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Homogeneous non-split superstrings of odd dimension 4

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Abstract. Let $\mathbf{L}_k$ be the holomorphic line bundle of degree $k \in \mathbb{Z}$ on the projective line. The tuples $(k_1k_2k_3k_4)$ for which there exists no homogeneous non-split supermanifolds $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}$ associated with the vector bundle $\mathbf{L}_{-k_1} \oplus \mathbf{L}_{-k_2} \oplus \mathbf{L}_{-k_3} \oplus \mathbf{L}_{-k_4}$ are classified.

For many types of the remaining tuples, there are listed cocycles that determine homogeneous non-split supermanifolds.

Proofs follow the lines indicated in the paper Bunegina V.A., Onishchik A.L., Homogeneous supermanifolds associated with the complex projective line. J. Math. Sci. V. 82 (1996) 3503–3527.

1 Introduction

In this paper, I summarize the results of classification (up to a diffeomorphism) of homogeneous complex (more precisely, almost complex, see [BGLS*], since the vanishing of the Nijenhuis tensor is never required) supermanifolds $\mathcal{M} := (M, \mathcal{O})$, where $M = \mathbb{CP}^1$ and $\dim \mathcal{M} = 1|n$. (*Comments with starred references are added by the editor of this Special Volume. D.L.*)

For the case where $\mathcal{M}$ is split, the classification is known, see [BuO1]: the non-diffeomorphic supermanifolds are in one-to-one correspondence with n-tuples of non-negative integers.

If $\mathcal{M}$ is non-split, the classification is considerably more complicated and reduces to computation of cohomology of split homogeneous supermanifolds with coefficients in the tangent sheaf.

For $n = 2$ and 3, V. A. Bunegina and A. L. Onishchik completely investigated the case, see [BuO1], [BuO2].

For $n = 4$, see below (summary of the results of [B1] – [B4], [BaO1], [BaO2]). For the method of the proof, see [BuO1], [BuO2].

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2 Results

As is known, any holomorphic bundle $\mathbf{E}$ over $\mathbb{CP}^1$ can be uniquely decomposed into a direct sum of line bundles: Grothendieck's theorem, see [HM*]. (For interesting applications of Linear Superalgebra (with elements of category theory) to the description of vector bundles over projective spaces, see the review [BG*]. For the latest results on non-splitness of supermanifolds whose retract is the Grassmann manifold, see [Vi*], [Vi1*], [Vi2*].) Let $\mathbf{L}_k$ be the holomorphic line bundle of degree $k \in \mathbb{Z}$.

Consider a holomorphic bundle

$$\mathbf{E} = \mathbf{L}_{-k_1} \oplus \mathbf{L}_{-k_2} \oplus \mathbf{L}_{-k_3} \oplus \mathbf{L}_{-k_4}, \text{ where } k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0.$$

If $\mathcal{M}$ is homogeneous, then the k_i must be non-negative, see [BuO1].

Let $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}$ designate the split supermanifold determined by $\mathbf{E}$.

Let us cover $\mathbb{CP}^1$ by two affine charts U_0 and U_1 with local coordinates x and $y = \frac{1}{x}$, respectively. Then, the transition functions on $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}$ in $U_0 \cap U_1$ are of the form

$$\begin{aligned} y &= x^{-1}, \\ \eta_i &= x^{-k_i}\xi_i \ \text{ for } \ i = 1, \dots, 4, \end{aligned}$$

where ξ_i and η_i are basis sections of $\mathbf{E}$ over U_0 and U_1, respectively.

Let M be a compact complex manifold. We will sometimes need general statements about the $m|n$-dimensional supermanifold $\mathcal{M} = (M, \mathcal{O})$. Let $\mathcal{I} \subset \mathcal{O}$ be the subsheaf of ideals generated by the subsheaf $\mathcal{O}_{\bar{1}}$. Consider the filtration of $\mathcal{O}$ by powers of $\mathcal{I}$:

$$\mathcal{O} = \mathcal{I}^0 \supset \mathcal{I} \supset \mathcal{I}^2 \supset \ldots \supset \mathcal{I}^n \supset \mathcal{I}^{n+1} = 0.$$

The graded sheaf $\operatorname{gr}\mathcal{O} = \oplus_{0\leq i\leq n} \operatorname{gr}_i\mathcal{O}$ with $\operatorname{gr}_i\mathcal{O} := \mathcal{I}^i/\mathcal{I}^{i+1}$ defines the split supermanifold $(M, \operatorname{gr}\mathcal{O})$ called the *retract* of $(M, \mathcal{O})$. Let $\mathcal{T}_{\text{gr}} := \oplus_{-1\leq p\leq 4}(\mathcal{T}_{\text{gr}})_p$ denote the graded tangent sheaf of the split supermanifold $(M, \mathcal{O}_{\text{gr}})$. Consider the subsheaf

$$\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}} = \exp((\mathcal{T}_{\text{gr}})_2 \oplus (\mathcal{T}_{\text{gr}})_4)$$

of the sheaf $\mathcal{A}ut\,\mathcal{O}_{\text{gr}}$. Thanks to a theorem due to Green ([G]), the set of supermanifolds with a given retract $(M, \mathcal{O}_{\text{gr}})$ is isomorphic to the set of orbits of the group $\operatorname{Aut}\mathbf{E}$ in $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$.

In what follows we often assume that $H^0(M, (\mathcal{T}_{\text{gr}})_2) = 0$. This is needed for existence of a bijection (see [O3, Section 3.2 pp.23–37])

$$H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}) \longleftrightarrow H^1(M, (\mathcal{T}_{\text{gr}})_2) \oplus H^1(M, (\mathcal{T}_{\text{gr}})_4).$$

2.1 Statement ([BaO2]). For $n \leq 5$, let $H^0(M, (\mathcal{T}_{\text{gr}})_2) = 0$ and let there be given subspaces $Q_{2p} \subset Z^1(\mathfrak{U}, (\mathcal{T}_{\text{gr}})_{2p})$, where $p = 1, 2$, such that every cohomology class in $H^1(M, (\mathcal{T}_{\text{gr}})_{2p})$ contains precisely one cocycle in Q_{2p}.

Then, every cohomology class in $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ can be represented by a unique cocycle of the form $z = \exp(u^2 + u^4)$, where $u^2 \in Q_2$ and $u^4 \in Q_4$.

Hereafter, the expression "$(M,\mathcal{O})$ is determined by the cocycle u^2+u^4" means that $(M,\mathcal{O})$ actually corresponds to the cocycle $z=\exp(u^2+u^4)$.

In [BuO1, Prop. 12], there is given a description of the algebra $\operatorname{End}\mathbf{E}$. Let us reformulate this Proposition in terms more convenient for us here. Any endomorphism $a\in\operatorname{End}\mathbf{E}$ can be considered as an endomorphism of the sheaf $\mathcal{E}$ of $\mathcal{F}$-modules. In U_0, we have

$$a(\xi_i)=\sum_{1\le j\le n} a_{ji}\xi_j \quad \text{for } i=1,\dots,n,$$

where $A=(a_{ji})$ is a matrix with elements $a_{ji}\in\mathcal{F}(U_0)$. The matrix A completely determines the endomorphism a, and $a\in\operatorname{Aut}\mathbf{E}$ if and only if A is invertible.

The following statement is needed in the classification of homogeneous non-split supermanifolds up to a diffeomorphism.

2.2 Statement ([BaO2]). The matrix $A=(a_{ji})$ over $\mathcal{F}(U_0)$ corresponds to an element in $\operatorname{End}\mathbf{E}$ if and only if

$$a_{ij}=\begin{cases}0 & \text{if } k_i<k_j,\\ \text{is a polynomial of degree} \le k_i-k_j & \text{if } k_i\ge k_j.\end{cases}$$

First, consider the tuples $(k,k,2,0)$ for $k\ge 2$. Let $\mathfrak{v}(\mathbb{CP}^1,\mathcal{O}_{\mathrm{gr}})$ be the Lie superalgebra of vector fields on $\mathbb{CP}^{1|4}_{kk20}$. Theorem 14 in [BaO1] implies the following

2.3. Lemma. *For a basis of* $H^1(\mathbb{CP}^1,(\mathcal{T}_{\mathrm{gr}})_q)$, *where* $q=1,2,4$, *we can take* (the classes of) *the following cocycles defining* $\mathcal{CP}^{1|4}_{kk20}$.

1) $q=1$ for $k=2$

$$\begin{array}{lllll} x^{-1}\xi_1\xi_2\partial_{\xi_2}, & x^{-1}\xi_1\xi_4\partial_{\xi_4}, & x^{-1}\xi_2\xi_1\partial_{\xi_1}, & x^{-1}\xi_2\xi_4\partial_{\xi_4}, & x^{-1}\xi_3\xi_1\partial_{\xi_1},\\ x^{-1}\xi_3\xi_4\partial_{\xi_4}, & x^{-1}\xi_1\xi_2\partial_{\xi_3}, & x^{-1}\xi_1\xi_3\partial_{\xi_2}, & x^{-1}\xi_2\xi_3\partial_{\xi_1}, & \\ x^{-r}\xi_1\xi_2\partial_{\xi_4}, & x^{-r}\xi_1\xi_3\partial_{\xi_4}, & x^{-r}\xi_2\xi_3\partial_{\xi_4} & \text{for } r=1,2,3 & \end{array}$$

for $k=3$

$$\begin{array}{l} x^{-1}\xi_3\xi_4\partial_{\xi_4},\ \ x^{-1}\xi_3\xi_1\partial_{\xi_1},\ \ x^{-1}\xi_1\xi_3\partial_{\xi_2},\ \ x^{-r}\xi_1\xi_3\partial_{\xi_4}\ (r=1,2,3,4),\ \ x^{-1}\xi_2\xi_3\partial_{\xi_1},\\ x^{-r}\xi_2\xi_3\partial_{\xi_4}\ (r=1,2,3,4),\ x^{-r}\xi_1\xi_2\partial_{\xi_3}\ (r=1,2,3),\ x^{-r}\xi_1\xi_2\partial_{\xi_4}\ (r=1,2,3,4,5),\\ x^{-r}\xi_1\xi_j\partial_{\xi_j}\ (j=2,3,4,\ r=1,2),\ x^{-r}\xi_2\xi_j\partial_{\xi_j}\ (j=1,3,4,\ r=1,2); \end{array}$$

for $k\ge 4$

$$\begin{array}{l} x^{-1}\xi_3\xi_4\partial_{\xi_4},\ \ x^{-1}\xi_3\xi_1\partial_{\xi_1},\qquad x^{-1}\xi_1\xi_3\partial_{\xi_2},\ \ x^{-r}\xi_1\xi_3\partial_{\xi_4}\ (r=1,\dots,k+1),\\ x^{-r}\xi_1\xi_2\partial_{\xi_3}\ (r=1,\dots,2k-3),\ \ x^{-1}\xi_2\xi_3\partial_{\xi_1},\ \ x^{-r}\xi_2\xi_3\partial_{\xi_4}\ (r=1,\dots,k+1),\\ x^{-r}\xi_1\xi_2\partial_{\xi_4}\ (r=1,\dots,2k-1),\ x^{-r}\xi_i\partial_x\ (i=1,2,\ j=1,\dots,k-3),\\ x^{-r}\xi_1\xi_j\partial_{\xi_j}\ (j=2,3,4,\ r=1,\dots,k-1),\ x^{-r}\xi_j\xi_4\partial_{\xi_3}\ (j=1,2,\ r=1,\dots,k-3),\\ x^{-r}\xi_2\xi_j\partial_{\xi_j}\ (j=1,3,4,\ r=1,\dots,k-1); \end{array}$$

2) $q=2$

for $k=2$

$$\begin{array}{llll} x^{-1}\xi_1\xi_2\partial_x, & x^{-r}\xi_1\xi_2\xi_j\partial_{\xi_j}\ (j=3,4,\ r=1,2,3), & x^{-1}\xi_1\xi_4\xi_2\partial_{\xi_2}, & x^{-1}\xi_2\xi_4\xi_1\partial_{\xi_1},\\ x^{-1}\xi_1\xi_3\partial_x, & x^{-r}\xi_1\xi_3\xi_j\partial_{\xi_j}\ (j=2,4,\ r=1,2,3), & x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}, & x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1},\\ x^{-1}\xi_2\xi_3\partial_x, & x^{-r}\xi_2\xi_3\xi_j\partial_{\xi_j}\ (j=1,4,\ r=1,2,3), & x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}, & x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3},\\ x^{-r}\xi_1\xi_2\xi_3\partial_{\xi_4}\ (r=1,2,3,4,5); \end{array}$$

for $k=3$

$$\begin{array}{lll} x^{-r}\xi_1\xi_2\partial_x\ (r=1,2,3), & x^{-r}\xi_1\xi_2\xi_j\partial_{\xi_j}\ (j=3,4,\ r=1,2,3,4,5), & x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\\ x^{-r}\xi_1\xi_3\partial_x\ (r=1,2), & x^{-r}\xi_1\xi_3\xi_j\partial_{\xi_j}\ (j=2,4,\ r=1,2,3,4), & x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1},\\ x^{-r}\xi_2\xi_3\partial_x\ (r=1,2), & x^{-r}\xi_2\xi_3\xi_j\partial_{\xi_j}\ (j=1,4,\ r=1,2,3,4), & x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\\ x^{-r}\xi_1\xi_2\xi_3\partial_{\xi_4}\ (r=1,\dots,7), & x^{-r}\xi_1\xi_4\xi_j\partial_{\xi_j}\ (j=2,3,\ r=1,2), & \\ x^{-r}\xi_1\xi_2\xi_4\partial_{\xi_3}\ (r=1,2,3), & x^{-r}\xi_2\xi_4\xi_j\partial_{\xi_j}\ (j=1,3,\ r=1,2); & \end{array}$$

for $k\geq 4$

$$\begin{array}{ll} x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}, & x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\\ x^{-r}\xi_1\xi_2\partial_x\ (r=1,\dots,2k-3), & x^{-r}\xi_1\xi_2\xi_j\partial_{\xi_j}\ (j=3,4,\ r=1,\dots,2k-1),\\ x^{-r}\xi_1\xi_3\partial_x\ (r=1,\dots,k-1), & x^{-r}\xi_1\xi_3\xi_j\partial_{\xi_j}\ (j=2,4,\ r=1,\dots,k+1),\\ x^{-r}\xi_2\xi_3\partial_x\ (r=1,\dots,k-1), & x^{-r}\xi_2\xi_3\xi_j\partial_{\xi_j}\ (j=1,4,\ r=1,\dots,k+1),\\ x^{-r}\xi_1\xi_2\xi_3\partial_{\xi_4}\ (r=1,\dots,2k+1), & x^{-r}\xi_1\xi_2\xi_4\partial_{\xi_3}\ (r=1,\dots,2k-3),\\ x^{-r}\xi_1\xi_4\partial_x\ (r=1,\dots,k-3), & x^{-r}\xi_1\xi_4\xi_j\partial_{\xi_j}\ (j=2,3,\ r=1,\dots,k-1),\\ x^{-r}\xi_2\xi_4\partial_x\ (r=1,\dots,k-3), & x^{-r}\xi_2\xi_4\xi_j\partial_{\xi_j}\ (j=1,3,\ r=1,\dots,k-1); \end{array}$$

4) $q=4$

$$x^{-r}\xi_1\xi_2\xi_3\xi_4\partial_x\ (r=1,\dots,2k-1).$$

Consider the exact sequence (see [BuO1])

$$0\to \operatorname{End}\mathbf{E}\to \mathfrak{v}(\mathbb{CP}^1,\mathcal{O}_{\mathrm{gr}})_0 \xrightarrow{\beta} \mathfrak{sl}_2(\mathbb{C})\to 0. \tag{1}$$

The subalgebra $\mathfrak{a}\subset\mathfrak{v}(\mathbb{CP}^1,\mathcal{O}_{\mathrm{gr}})_0$ splits the sequence (1) if β is an isomorphism with $\mathfrak{sl}_2(\mathbb{C})$ or, equivalently, $\mathfrak{v}(\mathbb{CP}^1,\mathcal{O}_{\mathrm{gr}})_0=\operatorname{End}\mathbf{E}\oplus\mathfrak{a}$ is a direct sum of Lie algebras. In [BuO1], it was shown that the supermanifold with retract $(\mathbb{CP}^1,\mathcal{O}_{\mathrm{gr}})$ is even-homogeneous if and only if a subalgebra $\mathfrak{a}$ splitting (1) can be lifted to it.

If this is the case, we will say that the $(\mathbb{CP}^1,\mathcal{O})$ is *even-homogeneous relative* $\mathfrak{a}$. Then, there exist only the following (up to an automorphism in $\operatorname{Aut}\mathbf{E}$) splitting subalgebras $\mathfrak{a}_i\simeq\mathfrak{sl}_2(\mathbb{C})$ (see [BuO1]) for the supermanifold $\mathcal{CP}^{1|4}_{kk20}$ (spanned by $\mathbf{e}$, $\mathbf{f}$, and $\mathbf{h}=[\mathbf{e},\mathbf{f}]$) for $k=2$

$\mathfrak{a}_1:\ \mathbf{e}=\partial_x,\quad \mathbf{f}=-x^2\partial_x-2x\left(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2}+\xi_3\partial_{\xi_3}\right);$

$\mathfrak{a}_2:\ \mathbf{e}=\xi_2\partial_{\xi_1}+\partial_x,\ \ \mathbf{f}=\xi_1\partial_{\xi_2}-x^2\partial_x-2x\left(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2}+\xi_3\partial_{\xi_3}\right);$

$\mathfrak{a}_3:\ \mathbf{e}=\xi_3\partial_{\xi_2}+\xi_2\partial_{\xi_1}+\partial_x,$
$\quad\mathbf{f}=2\xi_2\partial_{\xi_3}+2\xi_1\partial_{\xi_2}-x^2\partial_x-2x\left(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2}+\xi_3\partial_{\xi_3}\right);$

for $k\geq 3$ and $\nabla = k\xi_1\partial_{\xi_1} + k\xi_2\partial_{\xi_2} + 2\xi_3\partial_{\xi_3}$
$\mathfrak{a}_1$: $\mathbf{e} = \partial_x$, $\mathbf{f} = -x^2\partial_x - x\nabla$;
$\mathfrak{a}_2$: $\mathbf{e} = \xi_2\partial_{\xi_1} + \partial_x$, $\mathbf{f} = \xi_1\partial_{\xi_2} - x^2\partial_x - x\nabla$.

Designate by $H^1(\mathbb{CP}^1, (\mathcal{T}_{\mathrm{gr}}))^{\mathfrak{a}}$ the set of $\mathfrak{a}$-invariants.

2.4 Lemma ([B1]). *Let $n=4$, $k_1\geq k_2\geq k_3\geq k_4\geq 0$ and $H^0(\mathbb{CP}^1, (\mathcal{T}_{\mathrm{gr}})_2) = \{0\}$. Then,*

1) *$H^1(\mathbb{CP}^1, (\mathcal{T}_{\mathrm{gr}})_2)^{\mathfrak{s}} \neq \{0\}$ if and only if (k_1,k_2,k_3,k_4) is one of the following*

$$\begin{array}{l}(1,1,1,0),\quad (2,2,1,0),\quad (3,3,1,0),\quad (4,4,1,0),\quad (4,4,3,0),\\ (6,4,3,0),\quad (2,2,2,1),\quad ,(3,3,2,1),\quad (5,3,2,1),\quad (k+1,k,1,1)_{k\geq 1},\\ (k+1,k,2,0)_{k\geq 2},\ (k,k,2,0)_{k\geq 2},\ (k+3,k,2,2)_{k\geq 2},\ (k+1,k,2,2)_{k\geq 2},\\ (k,k,2,2)_{k\geq 2},\quad (k+3,k,3,1)_{k\geq 3},\quad (k+1,k,3,1)_{k\geq 3},\quad (k,k,3,1)_{k\geq 3},\\ (k+3,k,4,0)_{k\geq 4},\ (k+2,k,4,0)_{k\geq 4},\ (k+1,k,4,0)_{k\geq 4},\ (k,k,4,0)_{k\geq 4},\\ (k_2+k_3+k_4-2,k_2,k_3,k_4)_{k_3\geq 1}.\end{array}$$

2) *$H^1(\mathbb{CP}^1, (\mathcal{T}_{\mathrm{gr}})_4)^{\mathfrak{s}} \neq \{0\}$ if and only if $(k_1,k_2,k_3,k_4) = (1,1,1,1)$.*

2.5. Lemma. *For a basis of $H^1(\mathbb{CP}^1, (\mathcal{T}_{\mathrm{gr}})_2)^{\mathfrak{a}_i}$ we can take* (the classes of) *the following cocycles for the supermanifold $\mathcal{CP}^{1|4}_{kk20}$.*

for $k=2$

1) $i=1$:

$$\begin{array}{l}x^{-1}\xi_1\xi_2\partial_x + x^{-2}\left(\xi_1\xi_2\xi_3\partial_{\xi_3} + \xi_1\xi_2\xi_4\partial_{\xi_4}\right),\quad x^{-1}\xi_1\xi_4\xi_2\partial_{\xi_2},\quad x^{-1}\xi_2\xi_4\xi_1\partial_{\xi_1},\\ x^{-1}\xi_1\xi_3\partial_x + x^{-2}\left(\xi_1\xi_3\xi_2\partial_{\xi_2} + \xi_1\xi_3\xi_4\partial_{\xi_4}\right),\quad x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\quad x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3},\\ x^{-1}\xi_2\xi_3\partial_x + x^{-2}\left(\xi_2\xi_3\xi_1\partial_{\xi_1} + \xi_2\xi_3\xi_4\partial_{\xi_4}\right),\quad x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1};\end{array}$$

2) $i=2$: $x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3}$;

3) $i=3$: $x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3}$, $2x^{-3}\xi_1\xi_2\xi_3\partial_{\xi_3} + x^{-2}\xi_1\xi_3\xi_2\partial_{\xi_2} + x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1}$,
$2x^{-3}\xi_1\xi_2\xi_4\partial_{\xi_4} + x^{-2}\xi_1\xi_3\xi_4\partial_{\xi_4} + x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_4}$;

for $k=3$ and $k\geq 5$

1) $i=1$: $x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}$, $x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}$, $x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}$;

2) $i=2$: $x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}$, $x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}$;

for $k=4$

1) $i=1$:

$$\begin{array}{l}x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\quad x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\quad x^{-1}\xi_1\xi_4\partial_x + 2x^{-2}\xi_1\xi_4\xi_2\partial_{\xi_2} + x^{-2}\xi_1\xi_4\xi_3\partial_{\xi_3},\\ x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1},\quad x^{-1}\xi_2\xi_4\partial_x + 2x^{-2}\xi_2\xi_4\xi_1\partial_{\xi_1} + x^{-2}\xi_2\xi_4\xi_3\partial_{\xi_3};\end{array}$$

2) $i=2$: $x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}$, $x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}$.

Proof. follows from Theorem 15 in [BaO2]. □

The proof of the following Lemma is similar.

2.6. Lemma. *$H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_4)^{\mathfrak{a}} = \{0\}$ for any splitting subalgebra $\mathfrak{a}$ and supermanifold $\mathcal{CP}^{1|4}_{kk20}$.*

Let $\lambda_2 : \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr} \longrightarrow (\mathcal{T}_{\rm gr})_2$ be a sheaf homomorphism which to any germ of the automorphism a assigns the 2-component of $\log a$ in $(\mathcal{T}_{\rm gr})_2 \oplus (\mathcal{T}_{\rm gr})_4$.

Denote by $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})^{\mathfrak{a}}$ the set of classes that determine supermanifolds even-homogeneous relative $\mathfrak{a}$.

2.7. Proposition. *If $\mathfrak{a}$ splits the sequence (1), then λ_2^* maps $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})^{\mathfrak{a}}$ bijectively to $H^1(\mathbb{CP}^1, (\mathcal{T}_{\rm gr})_2)^{\mathfrak{a}}$.*

Proof. follows from Statement 2.1 and Lemma 2.6. □

Using the lifting condition of the vector field on non-split supermanifold described in [O2], and Lemma 2.6, we deduce that any supermanifold $\bar{0}$-homogeneous relative $\mathfrak{a}$ is determined by cocycles u^2 such that the class $[u^2]$ is $\mathfrak{a}$-invariant and $[u^2, u^2] = 0$. Since $[u^2, u^2] = 0$ is true for all cocycles in Lemma 2.5, the following Theorem holds.

2.8. Theorem. *For any splitting subalgebra $\mathfrak{a}$, the even-homogeneous relative $\mathfrak{a}$ supermanifolds are determined by the cocycles listed in Lemma 2.5.*

Thus, all even-homogeneous relative $\mathfrak{a}$ non-split supermanifolds with retract $\mathcal{CP}^{1|4}_{kk20}$ are described for any $k \geq 2$.

Let us find out if any of these even-homogeneous relative $\mathfrak{a}$ non-split supermanifolds are homogeneous. For this we use the following Proposition analogous to Proposition 15 in [BuO1]:

2.9. Proposition. *Under conditions of Proposition 2.7, let $(\mathbb{CP}^1, \mathcal{O})$ be even-homogeneous relative $\mathfrak{a}_i$ for one of $i = 1, 2, 3$.*

If $i = 1$, then $(\mathbb{CP}^1, \mathcal{O})$ is homogeneous if and only if the vector fields ∂_{ξ_j}, where $j = 1, \ldots, 4$ can be lifted to $(\mathbb{CP}^1, \mathcal{O})$;.

If $i = 2$, then $(\mathbb{CP}^1, \mathcal{O})$ is homogeneous if and only if the vector fields ∂_{ξ_j}, where $j = 1, 3, 4$, can be lifted to $(\mathbb{CP}^1, \mathcal{O})$.

If $i = 3$, then $(\mathbb{CP}^1, \mathcal{O})$ is homogeneous if and only if the vector fields ∂_{ξ_1} and ∂_{ξ_4} can be lifted to $(\mathbb{CP}^1, \mathcal{O})$.

Proposition 2.9 applied to the cocycles of Lemma 2.5 implies the classification (Theorems 2.10–2.20). For other 4-tuples, homogeneous non-split supermanifolds do not exist.

2.10. Theorem. *There does not exists homogeneous non-split supermanifolds with retract $\mathcal{CP}^{1|4}_{kk20}$ for any $k \geq 2$.*

2.11. Theorem. *For the following tuples* $(k_1k_2k_3k_4)$, *there does not exists homogeneous non-split supermanifolds* $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}$.

$(1,1,1,0)$, $(3,3,1,0)$, $(4,4,1,0)$, $(4,4,3,0)$, $(6,4,3,0)$, $(5,4,3,0)$, $(k+2,k,4,0)_{k\geq 4}$, $(3,3,2,1)$, $(5,3,2,1)$, $(k+3,k,4,0)_{k\geq 4}$, $(k+3,k,3,1)_{k\geq 3}$, $(k+1,k,4,0)_{k\geq 4}$, $(k+1,k,3,1)_{k\geq 3}$, $(k+1,k,2,0)_{k>2}$, $(k,k,4,0)_{k\geq 4}$, $(k,k,3,1)_{k\geq 3}$.

2.12. Theorem. *For every of the following tuples* $(k_1k_2k_3k_4)$, *there exists one homogeneous non-split supermanifold with retract* $\mathcal{CP}^{1|4}_{k_1k_2k_3k_4}$ *which can be represented, up to an isomorphism, by* (the classes of) *the following cocycles.*

$$\begin{array}{ll}
(2,2,1,0) & x^{-1}\xi_1\xi_4\xi_3\partial_{\xi_3};\\
(2,2,2,1) & x^{-1}\xi_1\xi_2\partial_x;\\
(3,2,2,0) & 2x^{-1}\xi_1\xi_3\partial_x+3x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1};\\
(4,3,2,1) & x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1};\\
(k+1,k,1,1)_{k>1} & x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1};\\
(k+3,k,2,2)_{k\geq 2} & 2x^{-1}\xi_3\xi_4\partial_x+(k+3)x^{-2}\xi_3\xi_4\xi_1\partial_{\xi_1}+kx^{-2}\xi_3\xi_4\xi_2\partial_{\xi_2};\\
(k+1,k,2,2)_{k>2} & 2x^{-1}\xi_3\xi_4\partial_x+(k+1)x^{-2}\xi_3\xi_4\xi_1\partial_{\xi_1}+kx^{-2}\xi_3\xi_4\xi_2\partial_{\xi_2};\\
(k,k,2,2)_{k\geq 4} & x^{-1}\xi_3\xi_4\partial_x.
\end{array}$$

2.13. Theorem. *If* $k_4\neq 0$, *then for every* $(k_2+k_3+k_4-2,k_2,k_3,k_4)_{k_3\geq 1}$ *which is different from* $(k,k,1,1)_{k\geq 1}$, *there exists one homogeneous non-split supermanifold which can be represented, up to an isomorphism, by the* (class of) *cocycle*

$$x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}.$$

2.14. Theorem. *There exist two homogeneous non-split supermanifolds with their retracts being* $\mathcal{CP}^{1|4}_{2222}$, *and which can be represented, up to an isomorphism, by* (the classes of) *the following cocycles.*

$$\begin{array}{l}
x^{-1}\xi_1\xi_2\partial_x+x^{-2}\xi_1\xi_2\xi_3\partial_{\xi_3}+x^{-2}\xi_1\xi_2\xi_4\partial_{\xi_4},\\
x^{-1}\xi_1\xi_2\partial_x+x^{-2}\xi_1\xi_2\xi_3\partial_{\xi_3}+x^{-2}\xi_1\xi_2\xi_4\partial_{\xi_4}+x^{-1}\xi_3\xi_4\partial_x+\\
\qquad\qquad +x^{-2}\xi_3\xi_4\xi_1\partial_{\xi_1}+x^{-2}\xi_3\xi_4\xi_2\partial_{\xi_2}.
\end{array}$$

2.15. Theorem. *There exist two homogenous non-split supermanifolds with their retracts being* $\mathcal{CP}^{1|4}_{3222}$, *and which can be represented, up to an isomorphism, by* (the classes of) *the cocycles*

$$\begin{array}{l}
x^{-1}\xi_2\xi_3\partial_x+x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1}+x^{-2}\xi_2\xi_3\xi_4\partial_{\xi_4},\\
2x^{-3}\xi_2\xi_3\xi_1\partial_{\xi_1}+x^{-2}\xi_2\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}.
\end{array}$$

2.16. Theorem. *There exist three homogeneous non-split supermanifolds with their retracts being* $\mathcal{CP}^{1|4}_{3322}$, *and which can be represented, up to an isomorphism, by* (the classes of) *the following cocycles.*

$$\begin{array}{l}
x^{-1}\xi_3\xi_4\partial_x,\quad x^{-2}\xi_1\xi_3\partial_x+x^{-1}\xi_2\xi_3\partial_x,\\
x^{-1}\xi_3\xi_4\partial_x+x^{-2}\xi_1\xi_3\partial_x+x^{-1}\xi_2\xi_3\partial_x.
\end{array}$$

2.17. Theorem. *There exist five homogeneous non-split supermanifolds with their retracts being* $\mathcal{CP}^{1|4}_{kk11}$, *where* $k>2$, *which can be represented, up to an isomorphism, by* (the classes of) *the following cocycles.*

$$
\begin{array}{l}
x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\quad x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\quad x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2},\\
x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}+x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1},\ x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}+x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}.
\end{array}
$$

2.18. Theorem. *There exist eight homogeneous non-split supermanifolds with their retracts being* $\mathcal{CP}^{1|4}_{2111}$, *which can be represented, up to an isomorphism, by* (the classes of) *the following cocycles.*

$$
\begin{array}{l}
x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1},\quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_4},\quad x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\\
x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1}+x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_4},\quad x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1}+x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2},\\
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_4}+x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\quad x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2},\\
x^{-1}\xi_2\xi_3\xi_1\partial_{\xi_1}+x^{-1}\xi_2\xi_4\xi_3\partial_{\xi_3}+x^{-1}\xi_3\xi_4\xi_2\partial_{\xi_2}.
\end{array}
$$

2.19. Theorem. *There exist nine homogeneous non-split supermanifolds with their retracts being* $\mathcal{CP}^{1|4}_{2211}$, *which can be represented, up to an isomorphism, by* (the classes of) *the following cocycles.*

$$
\begin{array}{l}
x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1},\\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1}-x^{-1}\xi_2\xi_4\xi_1\partial_{\xi_1},\quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}+x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\\
x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}+x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1}-x^{-1}\xi_2\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1},\\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1}-x^{-1}\xi_2\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1},\\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1}-x^{-1}\xi_2\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}+x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\\
x^{-2}\xi_2\xi_3\xi_1\partial_{\xi_1}-x^{-1}\xi_2\xi_4\xi_1\partial_{\xi_1}+x^{-1}\xi_3\xi_4\xi_1\partial_{\xi_1}+\\
\qquad\qquad +x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}+x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}.
\end{array}
$$

2.20 Theorem ([B5]). *There exist four homogeneous non-split superstrings and one* 1-*parameter family of homogeneous non-split superstrings with retract* $\mathcal{CP}^{1|4}_{1111}$, *which can be represented, up to an isomorphism, by* (the classes of) *the following cocycles.*

$$
\begin{array}{l}
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1},\quad x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}-x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2},\\
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}-x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}+x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3},\\
x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}-x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}+x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3}-x^{-1}\xi_1\xi_2\xi_3\partial_{\xi_4},\\
t(x^{-1}\xi_2\xi_3\xi_4\partial_{\xi_1}-x^{-1}\xi_1\xi_3\xi_4\partial_{\xi_2}+x^{-1}\xi_1\xi_2\xi_4\partial_{\xi_3}-x^{-1}\xi_1\xi_2\xi_3\partial_{\xi_4})+\\
\qquad\qquad +x^{-1}\xi_1\xi_2\xi_3\xi_4\partial_x,\ \textit{where}\ \ t\in\mathbb{C}^\times.
\end{array}
$$

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[1]Certain attributions in this paper are wrong. For example, the Lie superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ — the deformation of $\mathfrak{osp}(4|2)$ — was discovered by Kaplansky [Kap*], [Kapp*], see [KlE*]; later Kac redenoted $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ by $D(2, 1; \alpha)$; for reasons given in [CCLL*], this is an ill-chosen notation. For a reasonable and meaningful notation which gradually started to replace the initial *ad hoc* names of the exceptional Lie superalgebras given by Kaplansky and later ones, due to Kac, see [Sr*], [CCLL*].

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