \xi_2 \xi_3 \xi_4 \partial_{\xi_1}, \]

\[ x^{-r} \xi_2 \xi_3 \partial_x \ (r = 1, 2), \ x^{-r} \xi_2 \xi_3 \xi_j \partial_{\xi_j} \ (j = 1, 4, \ r = 1, 2, 3, 4), \ x^{-1} \xi_1 \xi_4 \xi_2 \partial_{\xi_2}, \]

\[ x^{-r} \xi_1 \xi_2 \xi_3 \partial_{\xi_4} \ (r = 1, \ldots, 7), \ x^{-r} \xi_1 \xi_4 \xi_j \partial_{\xi_j} \ (j = 2, 3, \ r = 1, 2), \]

\[ x^{-r} \xi_2 \xi_4 \partial_{\xi_3} \ (r = 1, 2, 3), \ x^{-r} \xi_2 \xi_4 \xi_j \partial_{\xi_j} \ (j = 1, 3, \ r = 1, 2); \]

for $k \geq 4$

\[ x^{-1} \xi_3 \xi_4 \xi_1 \partial_{\xi_1}, \ x^{-1} \xi_2 \xi_3 \xi_4 \partial_{\xi_1}, \ x^{-1} \xi_1 \xi_3 \xi_4 \partial_{\xi_2}, \]

\[ x^{-r} \xi_1 \xi_2 \xi_3 \partial_{\xi_4} \ (r = 1, \ldots, 2k - 3), \ x^{-r} \xi_1 \xi_2 \xi_j \partial_{\xi_j} \ (j = 3, 4, \ r = 1, \ldots, 2k - 1), \]

\[ x^{-r} \xi_1 \xi_3 \partial_{\xi_2} \ (r = 1, \ldots, k - 1), \ x^{-r} \xi_1 \xi_3 \xi_j \partial_{\xi_j} \ (j = 2, 4, \ r = 1, \ldots, k + 1), \]

\[ x^{-r} \xi_1 \xi_2 \partial_{\xi_3} \ (r = 1, \ldots, k - 1), \ x^{-r} \xi_1 \xi_2 \xi_j \partial_{\xi_j} \ (j = 1, 4, \ r = 1, \ldots, k + 1), \]

\[ x^{-r} \xi_1 \partial_{\xi_1} \partial_{\xi_2}, \ x^{-r} \xi_1 \partial_{\xi_1} \partial_{\xi_3} \ (r = 1, \ldots, 2k + 1), \ x^{-r} \xi_1 \partial_{\xi_2} \partial_{\xi_3} \ (r = 1, \ldots, 2k + 3), \]

\[ x^{-r} \xi_1 \partial_{\xi_4} \partial_{\xi_3} \ (r = 1, \ldots, k - 3), \ x^{-r} \xi_1 \partial_{\xi_4} \partial_{\xi_j} \ (j = 2, 3, \ r = 1, \ldots, k - 1), \]

\[ x^{-r} \xi_2 \partial_{\xi_1} \partial_{\xi_3} \ (r = 1, \ldots, k - 3), \ x^{-r} \xi_2 \partial_{\xi_1} \partial_{\xi_j} \ (j = 1, 3, \ r = 1, \ldots, k - 1); \]

4) $q = 4$

\[ x^{-r} \xi_1 \xi_2 \xi_3 \xi_4 \partial_{\xi_4} \ (r = 1, \ldots, 2k - 1). \]

Consider the exact sequence (see [BuO1])

\[ 0 \to \operatorname{End} E \to \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{gr})_0 \xrightarrow{\beta} \mathfrak{sl}_2(\mathbb{C}) \to 0. \tag{1} \]

The subalgebra $\mathfrak{a} \subset \mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{gr})_0$ splits the sequence (1) if $\beta$ is an isomorphism with $\mathfrak{sl}_2(\mathbb{C})$ or, equivalently, $\mathfrak{v}(\mathbb{CP}^1, \mathcal{O}_{gr})_0 = \operatorname{End} E \oplus \mathfrak{a}$ is a direct sum of Lie algebras. In [BuO1], it was shown that the supermanifold with retract $(\mathbb{CP}^1, \mathcal{O}_{gr})$ is even-homogeneous if and only if a subalgebra $\mathfrak{a}$ splitting (1) can be lifted to it.

If this is the case, we will say that the $(\mathbb{CP}^1, \mathcal{O})$ is even-homogeneous relative $\mathfrak{a}$. Then, there exist only the following (up to an automorphism in $\operatorname{Aut} E$) splitting subalgebras $\mathfrak{a}_i \simeq \mathfrak{sl}_2(\mathbb{C})$ (see [BuO1]) for the supermanifold $\mathbb{CP}^{114}_{kk20}$ (spanned by $e$, $f$, and $h = [e, f]$) for $k = 2$

\[
\begin{align*}
\mathfrak{a}_1 : & \quad e = \partial_x, \quad f = -x^2 \partial_x - 2x \left( \xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2} + \xi_3 \partial_{\xi_3} \right); \\
\mathfrak{a}_2 : & \quad e = \xi_2 \partial_{\xi_1} + \partial_x, \quad f = \xi_1 \partial_{\xi_2} - x^2 \partial_x - 2x \left( \xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2} + \xi_3 \partial_{\xi_3} \right); \\
\mathfrak{a}_3 : & \quad e = \xi_3 \partial_{\xi_2} + \xi_2 \partial_{\xi_1} + \partial_x, \quad f = 2\xi_2 \partial_{\xi_3} + 2\xi_1 \partial_{\xi_2} - x^2 \partial_x - 2x \left( \xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2} + \xi_3 \partial_{\xi_3} \right); \\
\end{align*}
\]
for \( k \geq 3 \) and \( \nabla = k \xi_1 \partial_{\xi_1} + k \xi_2 \partial_{\xi_2} + 2 \xi_3 \partial_{\xi_3} \)

\[ a_1 : e = \partial_x, \ f = -x^2 \partial_x - x \nabla; \]

\[ a_2 : e = \xi_2 \partial_{\xi_1} + \partial_x, \ f = \xi_3 \partial_{\xi_2} - x^2 \partial_x - x \nabla. \]

Designate by \( H^1(\mathbb{C}P^1, (T_{gr}))^a \) the set of \( a \)-invariants.

**2.4 Lemma ([B1]).** Let \( n = 4, \ k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0 \) and \( H^0(\mathbb{C}P^1, (T_{gr})_2) = \{0\} \). Then,

1) \( H^1(\mathbb{C}P^1, (T_{gr})_2)^a \neq \{0\} \) if and only if \((k_1, k_2, k_3, k_4)\) is one of the following

\[
(1, 1, 1, 0), \ (2, 2, 1, 0), \ (3, 3, 1, 0), \ (4, 4, 1, 0), \ (4, 4, 3, 0), \\
(6, 4, 3, 0), \ (2, 2, 2, 1), \ (3, 3, 2, 1), \ (5, 3, 2, 1), \ (k + 1, k, 1, 1)_{k \geq 1}, \\
(k + 1, k, 2, 0)_{k \geq 2}, \ (k, k, 2, 0)_{k \geq 2}, \ (k + 3, k, 2, 2)_{k \geq 2}, \ (k + 1, k, 2, 2)_{k \geq 2}, \\
(k, k, 2, 2)_{k \geq 2}, \ (k + 3, k, 3, 1)_{k \geq 3}, \ (k + 1, k, 3, 1)_{k \geq 3}, \ (k, k, 3, 1)_{k \geq 3}, \\
(k + 3, k, 4, 0)_{k \geq 4}, \ (k + 2, k, 4, 0)_{k \geq 4}, \ (k + 1, k, 4, 0)_{k \geq 4}, \ (k, k, 4, 0)_{k \geq 4}, \\
(k_2 + k_3 + k_4 - 2, k_2, k_3, k_4)_{k_i \geq 1}.
\]

2) \( H^1(\mathbb{C}P^1, (T_{gr})_4)^a \neq \{0\} \) if and only if \((k_1, k_2, k_3, k_4) = (1, 1, 1, 1)\).

**2.5. Lemma.** For a basis of \( H^1(\mathbb{C}P^1, (T_{gr})_2)^a \) we can take (the classes of) the following cocycles for the supermanifold \( \mathbb{C}P_{kk20}^{114} \)

for \( k = 2 \)

1) \( i = 1 + \)

\[
\begin{align*}
&x^{-1} \xi_1 \xi_2 \partial_x + x^{-2} (\xi_1 \xi_2 \xi_3 \partial_{\xi_3} + \xi_1 \xi_2 \xi_4 \partial_{\xi_4}), \ x^{-1} \xi_1 \xi_4 \xi_2 \partial_{\xi_2}, \ x^{-1} \xi_2 \xi_4 \xi_1 \partial_{\xi_1}, \\
&x^{-1} \xi_1 \xi_3 \partial_x + x^{-2} (\xi_1 \xi_3 \xi_2 \partial_{\xi_2} + \xi_1 \xi_3 \xi_4 \partial_{\xi_4}), \ x^{-1} \xi_3 \xi_4 \xi_1 \partial_{\xi_1}, \ x^{-1} \xi_1 \xi_2 \xi_3 \partial_{\xi_3}, \\
&x^{-1} \xi_2 \xi_3 \partial_x + x^{-2} (\xi_2 \xi_3 \xi_1 \partial_{\xi_1} + \xi_2 \xi_3 \xi_4 \partial_{\xi_4}), \ x^{-1} \xi_1 \xi_3 \xi_4 \partial_{\xi_2}, \ x^{-1} \xi_2 \xi_3 \xi_4 \partial_{\xi_3};
\end{align*}
\]

2) \( i = 2 : \)

3) \( i = 3 : \)

for \( k = 3 \) and \( k \geq 5 \)

1) \( i = 1 : \)

2) \( i = 2 : \)

for \( k = 4 \)

1) \( i = 1 : \)

\[
\begin{align*}
&x^{-1} \xi_3 \xi_4 \xi_1 \partial_{\xi_1}, \ x^{-1} \xi_1 \xi_3 \xi_4 \partial_{\xi_2}, \ x^{-1} \xi_2 \xi_3 \xi_4 \partial_{\xi_3}, \\
&x^{-1} \xi_3 \xi_4 \partial_x + 2 x^{-2} \xi_1 \xi_4 \xi_2 \partial_{\xi_2} + x^{-2} \xi_1 \xi_4 \xi_3 \partial_{\xi_3}, \\
&x^{-1} \xi_2 \xi_3 \partial_x + 2 x^{-2} \xi_2 \xi_4 \xi_1 \partial_{\xi_1} + x^{-2} \xi_2 \xi_4 \partial_{\xi_3};
\end{align*}
\]

2) \( i = 2 : \)

\[
\begin{align*}
&x^{-1} \xi_1 \xi_4 \partial_{\xi_2}, \ x^{-1} \xi_2 \xi_4 \partial_{\xi_3};
\end{align*}
\]

For proof follows from Theorem 15 in [BaO2]. \( \square \)
The proof of the following Lemma is similar.

**2.6. Lemma.** $H^1(\mathbb{C}P^1, (T_{gr})_4)^a = \{0\}$ for any splitting subalgebra $a$ and supermanifold $\mathcal{CP}^{114}_{kk20}$.

Let $\lambda_2 : \text{Aut}(2)O_{gr} \rightarrow (T_{gr})_2$ be a sheaf homomorphism which to any germ of the automorphism $a$ assigns the 2-component of $\log a$ in $(T_{gr})_2 \oplus (T_{gr})_4$.

Denote by $H^1(\mathbb{C}P^1, \text{Aut}(2)O_{gr})^a$ the set of classes that determine supermanifolds even-homogeneous relative $a$.

**2.7. Proposition.** If $a$ splits the sequence (1), then $\lambda_2^*$ maps $H^1(\mathbb{C}P^1, \text{Aut}(2)O_{gr})^a$ bijectively to $H^1(\mathbb{C}P^1, (T_{gr})_2)^a$.

*Proof.* follows from Statement 2.1 and Lemma 2.6. □

Using the lifting condition of the vector field on non-split supermanifold described in [O2], and Lemma 2.6, we deduce that any supermanifold 0-homogeneous relative $a$ is determined by cocycles $u^2$ such that the class $[u^2]$ is $a$-invariant and $[u^2, u^2] = 0$. Since $[u^2, u^2] = 0$ is true for all cocycles in Lemma 2.5, the following Theorem holds.

**2.8. Theorem.** For any splitting subalgebra $a$, the even-homogeneous relative $a$ supermanifolds are determined by the cocycles listed in Lemma 2.5.

Thus, all even-homogeneous relative $a$ non-split supermanifolds with retract $\mathcal{CP}^{114}_{kk20}$ are described for any $k \geq 2$.

Let us find out if any of these even-homogeneous relative $a$ non-split supermanifolds are homogeneous. For this we use the following Proposition analogous to Proposition 15 in [BuO1]:

**2.9. Proposition.** Under conditions of Proposition 2.7, let $(\mathbb{C}P^1, O)$ be even-homogeneous relative $a_i$ for one of $i = 1, 2, 3$.

If $i = 1$, then $(\mathbb{C}P^1, O)$ is homogeneous if and only if the vector fields $\partial_{\xi_j}$, where $j = 1, \ldots, 4$ can be lifted to $(\mathbb{C}P^1, O)$;

If $i = 2$, then $(\mathbb{C}P^1, O)$ is homogeneous if and only if the vector fields $\partial_{\xi_j}$, where $j = 1, 3, 4$, can be lifted to $(\mathbb{C}P^1, O)$.

If $i = 3$, then $(\mathbb{C}P^1, O)$ is homogeneous if and only if the vector fields $\partial_{\xi_1}$ and $\partial_{\xi_4}$ can be lifted to $(\mathbb{C}P^1, O)$.

Proposition 2.9 applied to the cocycles of Lemma 2.5 implies the classification (Theorems 2.10–2.20). For other 4-tuples, homogeneous non-split supermanifolds do not exist.

**2.10. Theorem.** There does not exists homogeneous non-split supermanifolds with retract $\mathcal{CP}^{114}_{kk20}$ for any $k \geq 2$. 
2.11. Theorem. For the following tuples \((k_1k_2k_3k_4)\), there does not exist homogeneous non-split supermanifolds \(\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}\):

\[
(1, 1, 1, 0), (3, 3, 1, 0), (4, 4, 1, 0), (4, 4, 3, 0), (5, 4, 3, 0), (k + 2, k, 4, 0)_{k \geq 4},
\]

\[
(3, 3, 2, 1), (5, 3, 2, 1), (k + 3, k, 4, 0)_{k \geq 4}, (k + 1, k, 3, 1)_{k \geq 3}, (k + 1, k, 4, 0)_{k \geq 4},
\]

\[
(k + 1, k, 3, 1)_{k \geq 3}, (k + 1, k, 2, 0)_{k > 2}, (k, k, 4, 0)_{k \geq 4}, (k, k, 3, 1)_{k \geq 3}.
\]

2.12. Theorem. For every of the following tuples \((k_1k_2k_3k_4)\), there exists one homogeneous non-split supermanifold with retract \(\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}\) which can be represented, up to an isomorphism, by \((\text{the classes of})\) the following cocycles.

\[
(2, 2, 1, 0) \quad x^{-1} \xi_1 \xi_4 \xi_3 \partial_{\xi_1};
\]

\[
(2, 2, 2, 1) \quad x^{-1} \xi_1 \xi_2 \partial_x;
\]

\[
(3, 2, 2, 0) \quad 2x^{-1} \xi_1 \xi_3 \partial_x + 3x^{-2} \xi_2 \xi_3 \xi_1 \partial_{\xi_1};
\]

\[
(4, 3, 2, 1) \quad x^{-1} \xi_1 \xi_4 \xi_3 \partial_{\xi_1};
\]

\[
(k + 1, k, 1, 1)_{k > 1} \quad x^{-1} \xi_1 \xi_4 \xi_4 \partial_{\xi_1};
\]

\[
(k + 3, k, 2, 2)_{k \geq 2} \quad 2x^{-1} \xi_3 \xi_4 \partial_x + (k + 3)x^{-2} \xi_3 \xi_4 \xi_1 \partial_{\xi_1} + kx^{-2} \xi_3 \xi_4 \xi_2 \partial_{\xi_2};
\]

\[
(k + 1, k, 2, 2)_{k \geq 2} \quad 2x^{-1} \xi_3 \xi_4 \partial_x + (k + 1)x^{-2} \xi_3 \xi_4 \xi_1 \partial_{\xi_1} + kx^{-2} \xi_3 \xi_4 \xi_2 \partial_{\xi_2};
\]

\[
(k, k, 2, 2)_{k \geq 4} \quad x^{-1} \xi_3 \xi_4 \partial_x.
\]

2.13. Theorem. If \(k_4 \neq 0\), then for every \((k_2 + k_3 + k_4 - 2, k_2, k_3, k_4)_{k \geq 1}\) which is different from \((k, k, 1, 1)_{k \geq 1}\), there exists one homogeneous non-split supermanifold which can be represented, up to an isomorphism, by \((\text{the class of})\) cocycle \(x^{-1} \xi_2 \xi_3 \xi_4 \partial_{\xi_1}\).

2.14. Theorem. There exist two homogeneous non-split supermanifolds with their retracts being \(\mathcal{CP}^{1|4}_{2222}, \text{and which can be represented, up to an isomorphism, by} \) \((\text{the classes of})\) the following cocycles.

\[
x^{-1} \xi_1 \xi_2 \partial_x + x^{-2} \xi_1 \xi_2 \xi_3 \partial_{\xi_3} + x^{-2} \xi_1 \xi_2 \xi_4 \partial_{\xi_4},
\]

\[
x^{-1} \xi_1 \xi_2 \partial_x + x^{-2} \xi_1 \xi_2 \xi_3 \partial_{\xi_3} + x^{-2} \xi_1 \xi_2 \xi_4 \partial_{\xi_4} + x^{-1} \xi_3 \xi_4 \partial_x +
\]

\[
+ x^{-2} \xi_3 \xi_4 \xi_1 \partial_{\xi_1} + x^{-2} \xi_3 \xi_4 \xi_2 \partial_{\xi_2}.
\]

2.15. Theorem. There exist two homogeneous non-split supermanifolds with their retracts being \(\mathcal{CP}^{1|4}_{3222}, \text{and which can be represented, up to an isomorphism, by} \) \((\text{the classes of})\) the cocycles

\[
x^{-1} \xi_3 \xi_4 \partial_x + x^{-2} \xi_2 \xi_3 \xi_1 \partial_{\xi_1} + x^{-2} \xi_2 \xi_3 \xi_4 \partial_{\xi_4},
\]

\[
2x^{-3} \xi_2 \xi_3 \xi_1 \partial_{\xi_1} + x^{-2} \xi_2 \xi_4 \xi_1 \partial_{\xi_1} + x^{-1} \xi_3 \xi_4 \xi_1 \partial_{\xi_1}.
\]

2.16. Theorem. There exist three homogeneous non-split supermanifolds with their retracts being \(\mathcal{CP}^{1|4}_{3322}, \text{and which can be represented, up to an isomorphism, by} \) \((\text{the classes of})\) the following cocycles.

\[
x^{-1} \xi_3 \xi_4 \partial_x, \quad x^{-2} \xi_1 \xi_3 \partial_x + x^{-1} \xi_2 \xi_3 \partial_x,
\]

\[
x^{-1} \xi_4 \xi_2 \partial_x + x^{-2} \xi_1 \xi_3 \partial_x + x^{-1} \xi_2 \xi_3 \partial_x.
\]
2.17. Theorem. There exist five homogeneous non-split supermanifolds with their retracts being $\mathbb{CP}_{k\mid k+11}^{1\mid 4}$, where $k > 2$, which can be represented, up to an isomorphism, by (the classes of) the following cocycles.

$$
\begin{aligned}
x^{-1}\xi_1\xi_2\xi_3\xi_4\partial_{\xi_1}, & \quad x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}, & \quad x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}, & \quad x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}, \\
x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2} & + x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_2}, & \quad x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}, & \quad x^{-1}\xi_3\xi_4\partial_{\xi_2}, & \quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_2}, \\
x^{-1}\xi_3\xi_4\partial_{\xi_2} & + x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1} & + x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2} & + x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_2}.
\end{aligned}
$$

2.18. Theorem. There exist eight homogeneous non-split supermanifolds with their retracts being $\mathbb{CP}_{2\mid 111}^{1\mid 4}$, which can be represented, up to an isomorphism, by (the classes of) the following cocycles.

$$
\begin{aligned}
x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1}, & \quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}, & \quad x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}, \\
x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}, & \quad x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_3\xi_4\partial_{\xi_2}, \\
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1} & + x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}, & \quad x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1} & + x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2}, \\
x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_2\xi_3\xi_3\partial_{\xi_1} & + x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2}.
\end{aligned}
$$

2.19. Theorem. There exist nine homogeneous non-split supermanifolds with their retracts being $\mathbb{CP}_{2\mid 211}^{1\mid 4}$, which can be represented, up to an isomorphism, by (the classes of) the following cocycles.

$$
\begin{aligned}
x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}, & \quad x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1}, \\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1} & - x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1}, & \quad x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}, \\
x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1} & + x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}, \\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1} & - x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_2}, \\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1} & - x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_2} & + x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2}, \\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1} & - x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1} & + x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_2} & + x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}.
\end{aligned}
$$

2.20 Theorem ([B5]). There exist four homogeneous non-split superstrings and one 1-parameter family of homogeneous non-split superstrings with retract $\mathbb{CP}_{1\mid 1111}^{4\mid 4}$, which can be represented, up to an isomorphism, by (the classes of) the following cocycles.

$$
\begin{aligned}
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}, & \quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1} & - x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}, \\
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1} & - x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2} & + x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3}, \\
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1} & - x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2} & + x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3}, \\
t(x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1} - x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2} & + x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3} - x^{-1}\xi_1\xi_2\xi_3\partial_{\xi_4} & + x^{-1}\xi_1\xi_2\xi_3\partial_{\xi_2} & + x^{-1}\xi_1\xi_2\xi_3\partial_{\xi_2} & + x^{-1}\xi_1\xi_2\xi_3\partial_{\xi_2}.
\end{aligned}
$$

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References


¹Certain attributions in this paper are wrong. For example, the Lie superalgebra $Γ(σ_1, σ_2, σ_3)$ — the deformation of $osp(4|2)$ — was discovered by Kaplansky [Kap*], [Kapp*], see [KIE*]; later Kac redenoted $Γ(σ_1, σ_2, σ_3)$ by $D(2, 1; κ); for reasons given in [CCLL*], this is an ill-chosen notation. For a reasonable and meaningful notation which gradually started to replace the initial ad hoc names of the exceptional Lie superalgebras given by Kaplansky and later ones, due to Kac, see [Sr*], [CCLL*].


