Communications in Mathematics **1** (2022), no. 30, 201–228
DOI: https://doi.org/10.46298/cm.9895

Chebyshev-quasilinearization method for solving fractional singular nonlinear Lane-Emden equations

Amir Mohammadi, Ghader Ahmadnezhad, Nasser Aghazadeh

Abstract. In this paper, we propose a method for solving some classes of the singular fractional nonlinear Lane-Emden type equations. The method is proposed by utilizing the second-kind Chebyshev wavelets in conjunction with the quasilinearization technique. The operational matrices for the second-kind Chebyshev wavelets are used. The method is tested on the fractional standard Lane-Emden equation, the fractional isothermal gas spheres equation, and some other examples. We compare the results produced by the present method with some well-known results to show the accuracy and efficiency of the method.

1 Introduction

Fractional ordinary and partial differential equations have found many applications in many physical, chemical and engineering problems. These equations provide a better description than the integer order of derivatives due to having a fractional derivative for describing fluid mechanics and viscoelastic theory. The most important advantage of fractional derivatives in describing physical phenomena is the more precise modeling that is ignored for the integer order derivatives of these cases. Modeling and mathematical simulation of physical phenomena and processes, based on their characteristics, leads to the

MSC 2020: 34A08, 33F05, 65L60
Keywords: Fractional differential equation, Singular Lane-Emden equation, Numerical solution, Second-kind Chebyshev wavelets, Quasilinearization.
Affiliation:
Amir Mohammadi – Department of applied mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
E-mail: amirmohammadi1945@yahoo.com
Ghader Ahmadnezhad – Department of applied mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
E-mail: hazharahmadnezhad@gmail.com
Nasser Aghazadeh – Department of applied mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
E-mail: aghazadeh@azaruniv.ac.ir

creation of fractional differential equations and the necessity of solving such equations, but the important point is that most of the fractional equations do not always have well-known exact solutions. There are a variety of numerical methods that provide approximate solutions for these equations, such as the Adomian Decomposition Method (ADM) [5], [42], [57], the Homotopy Perturbation Method (HPM) [27], [33], the Variational Iteration Method (VIM) [12], [32], the generalized Differential Transform Method (DTM) [11], [35], and collocation methods [2], [43]. In the meantime, spectral methods have been widely used for numerical solutions of fractional differential equations due to excellent error properties. The collocation method, the Galerkin, and Tau methods are three commonly used methods in the spectral scheme. Collocation methods have successfully been used to simulate numerically many problems in science and engineering, see [44], [48], [64]. In recent years, especially in the last two decades, the application of wavelets has greatly expanded in solving the fractional differential equations [52], [57], [63], [64]. Recently, the operational matrices of Chebyshev, Legendre, and Haar wavelets have been used in numerically solving many of the fractional differential equations [12], [38], [48], [52], [54], [61], [63].

Many problems arising in the field of mathematical physics and astrophysics can be modeled by the Lane-Emden type initial value problems. In this work, we consider following form of the fractional singular nonlinear Lane-Emden equations:

$$D_x^{\alpha}u(x) + \frac{2}{x}D_x^{\beta}u(x) + f(x,u(x)) = 0, \quad x > 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \tag{1}$$

with initial conditions:

$$u(0) = A, \qquad u^{'}(0) = B, \tag{2}$$

where A and B are constants, $f(x,u)$ is the nonlinear function of u, x and u are the independent and dependent variables respectively. For $\alpha = 2$ and $\beta = 1$, we have classical Lane-Emden type equations which are nonlinear ordinary differential equations which are categorized as singular initial value problems. The Lane-Emden equation was first studied by astrophysicists Jonathan Homer Lane and Robert Emden, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [7], [26], [58], [60]. Due to the presence of a singularity in $x = 0$, solving such equations is associated with difficulties. It should be mentioned that for non-fractional equation there is a proof that one can obtain analytic solution around fixed-singularity $x = 0$ [25]. There are methods for solving equation (1) with (2), such as the collocation methods [39], [40], tangent chord method [9], finite difference methods [4], [17], spline finite difference methods [16], B-Spline method [18], spline method [24], Chebyshev economization method [21], Cubic spline method [20], [22], Adomian decomposition method [10], [14], Adomian decomposition method with Green's function [49], [50], variational iteration method [59], [60], the optimal variational iteration method [47], homotopy analysis method [6] and the references cited therein.

Wavelets, as a well-known base set, are used to solve fractional differential equations; the use of wavelets, especially orthogonal wavelets, has been widely used to solve differential equations in the last two decades [29], [53]. The second-kind Chebyshev wavelets have been very much considered due to their useful properties and their ability to solve different types of fractional differential equations [64]. Babolian and Fattahzadeh use the Chebyshev wavelet operational matrix for numerical solution of differential equations [1], Dehghan use the Chebyshev finite difference methods for Fredholm integro-differential equation [8], Zhu and Fan solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet [62] and Iqbal and et all use the Chebyshev wavelets method for solving fractional differential equations [15].

The aim of this work is applying the second-kind Chebyshev wavelet collocation method combined with the quasilinearization technique for solving fractional differential equations with a singularity at the point $x = 0$. The quasilinearization technique was introduced by Bellman and Kalaba [19]. This technique indeed is a generalization of the Newton-Raphson method to solve nonlinear ordinary and partial differential equations. In this work, we convert the nonlinear singular Lane-Emden equation to a linear equation, then we solve this linear equation by the second-kind Chebyshev wavelet collocation method. Operational matrices of fractional integration are utilized to obtain approximate solutions. We compare our approximate solutions with other results introduced in [13], [23], [31], [34], [41], [45], [48], [54], [61].

2 Preliminaries and some notations

2.1 Fractional integral and derivative

In this section, we present some definitions, notations and preliminaries of the fractional calculus theory which will be used in this work [37].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function is defined as:

$$J^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau, \qquad \alpha, t > 0,$$
$$J^0 u(t) = u(t).$$

The properties of the operator J^{α} are given as follows:

(i) $J^{\alpha}J^{\beta}u(t) = J^{\alpha+\beta}u(t)$,

(ii) $J^{\beta}J^{\alpha}u(t) = J^{\alpha}J^{\beta}u(t)$,

(iii) $J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$.

Definition 2.2. The fractional derivative of $u(t)$ in the Caputo sense is defined as:

$$D^{\alpha}u(t) = \begin{cases} \frac{d^r u(t)}{dt^r} & \alpha = r \in N; \\ \frac{1}{\Gamma(r-\alpha)}\int_0^t \frac{u^{(r)}}{(1-\tau)^{\alpha-r+1}}d\tau, & 0 \leq r-1 < \alpha < r, \end{cases}$$

It be noticed that $u^{(r)}$ is the integer order differentiation of $u(t)$.

For instance, $u(t) = t^3$ we want to calculate of $D^{1.5}u(t)$, so $1 < \alpha < 2$ then $r = 2$,

$$D^{1.5}t^3 = \frac{1}{\Gamma(2-1.5)} \int_0^t \frac{6t}{(t-\tau)^{0.5}} d\tau = \frac{12t^{\frac{3}{2}}}{\sqrt{\pi}}$$

2.2 The second-kind of Chebyshev wavelets

The second-kind of Chebyshev wavelets $\psi_{n,m}(t) = \psi(k,n,m,t)$ have four arguments k, m, n, t, where k can assume any positive integer, $n = 1, 2, \ldots, 2^{k-1}$, m is the degree of the second-kind Chebyshev polynomials and t is the time. They are defined on the interval $[0, 1)$ as:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} U_m(2^k t - 2n + 1) & \frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}} \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

where $U_m(t)$'s are the second-kind Chebyshev polynomials of degree m which are orthogonal with respect to the weight function $w(t) = \sqrt{1-t^2}$ on the interval $[-1, 1]$ and satisfy the following recursive formula:

$$\begin{aligned} U_0(t) = 1, \quad U_1(t) = 2t, \\ U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \qquad m = 1, 2, 3, \ldots \end{aligned}$$

The weight function $\tilde{w}(t) = w(2t-1)$ has to be dilated and translated as

$$w_n(t) = w(2^k t - 2n + 1).$$

A function $f(x) \in L_2(R)$ defined over $[0, 1)$ can be expanded by the second-kind Chebyshev wavelets as:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \psi_{n,m}(x), \tag{4}$$

where

$$c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle.$$

If the infinite series in Eq (4) is truncated, then it can be written as:

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = C^T \Psi(x), \tag{5}$$

which the coefficient vector C and the second-kind Chebyshev wavelet function vector $\Psi(x)$ are $m' = 2^{k-1}M$ column vectors. For simplicity, Eq (5) can be written as:

$$f(x) \cong \sum_{i=1}^{m'} c_i \psi_i = C^T \Psi(x), \qquad \text{where } c_i = c_{n,m} \text{ and } \psi_i(t) = \psi_{n,m}(t). \tag{6}$$

The index i can be determined by the relation $i = M(n-1)+m+1$; thus, we have:

$$C = [c_1, c_2, c_3, \dots, c_{m'}]^T \qquad \text{and} \qquad \Psi(t) = [\psi_1, \psi_2, \psi_3, \dots, \psi_{m'}]^T.$$

By taking the collocation points as following $x_i = \frac{2i-1}{2^k M}, i = 1, 2, 3, \dots, 2^{k-1}M$, we define the second-kind Chebyshev wavelets matrix $\Phi(x)_{m'\times m'}$ as:

$$\Phi_{m'\times m'} = \Big[\Psi(\frac{1}{2m'}), \Psi(\frac{3}{2m'}), \dots, \Psi(\frac{2m'-1}{2m'})\Big],$$

where $m' = 2^{k-1}M$. For example, when $M = 4$ and $k = 2$, the second-kind Chebyshev wavelets matrix is expressed as:

$$\Phi_{8\times 8} = \begin{pmatrix} 1.5958 & 1.5958 & 1.5958 & 1.5958 & 0 & 0 & 0 & 0 \\ -2.3937 & -2.3937 & -2.3937 & -2.3937 & 0 & 0 & 0 & 0 \\ 1.9947 & 1.9947 & 1.9947 & 1.9947 & 0 & 0 & 0 & 0 \\ -0.5984 & -0.5984 & -0.5984 & -0.5984 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.5958 & 1.5958 & 1.5958 & 1.5958 \\ 0 & 0 & 0 & 0 & -2.3937 & -2.3937 & -2.3937 & -2.3937 \\ 0 & 0 & 0 & 0 & 1.9947 & 1.9947 & 1.9947 & 1.9947 \\ 0 & 0 & 0 & 0 & -0.5984 & -0.5984 & -0.5984 & -0.5984 \end{pmatrix}.$$

2.3 The fractional integral of the second-kind Chebyshev wavelets

In this section, a fractional integral formula of the Chebyshev wavelets in the Riemann-Liouville sense is derived by means of the shifted second-kind Chebyshev polynomials U_m^*, which plays an important role in dealing with the time fractional equations.

Theorem 2.3. *The fractional integral of a Chebyshev wavelet defined on the interval* $[0,1]$ *with compact support* $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$ *is given by:*

$$I^\alpha \psi_{n,m}(x) = \begin{cases} 0, & x < \frac{2n-2}{2^k} \\ 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\Bigg[\displaystyle\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r} \frac{2^i i!(i+m+1)!(-1)^{i+j+r}2^{k(r-j)}}{(2i+1)!j!(m-i)!(i-r)!\Gamma(\alpha-j+r+1)} & \\ \times\big(x-\frac{2n-2}{2^k}\big)^{\alpha-j+r}\Bigg], & \frac{2n-2}{2^k} \le x \le \frac{2n}{2^k} \\ 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\Bigg[\displaystyle\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r} \frac{2^i(-1)^{i+r}i!(i+m+1)!2^{k(r-j)}}{(2i+1)!j!(m-i)!(i-r)!\Gamma(\alpha-j+r+1)} & \\ \times\Big((-1)^j\big(x-\frac{2n-2}{2^k}\big)^{\alpha-j+r} - \big(x-\frac{2n}{2^k}\big)^{\alpha-j+r}\Big)\Bigg], & x > \frac{2n}{2^k} \end{cases} \tag{7}$$

Proof. The general form of the second kind Chebyshev polynomials is:

$$U_m(x) = \sum_{i=0}^{m}\sum_{r=0}^{i} \frac{(-1)^{i+r}2^i(m+i+1)!i!}{r!(i-r)!(m-i)!(2i+1)!}x^r. \tag{8}$$

We now derive the operator I^α for $\Psi(t)$

$$I^\alpha \Psi(x) = P_x^\alpha.$$

To obtain $I^\alpha \psi_{n,m}(t)$, we use the Laplace transform. We get the following relation for the second kind Chebyshev wavelets

$$\psi_{n,m}(x) = 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} \left(\nu_{\frac{2n-2}{2^k}}(x) U_m(2^k x - (2n-1)) - \nu_{\frac{2n}{2^k}}(x) U_m(2^k x - (2n-1)) \right), \tag{9}$$

where $\nu_c(x)$ is the unit step function defined as

$$\nu_c(x) = \begin{cases} 1, & x \geq c, \\ 0, & x < c. \end{cases}$$

By taking the Laplace transform from Eq (9), we get

$$\begin{aligned} \mathcal{L}\{\psi_{n,m}(x)\} = {} & 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} \mathcal{L}\Bigg\{ \nu_{\frac{2n-2}{2^k}}(x) U_m(2^k(x - \frac{2n-2}{2^k}) - 1) \\ & - \nu_{\frac{2n}{2^k}}(t) U_m(2^k(x - \frac{2n}{2^k}) + 1) \Bigg\} \\ = {} & 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} e^{-\frac{2n-2}{2^k}s} \mathcal{L}\Big\{ U_m(2^k x - 1) \Big\} \\ & - 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} e^{-\frac{2n}{2^k}s} \mathcal{L}\Big\{ U_m(2^k x + 1) \Big\}. \end{aligned} \tag{10}$$

From the definition of $U_m(x)$ in Eq (8), we have

$$\begin{aligned} \mathcal{L}\{\psi_{n,m}(x)\} = {} & 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} e^{-\frac{2n-2}{2^k}s} \mathcal{L}\Big\{ \sum_{i=0}^{m} \sum_{r=0}^{i} \frac{(-1)^{i+r} 2^i (m+i+1)! i!}{r!(i-r)!(m-i)!(2i+1)!} (2^k x - 1)^r \Big\} \\ & - 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} e^{-\frac{2n}{2^k}s} \mathcal{L}\Big\{ \sum_{i=0}^{m} \sum_{r=0}^{i} \frac{(-1)^{i+r} 2^i (m+i+1)! i!}{r!(i-r)!(m-i)!(2i+1)!} (2^k x + 1)^r \Big\}. \end{aligned}$$

For simplicity, let $T^{i,m,r} = \frac{(-1)^{i+r}2^i(m+i+1)!i!}{r!(i-r)!(m-i)!(2i+1)!}$, and we get

$$\begin{aligned}
\mathcal{L}\{\psi_{n,m}(x)\} &= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}e^{-\frac{2n-2}{2^k}s}\mathcal{L}\Big\{\sum_{i=0}^{m}\sum_{r=0}^{i}T^{i,m,r}(2^kx-1)^r\Big\} \\
&\quad - 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}e^{-\frac{2n}{2^k}s}\mathcal{L}\Big\{\sum_{i=0}^{m}\sum_{r=0}^{i}T^{i,m,r}(2^kx+1)^r\Big\} \\
&= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}e^{-\frac{2n-2}{2^k}s}\mathcal{L}\Big\{\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}T^{i,m,r}\frac{r!}{j!(r-j)!}(-1)^j2^{k(j-r)}x^{r-j}\Big\} \\
&\quad - 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}e^{-\frac{2n}{2^k}s}\mathcal{L}\Big\{\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}T^{i,m,r}\frac{r!}{j!(r-j)!}2^{k(r-j)}x^{r-j}\} \\
&= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}e^{-\frac{2n}{2^k}s}\left\{\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}\frac{r!}{j!}T^{i,m,r}2^{k(r-j)}(e^{\frac{2s}{2^k}}(-1)^j-1)\frac{1}{s^{r-j+1}}\right\}.
\end{aligned}$$

By using the Riemann-Liouville fractional integral operator of order α:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1} * f(x),$$

where $x^{\alpha-1} * f(x)$ is convolution product of $x^{\alpha-1}$ and $f(x)$, we get

$$\begin{aligned}
\mathcal{L}\{I^\alpha\psi_{n,m}(x)\} &= \mathcal{L}\{\frac{x^{\alpha-1}}{\Gamma(\alpha)}\}\mathcal{L}\{\psi_{n,m}(x)\} \\
&= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}e^{-\frac{2n}{2^k}s}\left\{\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}\frac{r!}{j!}T^{i,m,r}2^{k(r-j)}(e^{\frac{2s}{2^k}}(-1)^j-1)\frac{1}{s^{r-j+1+\alpha}}\right\}. \quad (11)
\end{aligned}$$

Taking the inverse Laplace transform of Eq (11) yields

$$\begin{aligned}
I^\alpha\psi_{n,m}(x) &= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\left\{\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}T^{i,m,r}\frac{r!}{j!}2^{k(r-j)}\mathcal{L}^{-1}\left\{\frac{e^{-\frac{2n-2}{2^k}s}}{s^{r-j+1+\alpha}}(-1)^j\right.\right. \quad (12) \\
&\quad \left.\left.-\frac{e^{-\frac{2n}{2^k}s}}{s^{r-j+1+\alpha}}\right\}\right\} \\
&= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\left\{\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}T^{i,m,r}\frac{r!}{j!}2^{k(r-j)}\left(\frac{\nu_{\frac{2n-2}{2^k}}(-1)^j(x-\frac{2n-2}{2^k})^{r-j+\alpha}}{\Gamma(r-j+1+\alpha)}\right.\right. \\
&\quad \left.\left.-\frac{\nu_{\frac{2n}{2^k}}(x-\frac{2n}{2^k})^{r-j+\alpha}}{\Gamma(r-j+1+\alpha)}\right)\right.
\end{aligned}$$

By using Eq (12), we have

$$I^{\alpha}\psi_{n,m}(x)=\begin{cases}0, & x<\frac{2n-2}{2^k}\\ 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\Bigg[\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}\frac{2^i i!(i+m+1)!(-1)^{i+j+r}2^{k(r-j)}}{(2i+1)!j!(m-i)!(i-r)!\Gamma(\alpha-j+r+1)} & \\ \quad\times\left(x-\frac{2n-2}{2^k}\right)^{\alpha-j+r}\Bigg], & \frac{2n-2}{2^k}\le x\le\frac{2n}{2^k}\\ 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\Bigg[\sum_{i=0}^{m}\sum_{r=0}^{i}\sum_{j=0}^{r}\frac{2^i(-1)^{i+r}i!(i+m+1)!2^{k(r-j)}}{(2i+1)!j!(m-i)!(i-r)!\Gamma(\alpha-j+r+1)} & \\ \quad\times\left((-1)^j\left(x-\frac{2n-2}{2^k}\right)^{\alpha-j+r}-\left(x-\frac{2n}{2^k}\right)^{\alpha-j+r}\right)\Bigg], & x>\frac{2n}{2^k}.\end{cases}\tag{13}$$

The proof is completed. □

For instance, for $k=2, M=4, x=0.6, \alpha=0.9$, we obtain

$$I^{0.9}\Psi_{8\times1}(0.6)=\begin{pmatrix}0.838817891721642\\0.045706956934399\\0.290734994150959\\0.021626272477045\\0.208881853762857\\-0.329813453309774\\0.323368822612918\\-0.217309447751042\end{pmatrix},$$

where $\Psi_{8\times1}=(\psi_{1,0}(x),\psi_{1,1}(x),\psi_{1,2}(x),\psi_{1,3}(x),\psi_{2,0}(x),\psi_{2,1}(x),\psi_{2,2}(x),\psi_{2,3}(x))^T$.

We can obtain the fractional order integration matrix $P^{\alpha}_{m'\times m'}=I^{\alpha}\psi_{n,m}(x)$ by substituting the collocation points in Eq (13) as

$$P^{\alpha}_{2^{k-1}M\times2^{k-1}M}=\begin{pmatrix}I^{\alpha}\psi_{1,0}(x(1)) & I^{\alpha}\psi_{1,0}(x(2)) & \dots & I^{\alpha}\psi_{1,0}(x(2^{k-1}M))\\ I^{\alpha}\psi_{1,1}(x(1)) & I^{\alpha}\psi_{1,1}(x(2)) & \dots & I^{\alpha}\psi_{1,1}(x(2^{k-1}M))\\ \vdots & \vdots & \ddots & \vdots\\ I^{\alpha}\psi_{2^{k-1},M}(x(1)) & I^{\alpha}\psi_{2^{k-1},M}(x(2)) & \dots & I^{\alpha}\psi_{2^{k-1},M}(x(2^{k-1}M))\end{pmatrix}.$$

For instance, we fix $k = 2, M = 4$ and $\alpha = 0.9$, then we have:

$$P^{0.9}_{8\times 8} = \begin{pmatrix} 0.1368 & 0.3678 & 0.5825 & 0.7885 & 0.8517 & 0.8165 & 0.7939 & 0.7771 \\ -0.2377 & -0.4452 & -0.3985 & -0.1245 & 0.0545 & 0.0337 & 0.0246 & 0.0194 \\ 0.2789 & 0.2423 & 0.0032 & 0.0615 & 0.2996 & 0.2783 & 0.2680 & 0.2612 \\ -0.2570 & -0.0232 & -0.0530 & -0.2259 & 0.0274 & 0.0148 & 0.0104 & 0.0081 \\ 0 & 0 & 0 & 0 & 0.1368 & 0.3678 & 0.5825 & 0.7885 \\ 0 & 0 & 0 & 0 & -0.2377 & -0.4452 & -0.3985 & -0.1245 \\ 0 & 0 & 0 & 0 & 0.2789 & 0.2423 & 0.0032 & 0.0615 \\ 0 & 0 & 0 & 0 & -0.2570 & -0.0232 & -0.0530 & -0.2259 \end{pmatrix}.$$

3 Procedure of implementation

In this section, we describe our method to solve the singular and nonlinear fractional Lane-Emden equations. We first convert nonlinear equations to linear equations by the quasilinearization technique, then using the second-kind Chebyshev wavelets collocation method to solve the equations obtained in the previous step. We describe the procedure of implementation in more details, which enable the readers to understand the method more effectively.

Consider the following form of Lane-Emden equation:

$$D^{\alpha}_{x} u(x) + \frac{2}{x} D^{\beta}_{x} u(x) + f(x, u(x)) = 0, \quad x > 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \tag{14}$$

with initial conditions:

$$u(0) = A, \qquad u^{'}(0) = B, \tag{15}$$

for applying the quasilinearization technique [3], [30], let an initial approximation of the function $u(x)$, for this we use $u_0(x)$ as initial approximation, it may be $u_0(x) = A$. The function f can now be expanded around the function $u_0(x)$ by using of the Taylor series expansion:

$$f(x, u(x)) = f(x, u_0(x)) + (u(x) - u_0(x))(f_{u_0}(x, u_0(x))), \tag{16}$$

in series expansion we ignore the second and other higher terms. Using Eq (16) in Eq (14):

$$D^{\alpha}_{x} u(x) + \frac{2}{x} D^{\beta}_{x} u(x) + f(x, u_0(x)) + (u(x) - u_0(x))(f_{u_0}(x, u_0(x))) = 0, \tag{17}$$

solving Eq (17) for $u(x)$, we obtain it and call $u_1(x)$. By continuing this process, we can get $u_2(x)$. Then, recurrence relation is:

$$D^{\alpha}_{x} u_{r+1}(x) + \frac{2}{x} D^{\beta}_{x} u_{r+1}(x) + f(x, u_r(x)) + (u_{r+1}(x) - u_r(x))(f_{u_r}(x, u_r(x))) = 0, \tag{18}$$

where $u_r(x)$ is known and we can obtain $u_{r+1}(x)$. Eq (18) is a linear equation with the following conditions:

$$u_{r+1}(0) = A, \qquad u'_{r+1}(0) = B. \tag{19}$$

Applying the Chebyshev wavelet method in Eq (18), we approximate the higher order derivative term by Chebyshev wavelet series as:

$$D_x^{\alpha} u_{r+1}(x) \approx D_x^{\alpha} u_{r+1}^{w}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} \psi_{n,m}(x). \tag{20}$$

Lower order derivatives are obtained by integrating (20) and use of initial conditions (19), we get:

$$u_{r+1}(x) \approx u_{r+1}^{w}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} P_x^{\alpha} \psi_{n,m}(x) + Bx + A, \tag{21}$$

$$D_x^{\beta} u_{r+1}(x) \approx D_x^{\beta} u_{r+1}^{w}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} P_x^{\alpha-\beta} \psi_{n,m}(x) + B \frac{x^{1-\beta}}{\Gamma(2-\beta)}. \tag{22}$$

Using Eqs ((21),(22),(20)) in Eq (18), we obtain:

$$\begin{aligned} &\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} \psi_{n,m}(x) + \frac{2}{x}\Big(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} P_x^{\alpha-\beta} \psi_{n,m}(x) + B \frac{x^{1-\beta}}{\Gamma(2-\beta)} \Big) \\ &\qquad + f(x, u_r(x)) + \Big((\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} P_x^{\alpha} \psi_{n,m}(x) \\ &\qquad + Bx + A) - u_r(x) \Big) (f_{u_r}(x, u_r(x))) \approx 0. \end{aligned} \tag{23}$$

By using collocation points $x_i = \frac{i-0.5}{2^{k-1}M}$, replacing $\approx$ by $=$, and solving this linear system for c^{r+1}, which is a coefficients vector, and substituting c^{r+1} in Eq (21), we get solution $u_{r+1}(x)$ at the collocation points. Suppose $u_0(x) = A$ as an initial approximation, we get a linear fractional differential equation in $u_1(x)$ by substituting $r = 0$ in Eq (18), where is solved by above procedure. Similarly for $r = 1$ we obtain $u_2(x)$ and so on.

4 Convergence of Chebyshev Wavelet Quasilinearization Method

We derive an error estimate of the Chebyshev wavelet quasilinearization approximations to an arbitrary unknown function.

Theorem 4.1. *Let $r, k, M \to \infty$, then the series solution*

$$u(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x)$$

converges to u(x).

Proof. Let $L^2[0,1)$ be the Hilbert space and $\psi_{n,m}$ forms a basis of $L^2[0,1)$. Let us consider

$$u_{r+1}(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} \psi_{n,m}(x), \tag{24}$$

where $c_{n,m}^{r+1} = \langle u_{r+1}(x), \psi_{n,m}(x) \rangle$. Let $S_{k,M}^{r+1}$ be a sequence of partial sums of $c_{n,m}^{r+1}\psi_{n,m}(x)$, we prove that $S_{k,M}^{r+1}$ is a Cauchy sequence in $L^2[0,1)$ and then we show that $S_{k,M}^{r+1}$ converges to u_{r+1}, when $k, M \to \infty$.

We show that $S_{k,M}^{r+1}$ is a Cauchy sequence. Let $S_{k,M}^{r+1}$ be arbitrary sums of $c_{n,m}^{r+1}\psi_{n,m}(x)$ with $k > k', M > M'$.

$$\begin{aligned} \|S_{k,M}^{r+1} - S_{k',M'}^{r+1}\|^2 &= \Big\| \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1}\psi_{n,m}(x) - \sum_{n=1}^{2^{k'-1}} \sum_{m=0}^{M'-1} c_{n,m}^{r+1}\psi_{n,m}(x) \Big\|^2 \\ &= \Big\| \sum_{n=2^{k'-1}+1}^{2^{k-1}} \sum_{m=M'}^{M-1} c_{n,m}^{r+1}\psi_{n,m}(x) \Big\|^2 \\ &= \Big\langle \sum_{n=2^{k'-1}+1}^{2^{k-1}} \sum_{m=M'}^{M-1} c_{n,m}^{r+1}\psi_{n,m}(x), \sum_{i=2^{k'-1}+1}^{2^{k-1}} \sum_{j=M'}^{M-1} c_{i,j}^{r+1}\psi_{i,j}(x) \Big\rangle \\ &= \sum_{n=2^{k'-1}+1}^{2^{k-1}} \sum_{m=M'}^{M-1} \sum_{i=2^{k'-1}+1}^{2^{k-1}} \sum_{j=M'}^{M-1} c_{n,m}^{r+1} c_{i,j}^{\prime r+1} \big\langle \psi_{n,m}(x), \psi_{i,j}(x) \big\rangle \\ &= \sum_{n=2^{k'-1}+1}^{2^{k-1}} \sum_{m=M'}^{M-1} |c_{n,m}^{r+1}|^2. \end{aligned} \tag{25}$$

From the Bessel's inequality, we have $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} |c_{n,m}^{r+1}|^2$ that is convergent and

$$\|S_{k,M}^{r+1} - S_{k',M'}^{r+1}\|^2 \to 0,$$

when $k, k', M, M' \to \infty$. This implies that $S_{k,M}^{r+1}$ is a Cauchy sequence and it converges to, say, $y_{r+1}(x) \in L^2[0,1)$. We need to show that $u_{r+1}(x) = y_{r+1}(x)$,

$$\begin{aligned} \langle y_{r+1}(x) - u_{r+1}(x), \psi_{n,m}(x) \rangle &= \langle y_{r+1}(x), \psi_{n,m}(x) \rangle - \langle u_{r+1}(x), \psi_{n,m}(x) \rangle \\ &= \lim_{k,M \to \infty} \langle S_{k,M}^{r+1}, \psi_{n,m}(x) \rangle - c_{n,m}^{r+1} \\ &= c_{n,m}^{r+1} - c_{n,m}^{r+1} \\ &= 0. \end{aligned} \tag{26}$$

Hence $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1}\psi_{n,m}(x)$ converges to $u_{r+1}(x)$ as $k, M \to \infty$. Now we show that $u_{r+1}(x) \to u(x)$, when $r \to \infty$. According to the convergence of quasilinearization technique [3], we have

$$\max_x |u_{r+1} - u_r| \leq \frac{\frac{s}{8}}{1 - \frac{p}{4}} (\max_x(|u_r - u_{r-1}|))^2, \tag{27}$$

where s, p are positive finite constants and are given in [3]. From Eq (27), we conclude that $u_{r+1}(x) \to u(x)$, when $r \to \infty$, if there is convergence at all. $\square$

5 Numerical results and examples

In this section, we implement the CWCQM (Chebyshev wavelets collocation quasi-linearization method) as discussed in section (3) to some of the singular and nonlinear fractional Lane-Emden differential equations. We define the maximum absolute error of L_∞:

$$L_\infty = \max |u_{Exact}(x) - u_{CWCQM}(x)|, \qquad x \in [0,1]. \tag{28}$$

Example 5.1. Consider the fractional standard Lane-Emden equation that is used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [54], [61], [41], [45]:

$$D_x^\alpha u(x) + \frac{2}{x} D_x^\beta u(x) + u(x)^n = 0, \tag{29}$$

with initial conditions:

$$u(0) = 1, u^{'}(0) = 0.$$

Exact solutions for $n = 0, 1, 5$, when $\alpha = 2, \beta = 1$ are given in [36].

Case(1):

If $n = 1$, we have:

$$D_x^\alpha u(x) + \frac{2}{x} D_x^\beta u(x) + u(x) = 0, \tag{30}$$

with initial conditions, $u(0) = 1, u^{'}(0) = 0$. Exact solution for $\alpha = 2, \beta = 1$ is $u(x) = \frac{\sin(x)}{x}$.

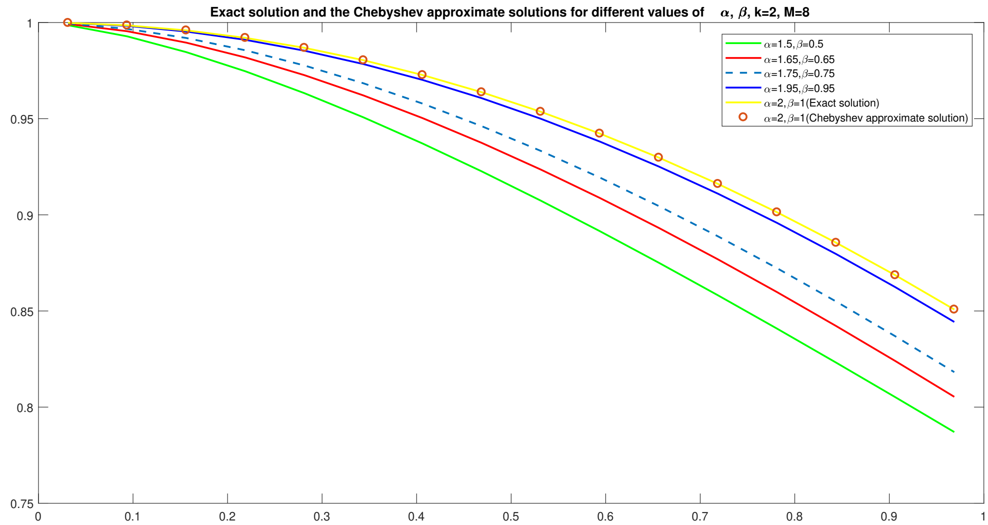
We implemented the presented method on Eq (30). We plot in Figure 1a Chebyshev approximate solutions for different values of α, β, as can be seen, when the values α, β tend to $2, 1$ respectively, the approximate solutions approach to the exact solution. Figure 1b shows that the approximate solution obtained using the method described above is a good approximation of the exact solution of the Eq (30). Table 1 shows the comparison of the error analysis for various methods, such as Adomian Decomposition Method (ADM), Haar Wavelet Collocation Method (HWCM), Haar Wavelet Adaptive Grid Method (HWAGM), Haar Wavelet Collocation Adomian Method (HWCAM). According to the Table 1, we conclude that results produced by the present method are better than the other methods results.

Case(2):

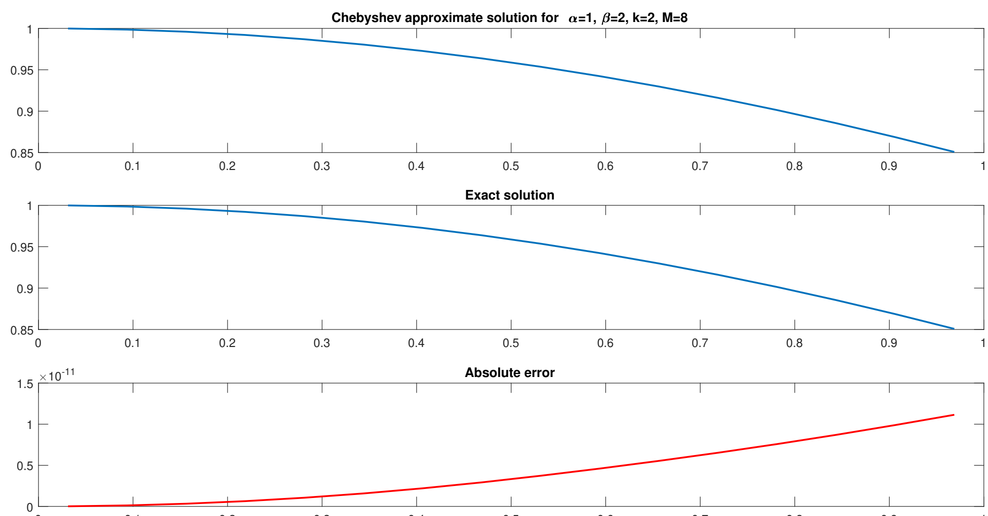
If $n = 5$, we have Eq (29) as:

$$D_x^\alpha u(x) + \frac{2}{x} D_x^\beta u(x) + u(x)^5 = 0, \tag{31}$$

with initial conditions, $u(0) = 1, u^{'}(0) = 0$. Exact solution for the case where $\alpha = 2$ and $\beta = 1$ is $u(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}$.



(a)



(b)

Figure 1: (1a) u_{CWCQM} for different values of α, β with $k = 2, M = 8$, (1b) Chebyshev approximate solution, exact solution and absolute error for $\alpha = 2, \beta = 1, k = 2, M = 8$ in Example (5.1) $n = 1$.

N	E_{ADM}[54] L_∞	E_{HWCM}[45] L_∞	E_{HWAGM}[23] L_∞	E_{HWCAM}[41] L_∞	E_{CWCQM} L_∞
8	$6.4356e-03$	$1.8562e-05$	$5.0856e-06$	$7.2156e-05$	$6.7870e-08$
16	$7.3373e-03$	$5.0012e-06$	$1.0012e-06$	$4.3274e-05$	$1.1146e-11$
32	$7.8221e-03$	$1.2932e-06$	$3.0938e-07$	$1.0015e-05$	$9.6485e-12$
64	$8.0733e-03$	$3.2854e-07$	$9.1857e-08$	$9.7345e-06$	$7.2153e-12$

Table 1: Comparison of the error analysis in Example 5.1, $n = 1$, $\alpha = 2, \beta = 1$

The implementation of the Chebyshev wavelets collocation quasilinearization method (CWCQM) to Eq (31) is as follows:

Let us assume that:

$$f(x, u(x)) = u(x)^5, \tag{32}$$

we apply quasilinearization technique in Eq (32) as follows:

$$f(x, u_{r+1}(x)) = u_r(x)^5 + (u_{r+1}(x) - u_r(x)) f'_{u_r(x)}(x, u_r(x)),$$

so:

$$f(x, u_{r+1}(x)) = u_r(x)^5 + (u_{r+1}(x) - u_r(x))(5u_r(x)^4), \tag{33}$$

by putting Eq (33) in Eq (31), we obtain:

$$D_x^\alpha u_{r+1}(x) + \frac{2}{x} D_x^\beta u_{r+1}(x) + u_r(x)^5 + (u_{r+1}(x) - u_r(x))(5u_r(x)^4) = 0, \tag{34}$$

for solving Eq (34) by the Chebyshev wavelet collocation method, we assume that:

$$D_x^\alpha u_{r+1}(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} \psi_{n,m}(x). \tag{35}$$

Lower order derivatives are obtained by integrating Eq (35) and use of initial conditions, we get:

$$D_x^\beta u_{r+1}(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} P_x^{\alpha-\beta} \psi_{n,m}(x) + B \frac{x^{1-\beta}}{\Gamma(2-\beta)}, \tag{36}$$

and

$$u_{r+1}(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{r+1} P_x^{\alpha} \psi_{n,m}(x) + Bx + A, \tag{37}$$

using Eqs ((37),(36),(35)) in Eq (34), replacing $\approx$ by $=$, and using collocation points $x_j = \frac{j-0.5}{2^{k-1}M}, j = 1, 2, \ldots, 2^{k-1}M$ to obtain:

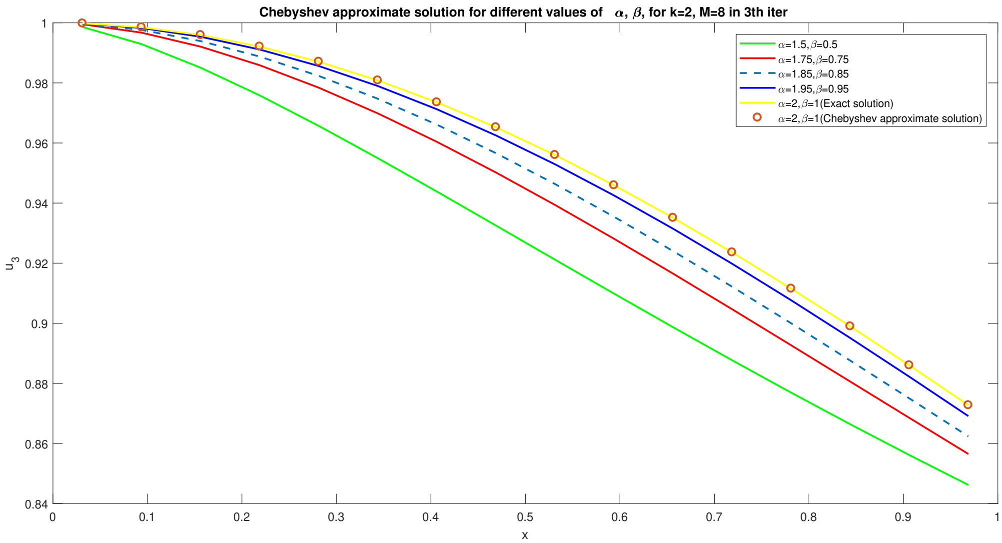
$$\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}^{r+1}\psi_{n,m}(x_j) + \frac{2}{x_j}\Big(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}^{r+1}P_{x_j}^{\alpha-\beta}\psi_{n,m}(x_j) + B\frac{x_j^{1-\beta}}{\Gamma(2-\beta)}\Big) + u_r(x_j)^5 + \Big((\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}^{r+1}P_{x_j}^{\alpha}\psi_{n,m}(x_j) + Bx_j + A) - u_r(x_j)\Big)(5u_r(x_j)^4) = 0. \quad (38)$$

For $r = 0$, and using initial approximation $u_0(x) = A$, we get c^1-coefficients and putting them in Eq (37), we get the $u_1(x)$ approximate solution, similarly for $r = 1$, solving linear system of equations (38), we obtain c^2-coefficients and we get the $u_2(x)$ and so on.

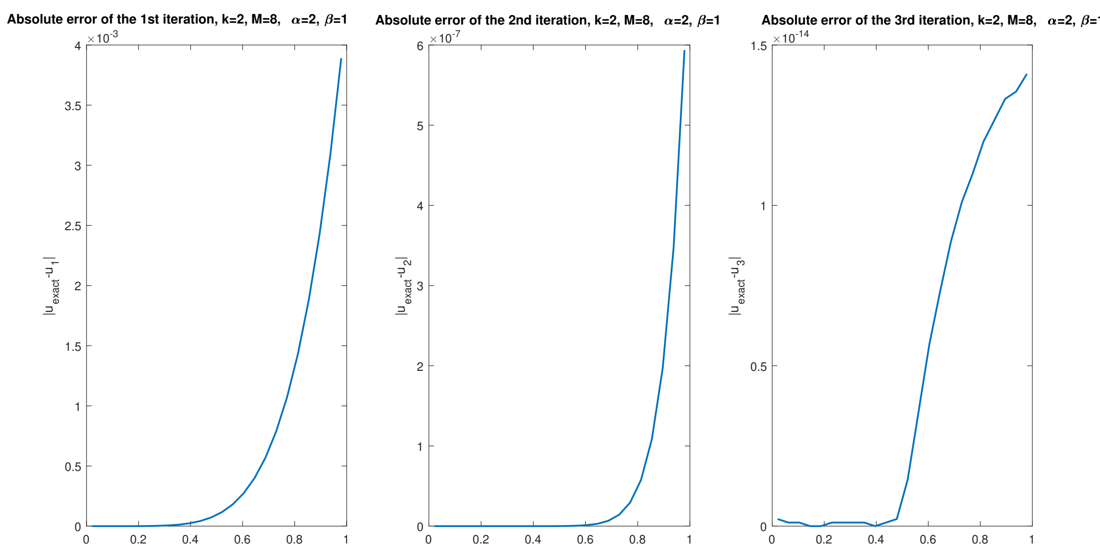
We implemented the presented method on Eq (31) for $k = 2, M = 8$. We plotted in Figure 2a the Chebyshev approximate solutions for different values of α, β, as can be seen, when the values α, β tend to $2, 1$ respectively, the approximate solutions approach the exact solution. Figure 2b shows that the absolute errors of 1st-3rd approximate solutions obtained using the method described above is a good approximation of the exact solution of the Eq (31), and indicate by increasing iterations, the absolute error decreasing. Table 2 shows the comparison of the error analysis for various methods, such as Adomian Decomposition Method (ADM) [54], Haar Wavelet Collocation Method (HWCM) [45], Haar Wavelet Adaptive Grid Method (HWAGM) [46], Haar Wavelet Collocation Adomian Method (HWCAM) [41]. According to the Table 2, we conclude that results produced by the present method are better than the other methods. In Tables 2, 4 and 6 N represents the dimension of the approximate solutions. For comparison our proposed method approximate solution with the Runge-Kutta of order 4, we set the Table 3, as can be seen our results much better than the Runge-Kutta's approximate solution. Also, we implemented the presented method on Eq (31) for $k = 1, M = 8$ and by solving the system (38) in the collocation points as $x_1 = \frac{1}{16}, x_2 = \frac{3}{16}, x_3 = \frac{5}{16}, \ldots, x_8 = \frac{15}{16}$, we get the $C_{n,m}^3$ coefficients as

$$C^3 = \begin{pmatrix} -0.194349084069466 \\ 0.070070088027989 \\ 0.007344030232925 \\ -0.004296362010761 \\ 0.000146805258182 \\ 0.000129579901339 \\ -0.000017329511545 \\ -0.000002478064867 \end{pmatrix},$$

replacing these values in Eq (37) and using the initial conditions, we get the following



(a)



(b)

Figure 2: (2a) u_{CWCQM} for different values of α, β with $k = 2, M = 8$, (2b) Absolute errors of different iterations for $\alpha = 2, \beta = 1, k = 2, M = 8$ (b) in Eq (31) $n = 5$.

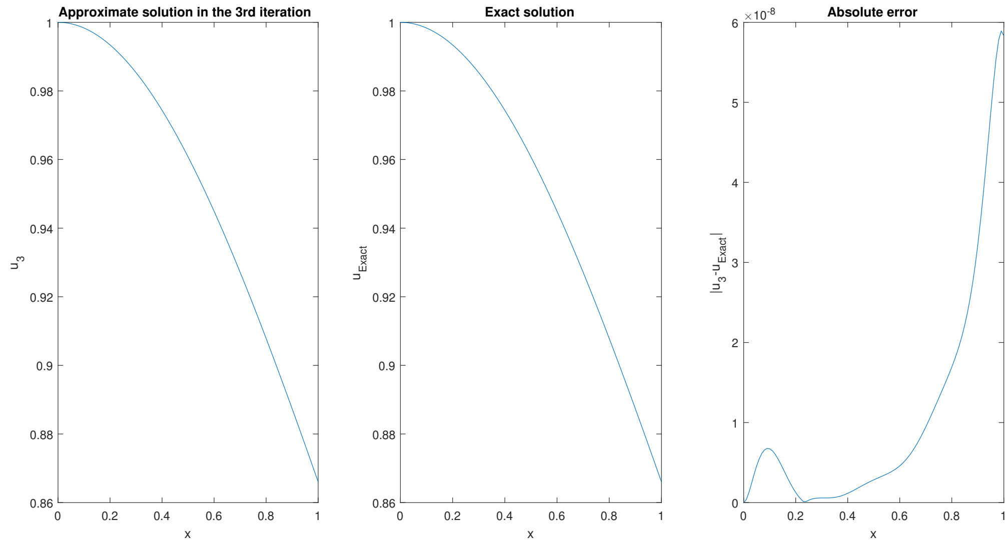


Figure 3: u_3, exact solution and the absolute error of the Example (5.1), $n = 5$ for $k = 1, M = 8$

approximate solution in the 3rd iteration

$$\begin{aligned} u_3(x) = &-0.00063629x^9 + 0.00143305x^8 + 0.00396832x^7 \\ &- 0.0149915x^6 + 0.001646x^5 + 0.0412078x^4 + 0.0000690821x^3 \\ &- 0.166671x^2 + 1. \end{aligned} \tag{39}$$

In the Figure 3, we plotted the Eq (39), exact solution and the absolute error in range $0 \leq x \leq 1$. Figure 3 shows that, the proposed method has a good performance dealing with the singularity in the point $x = 0$.

N	E_{ADM}[54] L_∞	E_{HWCM}[45] L_∞	E_{HWAGM}[23] L_∞	E_{HWCAM}[41] L_∞	E_{CWCQM} L_∞
8	$3.0591e-02$	$9.2374e-05$	$2.2745e-05$	$7.7048e-05$	$4.0104e-07$
16	$3.4652e-02$	$2.4231e-05$	$6.4165e-06$	$1.0573e-05$	$7.7340e-11$
32	$3.6814e-02$	$6.2101e-06$	$1.0121e-06$	$4.3726e-06$	$1.8416e-11$
64	$3.7929e-02$	$1.5723e-06$	$3.5372e-07$	$9.6483e-07$	$8.7315e-12$

Table 2: Comparison of the error analysis in Example (5.1), $n = 5$, $\alpha = 2, \beta = 1$

Example 5.2. Consider the following singular nonlinear fractional Lane-Emden equation:

$$D_x^\alpha u(x) + \frac{2}{x} D_x^\beta u(x) + 8e^{u(x)} + 4e^{\frac{u(x)}{2}} = 0, \qquad 0 < x \leq 1, \tag{40}$$

t	u_{CWCQM}	u_{Exact}	E_{CWCQM}	$u_{Runge-Kutta}$	$E_{Runge-Kutta}$
0.01562	$9.9996e-01$	$9.9996e-01$	$8.0000e-15$	$9.9996e-01$	$1.1263e-11$
0.04687	$9.9963e-01$	$9.9963e-01$	$3.0000e-15$	$9.9963e-01$	$4.0181e-10$
0.07812	$9.9898e-01$	$9.9898e-01$	0	$9.9898e-01$	$2.4146e-10$
0.10938	$9.9801e-01$	$9.9801e-01$	$1.0000e-15$	$9.9801e-01$	$2.0345e-10$
0.14063	$9.9672e-01$	$9.9672e-01$	$1.0000e-15$	$9.9672e-01$	$2.0641e-10$
0.17188	$9.9511e-01$	$9.9511e-01$	$2.0000e-15$	$9.9511e-01$	$3.6587e-10$
0.20313	$9.9319e-01$	$9.9319e-01$	$1.0000e-15$	$9.9319e-01$	$6.6344e-10$
0.23438	$9.9097e-01$	$9.9097e-01$	$7.0000e-15$	$9.9097e-01$	$9.6500e-10$
0.26563	$9.8844e-01$	$9.8844e-01$	$1.4000e-14$	$9.8844e-01$	$2.0501e-09$
0.29688	$9.8563e-01$	$9.8563e-01$	$4.0000e-14$	$9.8563e-01$	$6.4304e-13$
0.32813	$9.8252e-01$	$9.8252e-01$	$6.4000e-14$	$9.8252e-01$	$1.0550e-10$
0.35938	$9.7915e-01$	$9.7915e-01$	$8.2000e-14$	$9.7915e-01$	$5.3249e-09$
0.39063	$9.7550e-01$	$9.7550e-01$	$9.8000e-14$	$9.7550e-01$	$3.4451e-09$
0.42188	$9.7159e-01$	$9.7159e-01$	$1.1100e-13$	$9.7159e-01$	$3.6357e-09$
0.45313	$9.6744e-01$	$9.6744e-01$	$1.2100e-13$	$9.6744e-01$	$8.6139e-09$
0.48438	$9.6305e-01$	$9.6305e-01$	$1.3500e-13$	$9.6305e-01$	$7.2980e-09$
0.51563	$9.5843e-01$	$9.5843e-01$	$9.6000e-14$	$9.5843e-01$	$6.6067e-09$
0.54688	$9.5360e-01$	$9.5360e-01$	$1.8000e-14$	$9.5360e-01$	$1.2260e-08$
0.57813	$9.4856e-01$	$9.4856e-01$	$4.9000e-14$	$9.4856e-01$	$1.2013e-08$
0.60938	$9.4332e-01$	$9.4332e-01$	$1.0800e-13$	$9.4332e-01$	$9.9493e-09$
0.64063	$9.3790e-01$	$9.3790e-01$	$1.6200e-13$	$9.3790e-01$	$1.4632e-08$
0.67188	$9.3231e-01$	$9.3231e-01$	$2.1000e-13$	$9.3231e-01$	$1.7006e-08$
0.70313	$9.2656e-01$	$9.2656e-01$	$2.5300e-13$	$9.2656e-01$	$1.5520e-08$
0.73438	$9.2066e-01$	$9.2066e-01$	$2.8800e-13$	$9.2066e-01$	$1.5663e-08$
0.76563	$9.1463e-01$	$9.1463e-01$	$3.4200e-13$	$9.1463e-01$	$2.1368e-08$
0.79688	$9.0846e-01$	$9.0846e-01$	$4.0700e-13$	$9.0846e-01$	$2.1868e-08$
0.82813	$9.0218e-01$	$9.0218e-01$	$4.6500e-13$	$9.0218e-01$	$2.0381e-08$
0.85938	$8.9580e-01$	$8.9580e-01$	$5.1700e-13$	$8.9580e-01$	$2.1832e-08$
0.89063	$8.8932e-01$	$8.8932e-01$	$5.6400e-13$	$8.8932e-01$	$2.4842e-08$
0.92188	$8.8275e-01$	$8.8275e-01$	$6.0700e-13$	$8.8275e-01$	$2.5148e-08$
0.95313	$8.7611e-01$	$8.7611e-01$	$6.4400e-13$	$8.7611e-01$	$2.4392e-08$
0.98438	$8.6940e-01$	$8.6940e-01$	$6.7600e-13$	$8.6940e-01$	$2.5670e-08$

Table 3: Error analysis and comparison CWCQM and the Runge-Kutta of order 4 for Example (5.1), Eq (31) with $k = 3, M = 8$

with initial conditions: $u(0) = u^{'}(0) = 0$. Exact solution for the case where $\alpha = 2$ and $\beta = 1$ is $u(x) = -2ln(1 + x^2)$ [45].

t	u_{ADM}[54]	u_{VIM}[61]	u_{HWCM}[45]	u_{exact}[45]	u_{CWCQM}
$(\frac{1}{32})$	-0.001952	-0.001952	-0.001949	-0.001952	-0.001952
$(\frac{3}{32})$	-0.017501	-0.017501	-0.017504	-0.017501	-0.017501
$(\frac{5}{32})$	-0.048241	-0.048241	-0.048255	-0.048241	-0.048241
$(\frac{7}{32})$	-0.093483	-0.093483	-0.093513	-0.093483	-0.093483
$(\frac{9}{32})$	-0.152257	-0.152256	-0.152309	-0.152257	-0.152257
$(\frac{11}{32})$	-0.223376	-0.223378	-0.223454	-0.223376	-0.223376
$(\frac{13}{32})$	-0.305508	-0.305466	-0.305619	-0.305509	-0.305509
$(\frac{15}{32})$	-0.397247	-0.397080	-0.397399	-0.397253	-0.397253
$(\frac{17}{32})$	-0.497163	-0.496615	-0.497381	-0.497196	-0.497196
$(\frac{19}{32})$	-0.603819	-0.602281	-0.604194	-0.603967	-0.603967
$(\frac{21}{32})$	-0.715706	-0.711907	-0.716548	-0.716277	-0.716277
$(\frac{23}{32})$	-0.831008	-0.822627	-0.833257	-0.832944	-0.832944
$(\frac{25}{32})$	-0.947015	-0.930367	-0.953261	-0.952905	-0.952905
$(\frac{27}{32})$	-1.058866	-1.029114	-1.075621	-1.075224	-1.075224
$(\frac{29}{32})$	-1.157061	-1.109891	-1.199524	-1.199089	-1.199089
$(\frac{31}{32})$	-1.222860	-1.159399	-1.324275	-1.323804	-1.323804

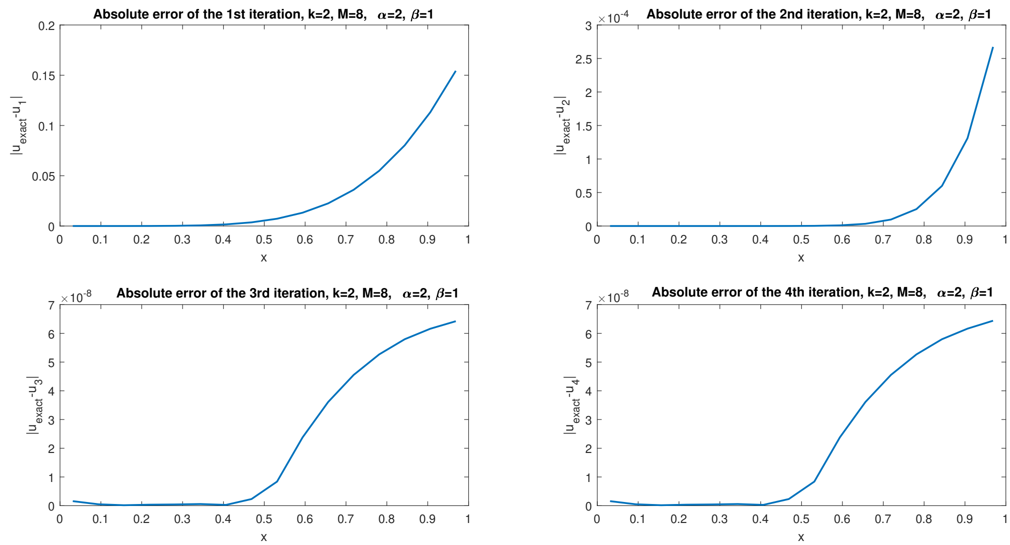
Table 4: Comparison of ADM, VIM, HWCM and CWCQM solutions with the Exact solution for $N = 16$ in Example 5.2, for $\alpha = 2, \beta = 1$

We plotted absolute errors of different iterations (1st-4th) in Figure 4a, as can be seen, by increasing iterations the absolute errors decrease, also Figure 4b shows that by increasing M approximate solutions are in a good coincidence with the exact solution, and the absolute errors are getting a decrease. The obtained numerical solution of the Example 5.2 is presented in comparison with the ADM, VIM, HWCM solutions and the exact solution in Table 4 for $(N = 16), k = 2, M = 8$. The error analysis for different values of α, β is given in Table 5, which shows that α, β-values tend to $2, 1$ respectively, L_∞ are getting decreases.

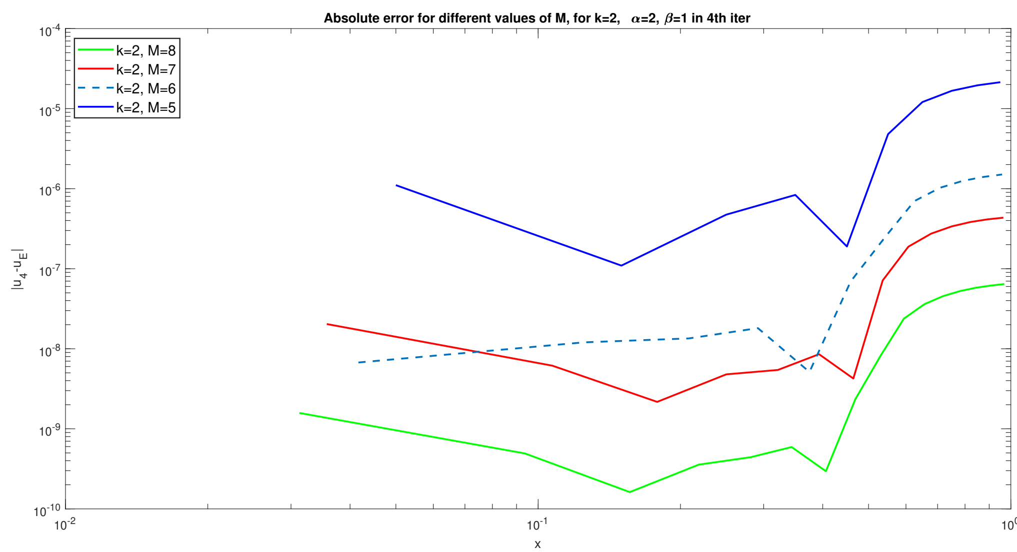
Example 5.3. Consider the fractional Lane-Emden equation:

$$D_x^\alpha u(x) + \frac{2}{x} D_x^\beta u(x) + e^{u(x)}(6 - 4x^2 e^{u(x)}) = 0, \qquad 0 < x \leq 1, \tag{41}$$

with initial conditions: $u(0) = -ln(4), u^{'}(0) = 0$. Exact solution for $\alpha = 2, \beta = 1$ is $u(x) = ln(\frac{1}{4+x^2})$ [48], [45].



(a)

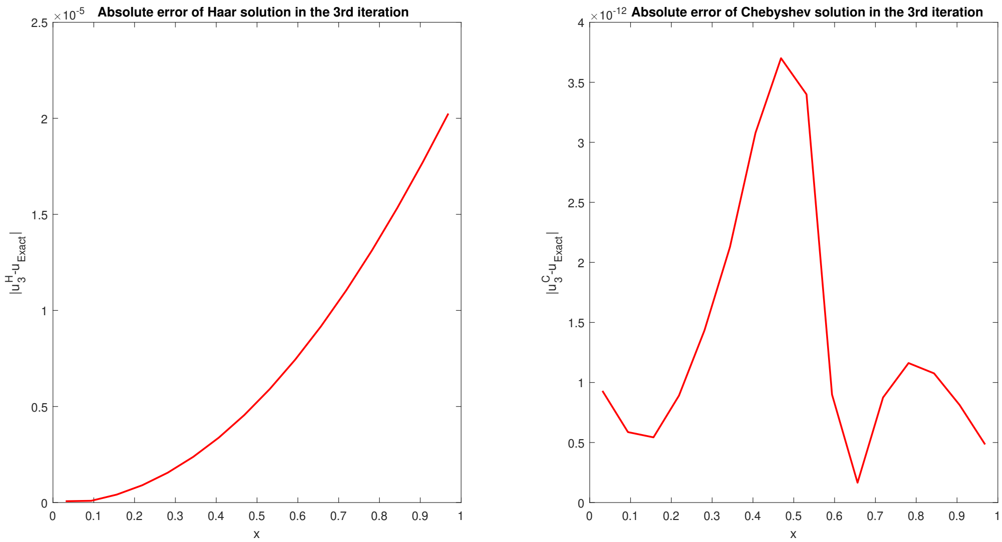


(b)

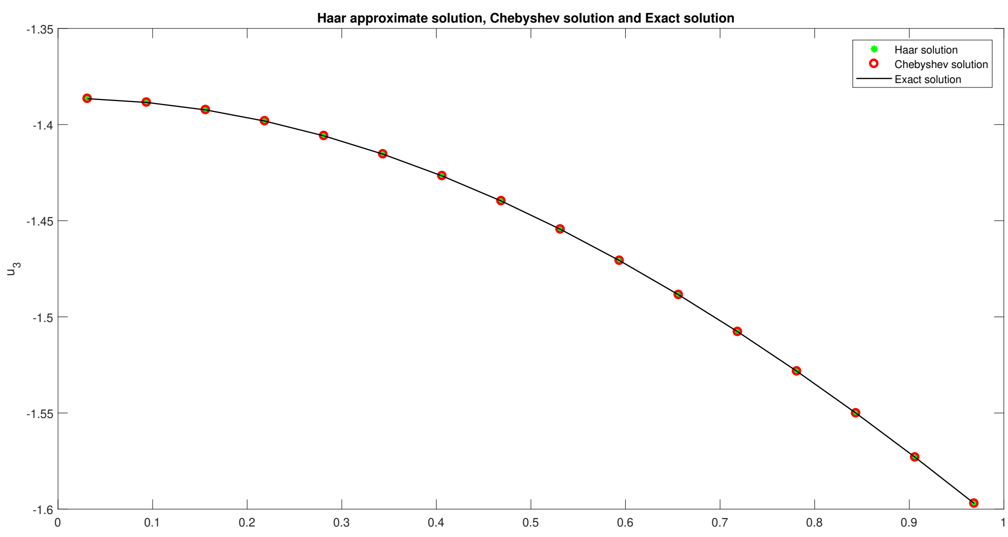
Figure 4: (4a) Absolute errors of different iterations for $\alpha = 2, \beta = 1, k = 2, M = 8$, (4b) Absolute error for different values of M with $k = 2, \alpha = 2, \beta = 1$ in Example 5.2

$\alpha = 1.7, \beta = 0.7$ L_∞	$\alpha = 1.85, \beta = 0.85$ L_∞	$\alpha = 1.95, \beta = 0.95$ L_∞	$\alpha = 2, \beta = 1$ L_∞
1.9768e-01	1.0128e-01	3.4345e-02	2.6207e-09

Table 5: Comparison of the error analysis in Example 5.2, for different values of α, β



(a)



(b)

Figure 5: (5a) Absolute errors of Haar and Chebyshev approximate solutions (3rd iteration), $J = 3$, for Haar, $k = 2, M = 8$ for Chebyshev approximate solutions, Haar and Chebyshev approximate solutions (3rd iteration) and Exact solution, (5b) $J = 3$, for Haar, $k = 2, M = 8$ for Chebyshev solutions in Example 5.3

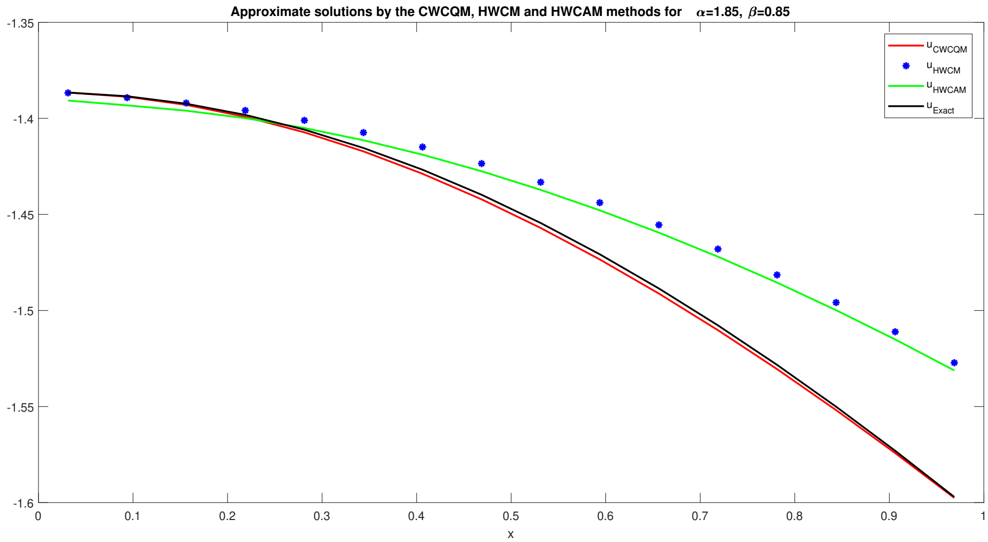


Figure 6: Approximate solutions of the CWCQM, HWCM and HWCAM methods for $\alpha = 1.85, \beta = 0.85$

N	8	16	32	64	128	256	512
HWCM[45] L_∞	8.94e-05	2.02e-05	5.22e-06	1.32e-06	3.34e-07	8.38e-08	2.10e-08
HWCM[48] L_∞	1.95e-03	4.89e-04	1.22e-04	3.06e-05	7.65e-06	1.91e-06	4.78e-07
CWCQM L_∞	5.32e-07	3.70e-12	1.25e-14	9.52e-16	4.23e-16	1.02e-16	8.62e-17

Table 6: Comparison of error analysis in Example 5.3 in the 3rd iteration, for $\alpha = 2, \beta = 1$

We implemented the present method for Eq (5.3) by using $u_0(x) = -ln(4)$ as initial approximation. Figure 5a shows the comparison of the absolute error for the Haar wavelet collocation method (HWCM) [45] and Chebyshev Wavelet Collocation Quasilinearization Method (CWCQM), as can be seen it had better approximate solutions than HWCM. We make a comparison between the results obtained by the CWCQM method and the Haar wavelet collocation method [45], [48] in Table 6. It can be seen from Table 6, the approximate solution obtained by the present method is in good coincidence accuracy with the exact solution in comparison of two other methods. In Figure 6, we plotted the approximate solutions of the CWCQM, HWCM and HWCAM methods for $\alpha = 1.85$ and $\beta = 0.85$, as can be seen the approximate solution obtained by the CWCQM has a better approximation than other methods.

Example 5.4. Consider the Lane-Emden equation of the fractional derivatives for a self-gravitating isothermal gas sphere [41]:

$$D_x^\alpha u(x) + \frac{2}{x} D_x^\beta u(x) - e^{u(x)} = 0, \qquad 0 < x \leq 1, \tag{42}$$

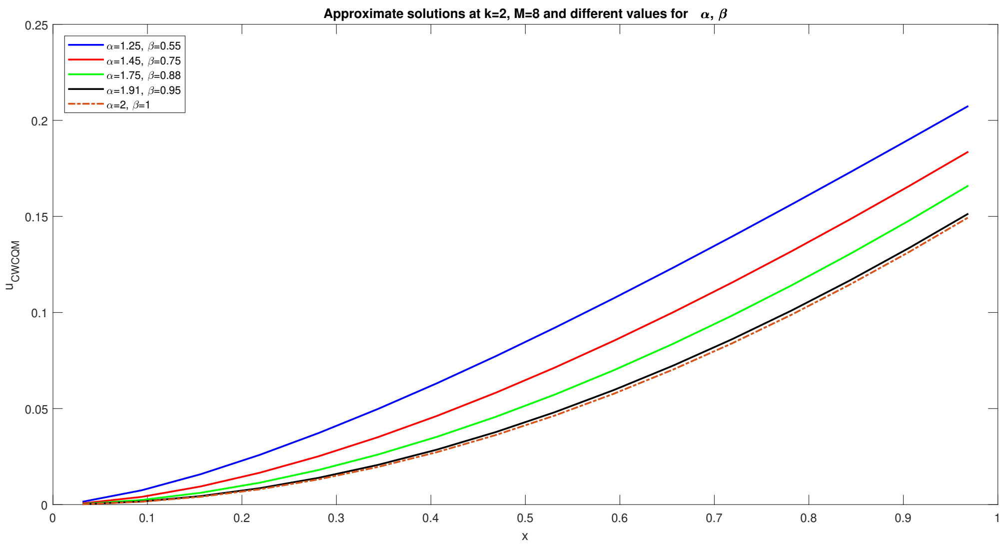


Figure 7: Approximate solutions of the isothermal gas spheres equation at different values of α, β in Example 5.4

with initial conditions: $u(0) = 0, u^{'}(0) = 0$, where $u(x)$ is the Newtonian gravitational potential function and x is the dimensionless radius [41]. We plotted the approximate solutions of different values of α, β obtained from the present method in Figure 7, as it can be seen, tending α, β to $2, 1$ respectively, approximate solutions approach the exact solution. Eq (42) is solved by the fractional approximation technique [31], a power series solution method [34], and Euler-transformed series [13]. In Table 7 exact solution represents the solution obtained from Runge-Kutta method of order 4 (this equation is devoid of an exact solution. To compare the proposed method with the others, we are going to consider the solution obtained by Runge-Kutta method of order 4 as the exact solution. It is noteworthy that it does not mean the Runge-Kutta method solution surpasses our method. We have just compared them). u_{HWCAM} is the obtained solution by the Haar Wavelet Collocation Adomian Method [41]. In [41], U. Saeed has solved Eq (42) by HWCAM in $J = 8, N = 512$ and the 23rd iteration. It can be seen from the Table 7, when $\alpha = 2, \beta = 1$, our approximate solutions are better than those obtained in [13], [31], [34], [41].

6 Conclusion

In this paper, we use the second-kind Chebyshev wavelet to obtain the approximate solutions of the Lane-Emden singular and nonlinear fractional differential equations. The matrices of the integral operator of the fractional order of the Chebyshev wavelets were used to convert the Lane-Emden equation to a linear system of algebraic equations. Solving the Lane-Emden singular and nonlinear fractional differential equations are difficult due to the singularity at the point $x = 0$. The proposed method is very convenient for

x	$u_{Rung-Kutta}$[41]	u_{HWCAM}[41]	u_{CWCQM}	$Mirza$[31]	$Nouh$[34]	$Hunter$[13]
0.1	0.001666	0.0016658	0.001666	0.0016	0.0166	0.0016
0.2	0.006653	0.0066534	0.006653	0.0066	0.0333	0.0065
0.3	0.014933	0.0149329	0.014933	0.0149	0.0500	0.0145
0.4	0.026455	0.0264555	0.026455	0.0266	0.0666	0.0253
0.5	0.041154	0.0411540	0.041154	0.0416	0.0833	0.0385
0.6	0.058944	0.0589441	0.058944	0.0598	0.1000	0.0536
0.7	0.079726	0.0797260	0.079726	0.0813	0.1166	0.0700
0.8	0.103386	0.1033861	0.103386	0.1060	0.1333	0.0870
0.9	0.129799	0.1297985	0.129799	0.1338	0.1500	0.1038
1.0	0.158828	0.1588277	0.158828	0.1646	0.1666	0.1198

Table 7: Comparison of the approximate solutions for the isothermal gas sphere equation when $\alpha = 2, \beta = 1$ in Example 5.4

solving the Lane-Emden fractional equations since the initial conditions are all considered during the process of constructing the approximate solutions. As was shown in Examples (5.1, 5.2, 5.3, 5.4), the method of the Chebyshev wavelet effectively and efficiently managed to solve the Lane-Emden equations of the fractional order. The results of this analysis are as follows:

1) The present method gives better accuracy in comparison with the other numerical methods [45], [48], [23]. Choose of small values of k, M gives better approximate solutions.

2) This method is applicable to all type of singular initial value problems.

3) This scheme is easy to implement in computer programs.

4) Unlike other methods (such as method [46]), this method does not require a long calculation to obtain a general order integration matrix.

References

[1] E. Babolian, F. FattahZadeh: Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration. *Appl. Math. Comput.* 188 (1) (2007) 417–426.

[2] E. Babolian, A.R. Vahidi, A. Shoja: An efficient method for nonlinear fractional differential equations: combination of the Adomian decomposition method and spectral method. *Indian J. Pure Appl. Math.* 45 (6) (2014) 1017–1028.

[3] R.E. Bellman, R.E. Kalaba: *Quasilinearization and nonlinear boundary-value problems*. Modern Analytic and Computional Methods in Science and Mathematics, Vol. 3. American Elsevier Publishing Co., New York (1965).

[4] M.M. Chawla, C.P. Katti: A finite-difference method for a class of singular two-point boundary-value problems. *IMA J. Numer. Anal.* 4 (4) (1984) 457–466.

[5] J.-F. Cheng, Y.-M. Chu: Solution to the linear fractional differential equation using Adomian decomposition method. *Math. Probl. Eng.* Article ID 587068 (2011) 14 pp.

[6] M. Danish, S. Kumar, S. Kumar: A note on the solution of singular boundary value problems arising in engineering and applied sciences: Use of OHAM. *Comput. Chem. Eng.* 36 (2012) 57–67.

[7] H.T. Davis: *Introduction to Nonlinear Differential and Integral Equations*. Dover Publications (2010).

[8] M. Dehghan, A. Saadatmandi: Chebyshev finite difference method for Fredholm integro-differential equation. *Int. J. Comput. Math.* 85 (1) (2008) 123–130.

[9] R.C. Duggan, A.M. Goodman: Pointwise bounds for a nonlinear heat conduction model of the human head. *Bltn. Mathcal. Biology* 48 (2) (1986) 229–236.

[10] A. Ebaid: A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method. *J. Comput. Appl. Math.* 235 (8) (2011) 1914–1924.

[11] V.S. Erturk, S. Momani, Z. Odibat: Application of generalized differential transform method to multi-order fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* 13 (8) (2008) 1642–1654.

[12] J.H. He, G.C. Wu, F. Austin: The variational iteration method which should be followed. *Nonl. Sci. Lett. A* 1 (1) (2010) 1–30.

[13] C. Hunter: Series solutions for polytropes and the isothermal sphere. *Mon. Not. R. Astron. Soc.* 328 (3) (2001) 839–847.

[14] M. Inc, M. Ergüt, Y. Cherruault: A different approach for solving singular two-point boundary value problems. *Kybernetes* 34 (7/8) (2005) 934–940.

[15] M.A. Iqbal, A. Ali, S.T. Mohyud-Din: Chebyshev Wavelets Method for Fractional Delay Differential Equations. *International Journal of Modern Applied Physics* 4 (1) (2013) 49–61.

[16] S.R.K. Iyengar, P. Jain: Spline finite difference methods for singular two point boundary value problems. *Numer. Math.* 50 (3) (1987) 363–376.

[17] P. Jamet: On the convergence of finite-difference approximations to one-dimensional singular boundary-value problems. *Numer. Math.* 14 (1969/1970) 355–378.

[18] M.K. Kadalbajoo, V. Kumar: B-spline method for a class of singular two-point boundary value problems using optimal grid. *Appl. Math. Comput.* 188 (2) (2007) 1856–1869.

[19] R. Kalaba: On nonlinear differential equations, the maximum operation and monotone convergence. *J. Math. Mech.* 8 (1959) 519–574.

[20] A.S.V. Ravi Kanth: Cubic spline polynomial for non-linear singular two-point boundary value problems. *Appl. Math. Comput.* 189 (2) (2007) 2017–2022.

[21] A.S.V. Ravi Kanth, Y.N. Reddy: A numerical method for singular two point boundary value problems via Chebyshev economization. *Appl. Math. Comput.* 146 (2-3) (2003) 691–700.

[22] A.S.V. Ravi Kanth, Y.N. Reddy: Cubic spline for a class of singular two-point boundary value problems. *Appl. Math. Comput.* 170 (2) (2005) 733–740.

[23] A.M.M. Khodier, A.Y. Hassan: One-dimensional adaptive grid generation. *Internat. J. Math. Math. Sci.* 20 (3) (1997) 577–584.

[24] M. Kumar: Higher order method for singular boundary-value problems by using spline function. *Appl. Math. Comput.* 192 (1) (2007) 175–179.

[25] R.A. Kycia, G. Filipuk: On the singularities of the Emden-Fowler type equations. *In: Current Trends in Analysis and Its Applications. Trends Math.* (2015) 93–99.

[26] H.J. Lane: On the theoretical temperature of the Sun; under the hypothesis of a gaseous mass maintaining its volume by its internal heat, and depending on the laws of gases as known to terrestrial experiment. *American Journal of Science and Arts* 50 (148) (1870) 57–74.

[27] X. Li, M. Xu, X. Jiang: Homotopy perturbation method to time-fractional diffusion equation with a moving boundary condition. *Appl. Math. Comput.* 208 (2) (2009) 434–439.

[28] Y. Li: Solving a nonlinear fractional differential equation using Chebyshev wavelets. *Commun. Nonlinear Sci. Numer. Simul.* 15 (9) (2010) 2284–2292.

[29] Y. Li, N. Sun, B. Zheng, Q. Wang, Y. Zhang: Wavelet operational matrix method for solving the Riccati differential equation. *Commun. Nonlinear Sci. Numer. Simul.* 19 (3) (2014) 483–493.

[30] V.B. Mandelzweig, F. Tabakin: Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs. *Comput. Phys. Comm.* 141 (2) (2001) 268–281.

[31] B.M. Mirza: Approximate analytical solutions of the Lane-Emden equation for a self-gravitating isothermal gas sphere. *Mon. Not. R. Astron. Soc.* 395 (4) (2009) 2288–2291.

[32] R. Yulita Molliq, M.S.M. Noorani, I. Hashim: Variational iteration method for fractional heat- and wave-like equations. *Nonlinear Anal. Real World Appl.* 10 (3) (2009) 1854–1869.

[33] S. Momani, Z. Odibat: Homotopy perturbation method for nonlinear partial differential equations of fractional order. *Phys. Lett. A* 365 (5-6) (2007) 345–350.

[34] M.I. Nouh: Accelerated power series solution of polytropic and isothermal gas spheres. *New Astronomy* 9 (6) (2004) 467–473.

[35] Z. Odibat, S. Momani, V.S. Erturk: Generalized differential transform method: Application to differential equations of fractional order. *Appl. Math. Comput.* 197 (2) (2008) 467–477.

[36] K. Parand, M. Dehghan, A.R. Rezaei, S.M. Ghaderi: An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method. *Comput. Phys. Comm.* 181 (6) (2010) 1096–1108.

[37] I. Podlubny: *Fractional Differential Equations*. Academic Press Inc., San Diego (1999).

[38] P. Rahimkhani, Y. Ordokhani, E. Babolian: A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations. *Numer. Algorithms* 74 (1) (2010) 223–245.

[39] G.W. Reddien: Projection methods and singular two point boundary value problems. *Numer. Math.* 21 (1973/74) 193–205.

[40] R.D. Russell, L.F. Shampine: Numerical methods for singular boundary value problems. *SIAM J. Numer. Anal.* 12 (1975) 13–36.

[41] U. Saeed: Haar Adomian method for the solution of fractional nonlinear Lane-Emden type equations arising in astrophysics. *Taiwanese J. Math.* 21 (5) (2017) 1175–1192.

[42] S. Saha Ray, R.K. Bera: An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method. *Appl. Math. Comput.* 167 (1) (2005) 561–571.

[43] C.N. Sam: *Numerical Solution of Partial Differential Equations with the Tau-collocation Method, Thesis (M.Phil.)*. City University of Hong Kong (2004).

[44] S.C. Shiralashetti, A.B. Deshi: An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations. *Nonlinear Dynam.* 83 (1-2) (2016) 293–303.

[45] S.C. Shiralashetti, A.B. Deshi, P.B. Mutalik Desai: Haar wavelet collocation method for the numerical solution of singular initial value problems. *Ain Shams Engineering Journal* 7 (2) (2016) 663–670.

[46] S.C. Shiralashetti, S. Kumbinarasaiah: Hermite wavelets operational matrix of integration for the numerical solution of nonlinear singular initial value problems. *Alexandria Engineering Journal* 57 (4) (2018) 2591–2600.

[47] R. Singh, N. Das, J. Kumar: The optimal modified variational iteration method for the Lane-Emden equations with Neumann and Robin boundary conditions. *Eur. Phys. J. Plus* 132 (2017) Article number 251.

[48] R. Singh, H. Garg, V. Guleria: Haar wavelet collocation method for Lane-Emden equations with Dirichlet, Neumann and Neumann-Robin boundary conditions. *J. Comput. Appl. Math.* 346 (2019) 150–161.

[49] R. Singh, J. Kumar: An efficient numerical technique for the solution of nonlinear singular boundary value problems. *Comput. Phys. Commun.* 185 (4) (2014) 1282–1289.

[50] R. Singh, J. Kumar, G. Nelakanti: Numerical solution of singular boundary value problems using Green's function and improved decomposition method. *J. Appl. Math. Comput.* 43 (1-2) (2013) 409–425.

[51] R. Singh, S. Singh, A.-M. Wazwaz: A modified homotopy perturbation method for singular time dependent Emden-Fowler equations with boundary conditions. *J. Math. Chem.* 54 (4) (2016) 918–931.

[52] L. Wang, Y. Ma, Z. Meng: Haar wavelet method for solving fractional partial differential equations numerically. *Appl. Math. Comput.* 227 (2014) 66–76.

[53] Y. Wang, L. Zhu: Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method. *Adv. Difference Equ.* (paper no. 27) (2017) 16 pp.

[54] A.-M. Wazwaz: Adomian decomposition method for a reliable treatment of the Emden-Fowler equation. *Appl. Math. Comput.* 161 (2) (2005) 543–560.

[55] A.-M. Wazwaz: Analytical solution for the time-dependent Emden-Fowler type of equations by Adomian decomposition method. *Appl. Math. Comput.* 166 (3) (2005) 638–651.

[56] A.-M. Wazwaz: A reliable iterative method for solving the time-dependent singular Emden-Fowler equations. *Cent. Eur. J. Eng.* 3 (1) (2013) 99–105.

[57] A.-M. Wazwaz: *Partial Differential Equations and Solitary Waves Theory*. Nonlinear Physical Science. Higher Education Press, Beijing; Springer, Berlin (2009).

[58] A.-M. Wazwaz: Solving the non-isothermal reaction-diffusion model equations in a spherical catalyst by the variational iteration method. *Chem. Phys. Lett.* 679 (2017) 132–136.

[59] A.-M. Wazwaz: The variational iteration method for solving nonlinear singular boundary value problems arising in various physical models. *Commun. Nonlinear Sci. Numer. Simul.* 16 (10) (2011) 3881–3886.

[60] A.-M. Wazwaz, R. Rach: Comparison of the Adomian decomposition method and the variational iteration method for solving the Lane-Emden equations of the first and second kinds. *Kybernetes* 40 (9-10) (2011) 1305–1318.

[61] A. Yıldırım, T. Öziş: Solutions of singular IVPs of Lane-Emden type by the variational iteration method. *Nonlinear Anal.* 70 (6) (2009) 2480–2484.

[62] F. Zhou, X. Xu: Numerical solution of the convection diffusion equations by the second kind Chebyshev wavelets. *Appl. Math. Comput.* 247 (2014) 353–367.

[63] F. Zhou, X. Xu: Numerical solution of time-fractional diffusion-wave equations via Chebyshev wavelets collocation method. *Adv. Math. Phys.* (article ID 2610804) (2017) 17 pp.

[64] L. Zhu, Q. Fan: Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet. *Commun. Nonlinear Sci. Numer. Simul.* 17 (6) (2012) 2333–2341.

Received: May 06, 2019
Accepted for publication: August 18, 2019
Communicated by: Diana Barseghyan