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Supermanifolds corresponding to the trivial vector bundle over torus

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Abstract. All supermanifolds whose retract $T^{m|n}$ is determined by the trivial bundle of rank n over the torus T^m are $\bar{0}$-homogeneous and only $T^{m|n}$ is homogeneous.

1 Preliminaries

1.1 Split and non-split supermanifolds The ground field is $\mathbb{C}$.

A *complex supermanifold* of dimension $m|n$ is a $\mathbb{Z}/2$-graded ringed space of the form $\mathcal{M} := (M, \mathcal{O})$, where M is a topological space and $\mathcal{O}$ is a sheaf of associative commutative superalgebras with unit on M, which is locally isomorphic to a superdomain in $\mathbb{C}^{m|n}$. For details, see [M], [O3], [BGLS*]. A *superdomain* in $\mathbb{C}^{m|n}$ is a pair $(U, \bigwedge_{\mathcal{F}}(\xi_1, \dots, \xi_n))$, where U is an open subset of $\mathbb{C}^m$, and $\mathcal{F}$ is the sheaf of holomorphic functions on $\mathbb{C}^m$. The coordinates $x_1, \dots, x_m$ in $U \subset \mathbb{C}^m$ and generators $\xi_1, \dots, \xi_n$ of the Grassmann algebra are identified with some sections of the sheaf $\mathcal{O}|_U$. They are called *local coordinates even* and *odd*, respectively.

Let $(M, \mathcal{F})$ be a complex manifold and $\mathcal{E}$ be a locally free analytic sheaf on it, i.e., $\mathcal{E}$ is a sheaf of holomorphic sections of some holomorphic vector bundle $\mathbf{E} \to M$. Then, $(M, \mathcal{O}_{\mathrm{gr}})$, where $\mathcal{O}_{\mathrm{gr}} = \bigwedge_{\mathcal{F}} \mathcal{E}$, is a complex supermanifold. A supermanifold is called *split* if it is isomorphic to a supermanifold of this form and is called *non-split* otherwise.

Let us show that every supermanifold is a deformation of a split supermanifold. Consider the subsheaf of ideals $\mathcal{J} = (\mathcal{O}_{\bar{1}})$, generated by odd elements. Denote $\mathcal{F} := \mathcal{O}/\mathcal{J}$. Then, $\mathcal{M}_{\mathrm{rd}} = (M, \mathcal{F})$ is a complex manifold called the *odd reduction* of $(M, \mathcal{O})$. The powers of $\mathcal{J}$ determine the following filtration:

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots \supset \mathcal{J}^{n+1} = 0. \tag{1}$$

The associated sheaf of graded algebras, $\operatorname{gr} \mathcal{O} = \bigoplus_{0 \le p \le n} \operatorname{gr}_p \mathcal{O}$, where $\operatorname{gr}_p \mathcal{O} = \mathcal{J}^p / \mathcal{J}^{p+1}$, is an analytic sheaf on the reduction $\mathcal{M}_{\mathrm{rd}}$. Actually, $\operatorname{gr} \mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{E} = \operatorname{gr}_1 \mathcal{O}$

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is locally free sheaf. Clearly, $(M, \operatorname{gr}\mathcal{O}) = (M, \mathcal{O}_{\mathrm{gr}})$ is a split supermanifold of the same dimension as $(M, \mathcal{O})$. We call it the *retract* of the supermanifold $(M, \mathcal{O})$. Obviously, a given supermanifold is split if and only if it is isomorphic to its retract.

Let $\pi_p : \mathcal{J}^p \to \mathrm{gr}_p\mathcal{O}$ be the canonical projection. Then, there is the exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}^{p+1} \longrightarrow \mathcal{J}^p \xrightarrow{\pi_p} \mathrm{gr}_p\mathcal{O} \longrightarrow 0. \tag{2}$$

A supermanifold $(M, \mathcal{O})$ is split if and only if there exists an isomorphism of superalgebra sheaves $h : \operatorname{gr}\mathcal{O} \to \mathcal{O}$, whose restriction $h_p : \mathrm{gr}_p\mathcal{O} \to \mathcal{J}^p$ splits the sequence (2), i.e., satisfies the condition $\pi_p \circ h_p = \mathrm{id}$. In general, this splitting exists in a neighborhood of any point in M. It can be given by means of local coordinates.

1.2 The tangent sheaf For an arbitrary supermanifold $(M, \mathcal{O})$ denote by $\mathcal{T} := \mathcal{D}er\ \mathcal{O}$ its *tangent sheaf* (or the sheaf of vector fields). It is the sheaf of derivations (over $\mathbb{C}$) of the structure sheaf $\mathcal{O}$. Note that the tangent sheaf is a sheaf of $\mathbb{Z}/2$-graded left $\mathcal{O}$-modules, and also a sheaf of Lie superalgebras. The sections of the tangent sheaf are called *holomorphic vector fields* on $(M, \mathcal{O})$. They form the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ of vector fields on $(M, \mathcal{O})$.

Note that the filtration (1) determines the filtration of the tangent sheaf

$$\mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \cdots \supset \mathcal{T}_{(n)} \supset \mathcal{T}_{(n+1)} = 0, \tag{3}$$

where

$$\mathcal{T}_{(p)} = \{v \in \mathcal{T} \mid v(\mathcal{O}) \subset \mathcal{J}^p,\ v(\mathcal{J}) \subset \mathcal{J}^{p+1}\},\ p \geq 0.$$

Since $(M, \mathcal{O}_{\mathrm{gr}})$ is split, its tangent sheaf $\mathcal{T}_{\mathrm{gr}}$ is a $\mathbb{Z}$-graded sheaf of Lie superalgebras

$$\mathcal{T}_{\mathrm{gr}} = \bigoplus_{-1 \leq p \leq n} (\mathcal{T}_{\mathrm{gr}})_p,$$

where

$$(\mathcal{T}_{\mathrm{gr}})_p := \mathcal{D}er_p\mathcal{O}_{\mathrm{gr}} = \{v \in \mathcal{T}_{\mathrm{gr}} \mid v((\mathcal{O}_{\mathrm{gr}})_q) \subset (\mathcal{O}_{\mathrm{gr}})_{q+p},\ q \in \mathbb{Z}\}.$$

This grading is compatible with the $\mathbb{Z}/2$-grading. The Lie superalgebra $\mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})$ of vector fields is a graded algebra with the $\mathbb{Z}$-grading compatible with the $\mathbb{Z}/2$-grading.

Since $\mathcal{F} \subset \mathcal{O}_{\mathrm{gr}}$, the tangent sheaf $\mathcal{T}_{\mathrm{gr}}$ is a $\mathbb{Z}$-graded analytic sheaf on M. This sheaf is locally free (see [O3]), and hence it is the sheaf of holomorphic sections of a $\mathbb{Z}$-graded holomorphic vector bundle **ST** over M (the *supertangent bundle*).

1.3 Sheaves of automorphisms and the classification theorem Let $(M, \mathcal{O})$ be an complex supermanifold. Denote by $\mathrm{Aut}(M, \mathcal{O})$ the group of automorphisms of $(M, \mathcal{O})$. By definition, $F \in \mathrm{Aut}(M, \mathcal{O})$ is a pair (f, φ), where $f : M \to M$ belongs to group $\mathrm{Bih}\, M$ of biholomorphic transformations of the manifold M and φ is an automorphism of the superalgebra sheaf $\mathcal{O}$ over f. Denote by $\mathcal{A}ut\,\mathcal{O}$ the sheaf of automorphisms of the structure sheaf $\mathcal{O}$ (mapping every stalk $\mathcal{O}_x$, where $x \in M$, onto itself). Moreover, for any $F = (f, \varphi) \in \mathrm{Aut}(M, \mathcal{O})$ the map $\mathrm{Int}\, F : a \mapsto \varphi \circ a \circ \varphi^{-1}$ is an automorphism of the group sheaf $\mathcal{A}ut\ \mathcal{O}$. Hence, we get the action Int of the group $\mathrm{Aut}(M,\mathcal{O})$ on $\mathcal{A}ut\ \mathcal{O}$ by automorphisms. The subsheaf

$$\mathcal{A}ut_{(2)}\mathcal{O} = \{a \in \mathcal{A}ut\,\mathcal{O} \mid a(f) - f \in \mathcal{J}^2,\ f \in \mathcal{O}\} \tag{4}$$

is invariant under this action.

Let $\mathbf{E}$ be a holomorphic vector bundle over $(M,\mathcal{F})$ and $\mathrm{Aut}\,\mathbf{E}$ the group of its automorphisms. Clearly, any element of this group gives rise to an automorphism of the split supermanifold $(M,\mathcal{O}_{\mathrm{gr}})$ corresponding to $\mathbf{E}$ which preserves the $\mathbb{Z}$-grading of the structure sheaf. Hence, we can identify $\mathrm{Aut}\,\mathbf{E}$ with a subgroup of $\mathrm{Aut}(M,\mathcal{O}_{\mathrm{gr}})$, consisting of automorphisms that preserve the $\mathbb{Z}$-grading. So the $\mathrm{Aut}(M,\mathcal{O}_{\mathrm{gr}})$-sheaves $\mathcal{A}ut\,\mathcal{O}_{\mathrm{gr}}$ and $\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}$ are also $\mathrm{Aut}\,\mathbf{E}$-sheaves.

1.1 Theorem ([G]). *Any supermanifold $(M,\mathcal{O})$ corresponds to an element of the set of 1-cohomology $H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. This correspondence gives rise to a bijection between the isomorphism classes of supermanifolds satisfying the above condition, and the orbits of the group $\mathrm{Aut}\,\mathbf{E}$ on $H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ under the natural $\mathrm{Aut}\,\mathbf{E}$-action.*

Let us describe the correspondence mentioned in Theorem 1.1. Let $(M,\mathcal{O})$ be a supermanifold with retract $(M,\mathcal{O}_{\mathrm{gr}})$. Then, we can choose an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of M such that there exist isomorphisms $h_i : \mathcal{O}_{\mathrm{gr}}|_{U_i} \to \mathcal{O}|_{U_i}$, where $i \in I$, with conditions $\pi_p \circ (h_i)_p = \mathrm{id}$ on $(\mathcal{O}_{\mathrm{gr}})_p|_{U_i}$ (see (2)). Setting $z_{ij} = h_i^{-1}h_j$, we get a 1-cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U},\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. Its class $\zeta \in H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ does not depend on the choice of h_i and corresponds to $(M,\mathcal{O})$.

1.4 A non-abelian complex Recall the construction of a non-abelian complex (see [O3], [O2]) which allows to express $H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ in terms of differential forms. Let $\Phi^{p,q}$ be the sheaf of smooth differential (p,q)-forms on M. First, we construct the Dolbeault–Serre resolution of the sheaf $\mathcal{O}_{\mathrm{gr}}$:

$$\widehat{\Phi} := \bigoplus_{p,q\geq 0} \widehat{\Phi}^{p,q}, \quad \widehat{\Phi}^{p,q} := \Phi^{0,q} \otimes (\mathcal{O}_{\mathrm{gr}})_p,$$

$$\overline{\partial}(\varphi \otimes u) = (\overline{\partial}\varphi) \otimes u \ \text{ for any } \ \varphi \in \Phi^{0,q},\ u \in (\mathcal{O}_{\mathrm{gr}})_p.$$

Then, regarding $\widehat{\Phi}$ as a sheaf of graded superalgebras with respect to the total degree $p+q$, we get a sheaf of graded Lie superalgebras $\widehat{\mathcal{T}} = \mathcal{D}er\ \widehat{\Phi}$. The sheaf $\widehat{\mathcal{T}}$ has the derivation $\overline{D} = \mathrm{ad}_{\overline{\partial}}$ of degree 1 (and of bidegree $(0,1)$). Denote

$$\mathcal{S} = \{u \in \widehat{\mathcal{T}} \mid u(\overline{f}) = u(d\overline{f}) = 0 \text{ for any } f \in \mathcal{F}\}.$$

This is a subsheaf of bigraded subalgebras, and $\overline{D}(\mathcal{S}) \subset \mathcal{S}$. As it was shown in [O3], the subsheaf $\mathcal{S}_{p,q}$ is naturally identified with the sheaf $\Phi^{0,q}\otimes(\mathcal{T}_{\mathrm{gr}})_p$ of $(0,q)$-forms with values in the vector bundle $\mathbf{ST}_p$, and $\overline{D} : \mathcal{S}_{p,q} \to \mathcal{S}_{p,q+1}$ goes over to the operator $\overline{\partial} : \varphi\otimes v \mapsto \overline{\partial}\varphi\otimes v$, where $\varphi \in \Phi^{0,q}$, and $v \in (\mathcal{T}_{\mathrm{gr}})_p$. Hence, the sequence

$$0 \longrightarrow \mathcal{T}_{\mathrm{gr}} \xrightarrow{i} \mathcal{S}_{*,0} \xrightarrow{\overline{D}} \mathcal{S}_{*,1} \xrightarrow{\overline{D}} \dots, \tag{5}$$

where i is a natural inclusion, is identified with the Dolbeault–Serre resolution of $\mathcal{T}_{gr}$. Set $S_{p,q} := \Gamma(M, \mathcal{S}_{p,q})$ and $S := \bigoplus_{p,q\geq 0} S_{p,q}$. Then, the bigraded Lie superalgebras $H^*(M, \mathcal{T})$ and $H(S, \overline{D})$ are isomorphic.

The desired non-abelian complex is the non-linear complex associated to the differential bigraded Lie superalgebra $(S, \overline{D})$. More precisely, denote by $\mathcal{F}^\infty$ the sheaf of differentiable complex-valued functions on M. Consider the sheaves $\mathcal{O}_{\mathrm{gr}}^\infty := \mathcal{F}^\infty \otimes \mathcal{O}_{\mathrm{gr}}$ and the group

$$\mathrm{PAut}_{(2)}\mathcal{O}_{\mathrm{gr}}^\infty := \{a \in \mathrm{Aut}\,\mathcal{O}_{\mathrm{gr}}^\infty \mid a(u) - u \in \bigoplus_{k\geq 2} (\mathcal{O}_{\mathrm{gr}}^\infty)_k,\ u \in \mathcal{O}_{\mathrm{gr}}^\infty\}.$$

The non-abelian complex is the triple $K = (K^0, K^1, K^2)$, where

$$K^0 := \mathrm{PAut}_{(2)}\mathcal{O}_{\mathrm{gr}}^\infty, \quad K^q := \bigoplus_{k\geq 1} S_{2k,q} \ \text{ for } \ q = 1, 2,$$

with the coboundary operators $\delta_i : K^i \to K^{i+1}$ for $i = 0, 1$, given by

$$\begin{aligned} &\delta_0(a) = \overline{\partial} - a\overline{\partial}a^{-1} \ \text{ for any } \ a \in K^0, \\ &\delta_1(u) = \overline{D}u - \tfrac{1}{2}[u, u] = -\tfrac{1}{2}[u - \overline{\partial}, u - \overline{\partial}] \ \text{ for any } \ u \in K^1. \end{aligned}$$

The gauge action ρ of K^0 on K^1 is given by

$$\rho(a)(u) = a(u - \overline{\partial})a^{-1} + \overline{\partial} \ \text{ for any } \ a \in K^0, u \in K^1.$$

Define $Z^1(K) := \{u \in K^1 \mid \delta_1 u = 0\}$ and $H^1(K) := Z^1(K)/\rho$. In [O3], it is proved that there is an isomorphism of pointed sets

$$\mu : \ H^1(K) \longrightarrow H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr}).$$

In order to describe this isomorphism, take a cocycle $w \in Z^1(K_{(1)})$ and an open cover $\mathfrak{U} = (U_i)$ on M such that $w = \delta_0(a_i)$, where $a_i \in \Gamma(U_i, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}^\infty)$. Then, we get the Čech cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$, where $z_{ij} = a_i^{-1}a_j$. We have $\mu(\omega) = \zeta$, where $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ and $\omega \in H^1(K)$ are the cohomology classes of the cocycles w and z, respectively.

Note that the group $\mathrm{Aut}\,\mathbf{E}$ acts on the complex K and on $H^1(K)$ in a natural way.

Using eq. (5), we can also construct a fine resolution of the tangent sheaf of any supermanifold with retract $(M, \mathcal{O}_{\mathrm{gr}})$. Consider the supermanifold $(M, \mathcal{O})$ with retract $(M, \mathcal{O}_{gr})$ that corresponds to the cohomology classes ω and ζ of cocycles $w \in Z^1(K)$ and $z = (z_{ij})$, as above. Twisting eq. (5) by z, we get the fine resolution

$$0 \longrightarrow \mathcal{T}_{\mathrm{gr}}^{\mathrm{Int}\, z} \xrightarrow{i} \mathcal{S}_{*,0}^{\mathrm{Int}\, z} \xrightarrow{\overline{D}} \mathcal{S}_{*,1}^{\mathrm{Int}\, z} \xrightarrow{\overline{D}} \dots . \tag{6}$$

Here any $v \in \mathcal{T}_{\mathrm{gr}}^{\mathrm{Int}\, z}$ is a family $v = (v^i)$, where $v^i \in \Gamma(U_i, \mathcal{T}_{\mathrm{gr}})$ and $v^i = z_{ij} \circ v^j \circ z_{ij}^{-1}$ in $U_i \cap U_j$. In the same way we express the sections of the sheaves $\mathcal{S}_{*,q}^{\mathrm{Int}\, z}$.

The correspondence $(v^i) \mapsto (h_i \circ v^i \circ h_i^{-1})$ gives an isomorphism $\mathcal{T}_{\rm gr}^{\rm Intz} \simeq \mathcal{T}$, and the correspondence $(v^i) \mapsto (a_i \circ v^i \circ a_i^{-1})$ gives an isomorphism $\mathcal{S}_{*,p}^{\rm Intz} \simeq \mathcal{S}_{*,p}$ for $p \geq 0$. Then, eq. (6) gives the following fine resolution of $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$:

$$0 \longrightarrow \mathcal{T} \xrightarrow{\tau} \mathcal{S}_{*,0} \xrightarrow{\overline{D}^w} \mathcal{S}_{*,1} \xrightarrow{\overline{D}^w} \dots, \tag{7}$$

where $\overline{D}^w := \overline{D} - \mathrm{ad}_w = \mathrm{ad}_{\overline{\partial}-w}$. Considering global sections, we get a complex $(S, \overline{D}^w)$ for calculating cohomology with values in the sheaf $\mathcal{T}$.

We give an explicit expression of τ. As we have seen in Subsection 1.3, the cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ of the cover $\mathfrak{U}$ can be represented in the form $z_{ij} = h_i^{-1}h_j$. But it can also be represented in the form $z_{ij} = a_i^{-1}a_j$. Then, we have $h_i^{-1}h_j = a_i^{-1}a_j$ in $U_i \cap U_j$, and $\varrho = a_i h_i^{-1} = a_j h_j^{-1}$ is an injective homomorphism $\mathcal{O} \to \mathcal{O}_{\rm gr}^{\infty}$. It follows that $\tau : \mathcal{T} = \mathcal{D}er\,\mathcal{O} \to \mathcal{S}_{*,0}$ is expressed by the formula $v \mapsto \varrho v \varrho^{-1}$.

1.2. Theorem. *The mapping $\tau : v \mapsto \varrho v \varrho^{-1}$ is an isomorphism of the graded Lie superalgebra $H^*(M, \mathcal{T})$ onto $H^*(\mathcal{S}, \overline{D}^w)$.*

In particular, we get the isomorphism $\tau : \mathfrak{v}(M, \mathcal{O}) \to \mathrm{Ker}\,\overline{D}^w \subset S_{*,0}$.

1.5 An application of the Hodge theory Suppose that M is compact. Then, we can develop the standard Hodge theory in the complex $(S, \overline{D})$ regarding it as the complex of $(0,*)$-forms with values in the bundle $\mathbf{ST}$, see [O3]. Endow M and $\mathbf{E}$ with smooth Hermitian metrics and consider the corresponding Hermitian metric on $\mathbf{ST}$. Denote by $\overline{D}^{\,*}$ the operator conjugate to $\overline{D}$ and by $\Box := [\overline{D}, \overline{D}^{\,*}]$ the Beltrami–Laplace operator. Their bidegrees are $(0,-1)$ and $(0,0)$, respectively. Then, we have the orthogonal decomposition

$$S = \mathbf{H} \oplus \overline{D}S \oplus \overline{D}^{\,*}S, \tag{8}$$

where $\mathbf{H} = \mathrm{Ker}\,\Box$ is the bigraded subspace of harmonic elements. Moreover,

$$\mathrm{id} = H + \Box G = H + \overline{D}\,\overline{D}^{\,*}G + \overline{D}^{\,*}\,\overline{D}\,G,$$

where H is the projection onto $\mathbf{H}$ in eq. (8) and G is the Green operator. It is well known that

$$\mathbf{H}_{p,q} \simeq H^{p,q}(S, \overline{D}) \simeq H^q(M, (\mathcal{T}_{\rm gr})_p) \quad \text{for any} \quad p, q \geq 0. \tag{9}$$

Consider now the nonlinear complex K. Denote

$$\mathbf{H}_{(1)} := \bigoplus_{p \geq 1} \mathbf{H}_{2p,1},$$
$$L_1 := \mathrm{Ker}\,\overline{D}^{\,*} \cap K^1, \quad \mathbf{K} := Z^1(K) \cap L_1,$$

and define also the subset $\mathbf{K}_0 \subset K^1$ consisting of the u such that

$$u - \frac{1}{2}\overline{D}^{\,*}G[u,u] = Hu. \tag{10}$$

1.3 Theorem ([O3]). *We have* $\mathbf{K} \subset \mathbf{K}_0 \subset L_1$. *The mapping* $H : \mathbf{K}_0 \to \mathbf{H}_{(1)}$ *is a bijection and maps* $\mathbf{K}$ *onto the connected algebraic subset* $\mathbf{V} \subset \mathbf{H}_{(1)} \simeq \bigoplus_{p\geq 1} H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p})$ *given by the equation*

$$H[\varphi(h), \varphi(h)] = 0,$$

where $\varphi : \mathbf{H}_{(1)} \to L_1$ *is inverse to* H.

The natural mapping $\mathbf{K} \to H^1(K) \simeq H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ *is onto.*

The set $\mathbf{K}$ is an analogue of the *Kuranishi family* of complex structures on a compact manifold. By Theorem 1.3 we can see that this family cuts every cohomology class, and hence it can be used for classification of supermanifolds with retract $(M, \mathcal{O}_{\mathrm{gr}})$.

1.6 Actions on supermanifolds Let $(M, \mathcal{O})$ be an arbitrary supermanifold. An *action* of a (real or complex) Lie group G on $(M, \mathcal{O})$ is a homomorphism $\Psi : G \to \mathrm{Aut}(M, \mathcal{O})$. For any $g \in G$ we have $\Psi(g) = (f(g), \psi(g))$, where $f : g \mapsto f(g) \in \mathrm{Bih}\, M$ is an (analytic) action of the group G on the complex manifold M and $\psi(g)$ is an automorphism of the sheaf $\mathcal{O}$ over $f(g)$.

Let $\mathbf{E}$ be a holomorphic vector bundle over a complex manifold M and G a Lie group. Suppose that $\mathbf{E}$ has a structure of the G-bundle, i.e., a homomorphism $\Phi : G \to \mathrm{Aut}\,\mathbf{E}$ satisfying the natural conditions of analiticity is given. Using the inclusion of $\mathrm{Aut}\,\mathbf{E}$ into $\mathrm{Aut}(M, \mathcal{O})$, we may consider Φ as an action on the split supermanifold $(M, \mathcal{O}_{gr})$ corresponding to the bundle $\mathbf{E}$. This action is $\mathbb{Z}$-graded, i.e., all $\varphi(g)$, where $g \in G$, preserve the $\mathbb{Z}$-grading of the structure sheaf. Conversely, any $\mathbb{Z}$-graded action of the group G on $(M, \mathcal{O}_{gr})$ extends an action on the vector bundle $\mathbf{E}$.

Let again $(M, \mathcal{O})$ be an arbitrary complex supermanifold, $(M, \mathcal{O}_{gr})$ its retract and $\mathbf{E}$ the corresponding vector bundle.

If $F = (f, \psi) \in \mathrm{Aut}(M, \mathcal{O})$, then the automorphism ψ of $\mathcal{O}$ over f preserves a filtration (1), and hence determines an automorphism φ of the $\mathbb{Z}$-graded sheaf $\mathcal{O}_{gr}$ over f. Here, φ is uniquely determined by the relation $\pi_p \circ \psi = \varphi \circ \pi_p$ on $\mathcal{J}^p$.

Define $\overline{F} = (f, \varphi) \in \mathrm{Aut}(M, \mathcal{O}_{gr})$ for every $F = (f, \psi) \in \mathrm{Aut}(M, \mathcal{O})$. Thus. we get a homomorphism $\mathrm{Aut}(M, \mathcal{O}) \to \mathrm{Aut}(M, \mathbf{E})$. It follows that any action $\Psi : G \to \mathrm{Aut}(M, \mathcal{O})$ induces a $\mathbb{Z}$-graded action $\Phi : G \to \mathrm{Aut}(M, \mathcal{O}_{gr})$. In this case, we say that the action Φ *lifts to the action* Ψ *on* $(M, \mathcal{O})$.

There is the following lifting criterion:

1.4 Theorem ([O2]). *Let* G *be a compact Lie group and suppose an analytic* $\mathbb{Z}$*-graded action* Ψ *of* G *on a split supermanifold* $(M, \mathcal{O}_{gr})$ *be given. Let* $(M, \mathcal{O})$ *be the supermanifold corresponding to a given class* $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$ *by Theorem 1.1. Then, the following conditions are equivalent:*

(i) *the action* Ψ *lifts to* $(M, \mathcal{O})$;

(ii) *the class* ζ *contains a* G*-invariant cocycle* $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$ *where* $\mathfrak{U}$ *is an open* G*-cover of* M;

(iii) *the class* $\mu_1^{-1}(\zeta) \in H^1(K)$ (see Theorem 1.1) *contains a* G*-invariant cocycle.*

Now we give definitions of homogeneous and $\bar{0}$-homogeneous supermanifolds. Let $(M,\mathcal{O})$ be a complex supermanifold. For any $x \in M$ we can define the *tangent space* $T_x(M,\mathcal{O}) := (m_x/m_x^2)^*$, where m_x is the maximal ideal of the local superalgebra $\mathcal{O}_x$.

There is a natural even linear mapping $\mathrm{ev}_x : \mathfrak{v}(M,\mathcal{O}) \to T_x(M,\mathcal{O})$. Namely, every $v \in \mathfrak{v}(M,\mathcal{O})$ determines a linear mapping $m_x \to \mathcal{O}_x$ with $v(m_x^2) \subset m_x$, and hence a linear mapping

$$m_x/m_x^2 \to \mathcal{O}_x/m_x = \mathbb{C},$$

i.e., an element $\mathrm{ev}_x(v) \in (m_x/m_x^2)^*$.

The subalgebra $\mathfrak{g} \subset \mathfrak{v}(M,\mathcal{O})$ is called *transitive* if $\mathrm{ev}_x : \mathfrak{g} \to T_x(M,\mathcal{O})$ is surjective for all $x \in M$ and if $\mathrm{ev}_x : \mathfrak{g}_{\bar{0}} \to T_x(M,\mathcal{O})_{\bar{0}} = T_x(M)$ is surjective for all $x \in M$, then it is called $\bar{0}$-*transitive*. A supermanifold $(M,\mathcal{O})$ is called *homogeneous* ($\bar{0}$-*homogeneous*) if there is a transitive ($\bar{0}$-transitive) subalgebra $\mathfrak{g} \subset \mathfrak{v}(M,\mathcal{O})$ of finite dimension. In the case when M is a compact we can replace $\mathfrak{g}$ by $\mathfrak{v}(M,\mathcal{O})$.

1.5 Theorem ([OP]). *If a supermanifold $(M,\mathcal{O})$ is homogeneous ($\bar{0}$-homogeneous), then $(M, \mathrm{gr}\,\mathcal{O})$ is homogeneous ($\bar{0}$-homogeneous).*

2 Supermanifolds associated with the complex torus

2.1 Complex tori Let $\Gamma \subset \mathbb{C}^m$ be a discrete subgroup of rank $2m$. Then, the manifold $T = \mathbb{C}^m/\Gamma$ is a *complex torus* of dimension m. Note that T is a compact complex commutative Lie group. There is a local coordinate system in a neighborhood of any point of the manifold T formed by the standard coordinates $z_1,\ldots,z_m$ in $\mathbb{C}^m$. Let us denote these coordinates on T also by $z_1\ldots,z_m$. The differential forms $dz_1,\ldots,dz_m$ are defined on T globally, since they are not changed if we add a complex number to the variable. Using duality between differential forms and vector fields, we get the vector fields $\partial_{z_1},\ldots,\partial_{z_m}$ which are defined globally, too. The tangent and the cotangent bundles over T are trivial, and dz_i, ∂_{z_i} are basis sections of these vector bundles.

2.1 Proposition ([GH]). *Let M be a compact Kähler manifold. A form $\alpha \in \Gamma(M,\Phi^{0,q})$ is harmonic if and only if α is an antiholomorphic form, i.e., $\partial\alpha = 0$.*

It is well known that $T = \mathbb{C}^m/\Gamma$ is a compact Kähler manifold with flat metrics induced by Hermitian metrics in $\mathbb{C}^m$ (see [GH]). We can represent any $\alpha \in \Gamma(M,\Phi^{0,q})$ in a form

$$\alpha = \sum_{1\le i_1<\ldots<i_q\le m} a_{i_1\ldots i_q}(z,\bar{z})d\bar{z}_{i_1}\ldots d\bar{z}_{i_q}, \tag{11}$$

where $a_{i_1\ldots i_q}(z,\bar{z})$ are smooth global defined functions, $z = (z_1,\ldots,z_m)$, $\bar{z} = (\bar{z}_1,\ldots,\bar{z}_m)$. Since any antiholomorphic function on T is constant, we have

2.2. Proposition. *A form $\alpha \in \Gamma(M,\Phi^{0,q})$ is harmonic if and only if*

$$\alpha = \sum_{1\le i_1<\ldots<i_q\le m} a_{i_1\ldots i_q}d\bar{z}_{i_1}\ldots d\bar{z}_{i_q}, \quad \textit{where } a_{i_1\ldots i_q} \in \mathbb{C}.$$

A form on T is called *T-invariant* if it is invariant under the action of the group T on itself by translations.

2.3. Proposition. *The spaces of harmonic and T-invariant $(0,q)$-forms on T coincide.*

Proof. Clearly, the forms $d\overline{z}_{i_1}\dots d\overline{z}_{i_q}$ are T-invariant. It follows that the form (11) is T-invariant if and only if $a_{i_1\dots i_q}\in\mathbb{C}$. Then, we apply Proposition 2.2.

□

2.2 Supermanifolds corresponding to the trivial bundle over the complex torus Let $\mathbf{E}=T\times\mathbb{C}^n$ be a trivial holomorphic vector bundle of rank n over T and $\xi_1,\dots,\xi_n$ be the standard basis of $\mathbb{C}^n$. Denote by $T^{m|n}=(T,\mathcal{O}_{gr})$ the split supermanifold corresponding to the bundle $\mathbf{E}$. The structure sheaf $\mathcal{O}_{\text{gr}}$ has the form $\mathcal{F}\otimes\bigwedge(\xi_1,\dots,\xi_n)$. The local coordinates $z_1,\dots,z_m$ on T are even coordinates on $T^{m|n}$, and $\xi_1,\dots,\xi_n$ are odd ones.

Consider the tangent sheaf $\mathcal{T}_{\text{gr}}=\mathcal{D}er\,\mathcal{O}_{\text{gr}}$. This sheaf is free over $\mathcal{F}$, or, equivalently, the bundle $\mathbf{ST}$ is trivial, and the basis of its sections is

$$\begin{gathered}\xi_{i_1}\dots\xi_{i_k}\partial_{z_j},\ \xi_{i_1}\dots\xi_{i_k}\partial_{\xi_l},\quad\text{where}\\ 1\le i_1<\dots<i_k\le n,\quad j=1,\dots,m,\ l=1,\dots,n.\end{gathered}\tag{12}$$

Hence, the elements of $S_{p,q}$ have the form

$$\alpha=\sum_{i_1<\dots<i_q}\Big(\sum_{\substack{j_1<\dots<j_p\\ i=1,\dots,m}}a^{i,i_1,\dots,i_q}_{j_1,\dots,j_p}(z,\overline{z})\xi_{j_1}\dots\xi_{j_p}\partial_{z_i}+\sum_{\substack{j_1<\dots<j_{p+1}\\ j=1,\dots,n}}b^{j,i_1,\dots,i_q}_{j_1,\dots,j_{p+1}}(z,\overline{z})\xi_{j_1}\dots\xi_{j_{p+1}}\partial_{\xi_j}\Big)d\overline{z}_{i_1}\dots d\overline{z}_{i_q},$$

where $a^{i,i_1,\dots,i_q}_{j_1,\dots,j_p}(z,\overline{z})$ and $b^{j,i_1,\dots,i_q}_{j_1,\dots,j_{p+1}}(z,\overline{z})$ are smooth globally defined functions on T. Thus, from Proposition 2.2 we get

2.4. Proposition. *The form $\alpha\in S_{p,q}$ is harmonic if and only if*

$$\alpha=\sum_{i_1<\dots<i_q}\Big(\sum_{\substack{j_1<\dots<j_p\\ i=1,\dots,m}}a^{i,i_1,\dots,i_q}_{j_1,\dots,j_p}\xi_{j_1}\dots\xi_{j_p}\partial_{z_i}+\sum_{\substack{j_1<\dots<j_{p+1}\\ j=1,\dots,n}}b^{j,i_1,\dots,i_q}_{j_1,\dots,j_{p+1}}\xi_{j_1}\dots\xi_{j_{p+1}}\partial_{\xi_j}\Big)d\overline{z}_{i_1}\dots d\overline{z}_{i_q},\tag{13}$$

where $a^{i,i_1,\dots,i_q}_{j_1,\dots,j_p}$, $b^{j,i_1,\dots,i_q}_{j_1,\dots,j_{p+1}}\in\mathbb{C}$.

2.5. Corollary. *If $\alpha,\beta\in\mathbf{H}$, then $[\alpha,\beta]\in\mathbf{H}$.*

Assigning to every cohomology class from $H^q(M,(\mathcal{T}_{\text{gr}})_p)$ the correspondent harmonic form from $\mathbf{H}_{p,q}$ (see (9)), we get an isomorphism of graded Lie superalgebras $H(M,\mathcal{T}_{\text{gr}})$ onto the subalgebra $\mathbf{H}\subset S$.

Since $\xi_{j_1}\dots\xi_{j_p}\partial_{z_i}$ and $\xi_{j_1}\dots\xi_{j_{p+1}}\partial_{\xi_j}$ are T-invariant, from Proposition 2.3 we get

2.6. Proposition. *Any harmonic form from S is T-invariant, and the other way round.*

2.7. Theorem. *We have $K_0 = \mathbf{H}_{(1)}$ and*

$$\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w] = 0\}.$$

Proof. Take $w \in \mathbf{K}_0$ and denote $h = Hw$. We write $h = \sum_{k\geq 1} h_{2k}$, and $w = \sum_{k\geq 1} w_{2k}$, where $h_{2k} \in \mathbf{H}_{2k,1}$, $w_{2k} \in \mathrm{S}_{2k,1}$. From (10) we get the following equations:

$$\begin{aligned}
&w_2 = h_2,\\
&w_4 - \tfrac{1}{2}\overline{D}^* G[w_2, w_2] = h_4,\\
&\dots\\
&w_{2k} - \tfrac{1}{2}\overline{D}^* G \sum_{1\leq s\leq k-1} [w_{2s}, w_{2(k-s)}] = h_{2k},\\
&\dots
\end{aligned}$$

We prove that $w_{2k} = h_{2k}$ by induction on k. For $k = 1$ this follows from the first equation. Suppose that $w_{2i} = h_{2i}$ for $1 \leq i \leq k-1$. By Corollary 2.5 we see that

$$h' = \sum_{1\leq s\leq k-1} [w_{2s}, w_{2(k-s)}] \in \mathbf{H}.$$

Since $\overline{D}^*$ and G commute and $\overline{D}^* h' = 0$, we get $w_{2k} = h_{2k}$.

So we have proved that $w = h \in \mathbf{H}_{(1)}$. By Theorem 3, $\mathbf{K}_0 = \mathbf{H}_{(1)}$, and $\varphi = \mathrm{id}$. Therefore, $\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w = 0\}$. □

2.3 Lie superalgebras of vector fields on supermanifolds with retract $T^{m|n}$

Consider holomorphic vector fields on the split supermanifold $T^{m|n}$. It is clear that any $v \in \mathfrak{v}_p(T^{m|n})$ is a linear combination of the fields

$$\begin{aligned}
&\xi_{j_1} \dots \xi_{j_p} \partial_{z_i} \text{ for } j_1 < \dots < j_p \text{ and } i = 1, \dots, m,\\
&\xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \text{ for } j_1 < \dots < j_{p+1} \text{ and } j = 1, \dots, n \text{ (see (12))}
\end{aligned}$$

with holomorphic coefficients. Since any holomorphic function on T is constant,

$$v = \sum_{1\leq i\leq m} \sum_{j_1<\dots<j_p} a^i_{j_1,\dots,j_p} \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{1\leq j\leq n} \sum_{j_1<\dots<j_{p+1}} b^j_{j_1,\dots,j_{p+1}} \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j}, \quad (14)$$

where $a^i_{j_1,\dots,j_p}$, $b^j_{j_1,\dots,j_{p+1}} \in \mathbb{C}$. Since

$$\mathfrak{v}(T^{m|n}) = \mathbf{H}_{p,0} = \{v \in S_{p,0} \mid \overline{D} v = 0\},$$

we see that (14) is a special case of the formula (13).

By Theorem 1.3, any supermanifold with retract $T^{m|n}$ can be described by a form from the Kuranishi family $\mathbf{K}$. By Theorem 2.7, $\mathbf{K} = \mathbf{V}$ consists of harmonic elements.

2.8. Theorem. *Let $(T, \mathcal{O})$ be a supermanifold with retract $T^{m|n}$ given by an element w in* $\mathbf{V}$. *Then, the mapping τ from* (1.7) *determines an isomorphism*

$$\mathfrak{v}(T, \mathcal{O}) \to \{v \in S_{*,0} \mid \overline{D}v = [w, v] = 0\} = \{v \in \mathfrak{v}_p(T^{m|n}) \mid [w, v] = 0\}.$$

Proof. We can write $w = w_2 + w_4 + \ldots$, where $w_{2k} \in \mathbf{H}_{2k,1}$. Let $v \in S_{*,0}$ and $\overline{D}v = [w, v]$. Then, $v = v_{-1} + v_0 + v_1 + \ldots$, where $v_i \in S_{i,0}$. The equation $\overline{D}v = [w, v]$ gives the finite system of equations:

$$\begin{array}{l} \overline{D}v_{-1} = 0, \\ \overline{D}v_0 = 0, \\ \overline{D}v_1 = [w_2, v_{-1}], \\ \overline{D}v_2 = [w_2, v_0], \\ \overline{D}v_3 = [w_2, v_1] + [w_4, v_{-1}], \\ \overline{D}v_4 = [w_2, v_2] + [w_4, v_0], \\ \ldots \end{array} \tag{15}$$

Let us prove that $\overline{D}v_k = 0$ for $k = -1, 0, \ldots$, by induction on k. For $k = -1, 0$ this follows from the first and the second equations of (15). If $\overline{D}v_{-1} = \ldots = \overline{D}v_{k-1} = 0$, then, using system (15), we see that $\overline{D}v_k$ is a sum of commutators of the fields $v_{-1}, \ldots, v_{k-2}$ with the forms w_{2l}. Since $w_{2l} \in \mathbf{H}_{2l,1}$, Corollary 2.5 shows that $\overline{D}v_k \in \mathbf{H}_{p,1}$. Hence $\overline{D}v_k = 0$.

Thus, we proved that the kernel of $\overline{D}_w = \overline{D} - \operatorname{ad} w$ in $S_{*,0}$ coincides with the subalgebra $\{v \in \mathfrak{v}_p(T^{m|n}) \mid [w, v] = 0\} \subset \mathfrak{v}_p(T^{m|n})$. Now our statement follows from Theorem 1.2. □

2.4 Homogeneous supermanifolds with retract $T^{m|n}$

2.9. Theorem. *Any supermanifold $(T, \mathcal{O})$ with retract $T^{m|n}$ is $\bar{0}$-homogeneous. It is homogeneous if and only if $(T, \mathcal{O}) \simeq T^{m|n}$.*

Proof. Let $(T, \mathcal{O})$ be an arbitrary supermanifold with retract $T^{m|n}$. From Theorem 2.7 we see that it is determined by a harmonic form $w \in \mathbf{V}$. By Proposition 2.6 w is invariant under the natural action of the group T. Hence, by Theorem 1.4 the action of the group T on $T^{m|n}$ lifts to an action on $(T, \mathcal{O})$. So $(T, \mathcal{O})$ is $\bar{0}$-homogeneous.

From eq. (12) we see that the Lie superalgebra $\mathfrak{v}(T^{m|n})$ is transitive. Then, $T^{m|n}$ is homogeneous. Let $(T, \mathcal{O})$ be the supermanifold with retract $T^{m|n}$ determined by a cocycle $w \in \mathbf{V} \subset \mathbf{H}_{(1)}$. Let $(T, \mathcal{O})$ be homogeneous. Take a point $x_0 \in T$, and denote by $\xi_1, \ldots, \xi_n$ the odd local coordinates in the neighborhood U of x_0 which correspond to the coordinates $\xi_1, \ldots, \xi_n$ on $T^{m|n}$ by the local spliting $h_U : \mathcal{O}_{\mathrm{gr}}|_U \to \mathcal{O}|_U$. Since $\mathrm{ev}_{x_0} : \mathfrak{v}(T, \mathcal{O})_{\bar{1}} \to T_{x_0}(T, \mathcal{O})_{\bar{1}}$ is surjective, then for any j such that $1 \leq j \leq n$, there exists a field $v_j \in \mathfrak{v}(T, \mathcal{O})_{\bar{1}}$ such that $v_j = \partial_{\xi_j} + v'_j$ in U, where $(v'_j)_{x_0} \in m_{x_0}\mathcal{T}_{x_0}$. We can assume that the neighborhood $U = U_i$ is included into the cover (U_i) which we used in the description of τ in Subsection 1.4. As was shown in Subsection 1.4, we have

$$\tau(v_j) = a_i(h_i^{-1}\partial_{\xi_j} h_i)a_i^{-1} + (a_i h_i^{-1})v'_j(h_i a_i^{-1}).$$

The sheaf $\mathcal{S}_{*,0} = \mathcal{T}^{\infty}_{\text{gr}} = \mathcal{D}er\,\mathcal{O}^{\infty}_{\text{gr}}$ has a filtration similar to (3):

$$\mathcal{T}^{\infty}_{\text{gr}} = \mathcal{T}^{\infty}_{\text{gr}(-1)} \subset \mathcal{T}^{\infty}_{\text{gr}(0)} \subset \mathcal{T}^{\infty}_{\text{gr}(1)} \subset \dots$$

Since $a_i \in \Gamma(U_i, \mathcal{A}ut_{(2)}\mathcal{O}^{\infty}_{\text{gr}})$, we have

$$a_i(h_i^{-1}\partial_{\xi_j} h_i)a_i^{-1} = a_i\partial_{\xi_j} a_i^{-1} = \partial_{\xi_j} + u_j,$$

where $u_j \in \Gamma(U_i, \mathcal{T}^{\infty}_{\text{gr}(1)})$. Clearly,

$$(a_i h_i^{-1}) v'_j (h_i a_i^{-1}) = v''_j$$

satisfies $(v''_j)_{x_0} \in m^{\infty}_{x}(\mathcal{T}^{\infty}_{\text{gr}})_{x_0}$, where $m^{\infty}_{x_0}$ is the maximal ideal of $(\mathcal{O}^{\infty}_{\text{gr}})_{x_0}$. Hence in U_i we have

$$\tau(v_j) = \partial_{\xi_j} + u_j + v''_j,$$

where $(u_j + v''_j)_{x_0} \in m^{\infty}_{x_0}(\mathcal{T}^{\infty}_{\text{gr}})_{x_0}$. By Theorem 2.8 $\tau(v_j) \in \mathfrak{v}(T^{m|n})_{\bar{1}}$.

So, if $\tilde{v}_j = \tau(v_j) - \partial_{\xi_j} \in \mathfrak{v}(T^{m|n})_{\bar{1}}$, then $\tilde{v}_{x_0} \in m_{x_0}(\mathcal{T}_{\text{gr}})_{x_0}$, and therefore,

$$\tilde{v}_j \in \bigoplus_{k \geq 0} \mathfrak{v}(T^{m|n})_{2k+1}.$$

By Theorem 2.8

$$[w, \partial_{\xi_j} + \tilde{v}_j] = 0 \quad \text{for} \quad j = 1, \dots, n. \tag{16}$$

We write w as $w = w_2 + w_4 + \dots$, where $w_{2k} \in \mathbf{H}_{2k,1}$, and prove that $w_{2k} = 0$ for $k = 1, 2, \dots$, by induction on k.

Considering the component of degree 1 of the left part of formula (16), we see that $[w_2, \partial_{\xi_j}] = 0$. Hence,

$$\begin{aligned} &[w_2, \partial_{\xi_j}](z_r) = \partial_{\xi_j}(w_2(z_r)) = 0 \quad \text{for } r = 1, \dots, m, \\ &[w_2, \partial_{\xi_j}](\xi_s) = \partial_{\xi_j}(w_2(\xi_s)) = 0 \quad \text{for } s = 1, \dots, n. \end{aligned}$$

From (13) we see that $w_2 = 0$.

Suppose we have proved that $w_2 = w_4 = \dots = w_{2k-2} = 0$. Considering the component of degree $2k-1$ of the left part of the formula (16), we get $[w_{2k}, \partial_{\xi_j}] = 0$ for all $j = 1, \dots, n$. As above, we can prove that $w_{2k} = 0$.

Thus, $w = 0$. Hence, $(T, \mathcal{O}) \simeq T^{m|n}$. □

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